<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>Notes on a certain class of analytic functions (Some inequalities concerned with the geometric function theory)</td>
</tr>
<tr>
<td>著者</td>
<td>Nishiwaki, Junichi; Owa, Shigeyoshi</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 (2014), 1878: 67-73</td>
</tr>
<tr>
<td>発行日</td>
<td>2014-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195602">http://hdl.handle.net/2433/195602</a></td>
</tr>
<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>テキストフォーマット</td>
<td>publisher</td>
</tr>
<tr>
<td>部門</td>
<td>京都大学</td>
</tr>
<tr>
<td>発行機関</td>
<td>京都大学</td>
</tr>
<tr>
<td>電子版</td>
<td>有</td>
</tr>
</tbody>
</table>
Notes on a certain class of analytic functions

Junichi Nishiwaki and Shigeyoshi Owa

Abstract

Let \( \mathcal{A} \) be the class of analytic functions \( f(z) \) in the open unit disk \( \mathbb{U} \). Furthermore, the subclass \( \mathcal{B} \) of \( \mathcal{A} \) concerned with the class of uniformly convex functions or the class \( \mathcal{S}_p \) is defined. By virtue of some properties of uniformly convex functions and the class \( \mathcal{S}_p \), an extreme function of the class \( \mathcal{B} \) and its power series are considered.

1 Introduction

Let \( \mathcal{A} \) be the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \). A function \( f(z) \in \mathcal{A} \) is said to be in the class of uniformly convex (or starlike) functions denoted by \( \mathcal{UCV} \) (or \( \mathcal{UST} \)) if \( f(z) \) is convex (or starlike) in \( \mathbb{U} \) and maps every circle or circular arc in \( \mathbb{U} \) with center at \( \zeta \) in \( \mathbb{U} \) onto the convex arc (or the starlike arc with respect to \( f(\zeta) \)). These classes are introduced by Goodman [1] (see also [2]). For the class \( \mathcal{UCV} \), it is defined as the one variable characterization by Rønning [4] and [5], that is, a function \( f(z) \in \mathcal{A} \) is said to be in the class \( \mathcal{UCV} \) if it satisfies

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}).
\]

It is independently studied by Ma and Minda [3]. But the one variable characterization for the class \( \mathcal{UST} \) is still open. Further, a function \( f(z) \in \mathcal{A} \) is said to be the corresponding class denoted by \( \mathcal{S}_p \) if it satisfies

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).
\]

This class \( \mathcal{S}_p \) was introduced by Rønning [4]. We easily know that the relation \( f(z) \in \mathcal{UCV} \) if and only if \( zf'(z) \in \mathcal{S}_p \). In view of these classes, we introduce the subclass \( \mathcal{B} \) of \( \mathcal{A} \) consisting

\[2010 \text{ Mathematics Subject Classification:} \text{ Primary 30C45}
\]

Keywords and Phrases: Analytic function, uniformly convex function, extreme function, power series.
of all functions \( f(z) \) which satisfy
\[
\Re \left( \frac{z}{f(z)} \right) > \left| \frac{z}{f(z)} - 1 \right| \quad (z \in \mathbb{U}).
\]

We try to derive some properties of functions \( f(z) \) belonging to the class \( \mathcal{B} \).

**Remark 1.1.** For \( f(z) \in \mathcal{B} \), we write \( w(z) = \frac{f(z)}{z} = u + iv \), then \( w \) lies in the domain which is the part of the complex plane which contains \( w = 1 \) and is bounded by a kind of teardrop-shape domain such that
\[
u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0.
\]

**Example 1.1.** Let us consider the function \( f(z) \in \mathcal{A} \) as given by
\[
f(z) = z + \frac{1}{\sqrt{2}}z^2.
\]
Then we easily see that the function \( f(z) \) is not univalent. And \( \frac{f(z)}{z} \) maps \( \mathbb{U} \) onto the circular domain which is 1 as the center and \( \frac{1}{\sqrt{2}} \) as the radius, that is, \( f(z) \in \mathcal{B} \).

## 2 An extreme function for the class \( \mathcal{B} \)

In this section, we would like to exhibit an extreme function of the class \( \mathcal{B} \) and its power series. For our results, we need to recall here some properties of the class \( \mathcal{S}_p \).

**Lemma 2.1.** (Rønning [4]). The extremal function \( f(z) \) for the class \( \mathcal{S}_p \) is given by
\[
\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2.
\]
By using the expansion of logarithmic part of \( \frac{zf'(z)}{f(z)} \) in Lemma 2.1, we get

**Lemma 2.2.** (Ma and Minda [3]). The power series of \( \frac{zf'(z)}{f(z)} \) is following
\[
\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2
\]
\[
= 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2k - 1} \right) z^n.
\]
The digamma function $\psi(z+1)$ is defined by

$$\psi(z+1) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \psi(z) + \frac{1}{z},$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{t}dt.$$

When $z$ is natural number, we obtain

$$\psi(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma \quad (n \in \mathbb{N}),$$

where $\gamma$ is Euler's constant and $-\gamma = \psi(1)$.

From Remark 1.1 and Lemma 2.1, we have the first result for the class $\mathcal{B}$.

**Theorem 2.1.** The extreme function $f(z)$ for the class $\mathcal{B}$ is given by

$$f(z) = \frac{z}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2}.$$

**Proof.** Let us consider the function $\frac{f(z)}{z}$ as given by

$$\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2}.$$

It suffices to show that $\frac{f(z)}{z}$ maps $\mathbb{U}$ onto the interior of the domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0,$$

implying that $\frac{f(z)}{z}$ maps the unit circle onto the boundary of the domain. Taking $z = e^{i\theta}$, we obtain that

$$\frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}} \right) \right)^2}$$

$$= \frac{1}{1 + \frac{2}{\pi^2} \left( \log i - \log \left( \tan \frac{\theta}{4} \right) \right)^2}.$$
\[
= \frac{1}{\frac{1}{2} + \frac{2}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 - i \frac{2}{\pi} \log \left( \tan \frac{\theta}{4} \right)}
\]

\[
= \frac{1}{\frac{1}{4} + \frac{6}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^4 + i \frac{2}{\pi} \log \left( \tan \frac{\theta}{4} \right)}
\]

Writing \( f(z) = u + iv \), we see that

\[
\log \left( \tan \frac{\theta}{4} \right) = \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}.
\]

Thus we have

\[
v = \frac{2}{\pi} \log \left( \tan \frac{\theta}{4} \right)
\]

\[
= \frac{1 + \frac{6}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^4}{\frac{1}{4} + \frac{6}{\pi^2} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^2 + \frac{4}{\pi^4} \left( \log \left( \tan \frac{\theta}{4} \right) \right)^4}.
\]

Therefore, we arrive that

\[
u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 = 0.
\]

This completes the proof of the theorem.

Considering the power series of the function \( f(z) \) in Theorem 2.1, we derive

**Theorem 2.2.** The power series of the extreme function for the class \( B \) is given by

\[
f(z) = \frac{z}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}
\]

\[
= z + \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{j=1}^{p} \left( \prod_{k=1}^{m_j} \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k - 1} \right) z^n \quad (m_j \in \mathbb{N}).
\]
Proof. Let us suppose that

\[
\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2} \left( \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2}
\]

as the proof of Theorem 2.1. Then from Lemma 2.2, we have

\[
\frac{f(z)}{z} = \frac{1}{1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2k-1} \right) z^n}
\]

\[
= 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2k-1} \right) z^n + \left( \frac{8}{\pi^2} \right)^2 \left\{ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2k-1} \right) z^n \right\}^2
\]

\[
- \left( \frac{8}{\pi^2} \right)^3 \left\{ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2k-1} \right) z^n \right\}^3 + \cdots
\]

\[
+ (-1)^n \left( \frac{8}{\pi^2} \right)^n \left\{ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2k-1} \right) z^n \right\}^n + \cdots
\]

\[
= 1 - \frac{8}{\pi^2} \left( \frac{1}{1} \sum_{k=1}^{1} \frac{1}{2k-1} \right) z
\]

\[
+ \left\{ -\frac{8}{\pi^2} \left( \frac{1}{2} \sum_{k=1}^{2} \frac{1}{2k-1} \right) + \left( \frac{8}{\pi^2} \right)^2 \left( \frac{1}{1} \sum_{k=1}^{1} \frac{1}{2k-1} \right) \left( \frac{1}{1} \sum_{k=1}^{1} \frac{1}{2k-1} \right) \right\} z^2
\]

\[
+ \left\{ -\frac{8}{\pi^2} \left( \frac{1}{3} \sum_{k=1}^{3} \frac{1}{2k-1} \right) + \left( \frac{8}{\pi^2} \right)^2 \left\{ \left( \frac{1}{1} \sum_{k=1}^{1} \frac{1}{2k-1} \right) \left( \frac{1}{2} \sum_{k=1}^{1} \frac{1}{2k-1} \right)
\]

\[
+ \left( \frac{1}{2} \sum_{k=1}^{1} \frac{1}{2k-1} \right) \left( \frac{1}{1} \sum_{k=1}^{1} \frac{1}{2k-1} \right) \right\} - \left( \frac{8}{\pi^2} \right)^3 \left( \frac{1}{1} \sum_{k=1}^{1} \frac{1}{2k-1} \right)^3 \right\} z^3
\]

\[
+ \cdots
\]

\[
+ \left\{ -\frac{8}{\pi^2} \sum_{j=1}^{\infty} \left( \frac{1}{m_j} \sum_{j=1}^{m_j} \frac{1}{2k-1} \right) + \left( \frac{8}{\pi^2} \right)^2 \sum_{j=1}^{\infty} \left( \frac{1}{m_j} \sum_{j=1}^{m_j} \frac{1}{2k-1} \right) \right\} z^2
\]

\[
+ \left( \frac{8}{\pi^2} \right)^3 \sum_{j=1}^{\infty} \left( \frac{1}{m_j} \sum_{j=1}^{m_j} \frac{1}{2k-1} \right) + \cdots + \left( \frac{8}{\pi^2} \right)^p \sum_{j=1}^{\infty} \left( \frac{1}{m_j} \sum_{j=1}^{m_j} \frac{1}{2k-1} \right)
\]

\[
+ \cdots + \left( \frac{8}{\pi^2} \right)^n \sum_{j=1}^{\infty} \left( \frac{1}{m_j} \sum_{j=1}^{m_j} \frac{1}{2k-1} \right) \right\} z^n + \cdots \quad (m_j \in \mathbb{N})
\]
\[= 1 - \frac{8}{\pi^2} \left( \frac{1}{1} \sum_{k=1}^{1} \frac{1}{2k-1} \right) z + \sum_{p=1}^{2} (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^{p} m_j = 2} \left( \prod_{j=1}^{p} \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^2 \]

\[
+ \sum_{p=1}^{3} (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^{p} m_j = 3} \left( \prod_{j=1}^{p} \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^3 + \ldots
\]

\[
+ \sum_{p=1}^{n} (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^{p} m_j = n} \left( \prod_{j=1}^{p} \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^n + \ldots
\]

\[= 1 + \sum_{n=1}^{\infty} \sum_{p=1}^{n} (-1)^p \left( \frac{8}{\pi^2} \right)^p \sum_{\sum_{j=1}^{p} m_j = n} \left( \prod_{j=1}^{p} \frac{1}{m_j} \sum_{k=1}^{m_j} \frac{1}{2k-1} \right) z^n.
\]

This completes the proof of the theorem. \(\square\)

By using digamma function in Theorem 2.2, we have

**Corollary 2.1.** The power series of the extreme function for the class \(B\) is rewritten as following

\[f(z) = z + \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} (-1)^p \left( \frac{8}{\pi^2} \right)^p \times
\]

\[
\sum_{\sum_{j=1}^{p} m_j = n-1} \left\{ \prod_{j=1}^{p} \frac{1}{m_j} \left( \psi(m_j + 1) - \frac{1}{2} \psi([m_j/2] + 1) - \frac{1}{2} \psi(1) \right) \right\} z^n \quad (m_j \in \mathbb{N}),
\]

where \([\cdot]\) is the Gauss symbol

**References**


Junichi Nishiwaki  
Department of Mathematics and Physics  
Setsunan University  
Neyagawa, Osaka 572-8508 Japan  
email:jerjun2002@yahoo.co.jp

Shigeyoshi Owa  
Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577-8502 Japan  
email:shige21@ican.zaq.ne.jp