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SPREADING-VANISHING DICHOTOMY IN NONLINEAR STEFAN PROBLEMS

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ABSTRACT. We report some recent results on nonlinear Stefan problems used to describe the spreading of an invasive species. Research on this rather new topic is fast progressing. It is hoped that this brief survey helps the interested reader to gain a better view of the current status of research in this area.

Dedicated to Professor Hiroshi Matano

1. INTRODUCTION

In this paper, we give a brief review of some recent results on nonlinear Stefan problems, which reveal an interesting spreading-vanishing phenomenon. We start from the first paper in this direction [8], and include several results whose proofs are contained in preprints only. We will end with some discussions of possible future directions along this line of research.

The problem considered in [8] has the following form:

$$
(1.1)\qquad \begin{cases} u_t - du_{xx} = u(a-bu), & t>0,\ 0<x<h(t), \\ u_x(t,0)=0,\ u(t,h(t))=0, & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0)=h_0, u(0,x)=u_0(x), & 0\le x\le h_0, \end{cases}
$$

where $x=h(t)$ is the moving boundary to be determined together with $u(t,x)$, h_0, μ, d, a and b are given positive constants, and the initial function $u_0(x)$ satisfies

$$
(1.2)\qquad u_0\in C^2([0,h_0]),\ u_0'(0)=u_0(h_0)=0, \quad u_0>0 \text{ in } [0,h_0).
$$

The intention of [8] was to better understand the spreading of invasive species. A systematic discussion of earlier mathematical models for ecological invasion can be found in [22]. If $u(t,x)$ stands for the population density of an invasive species, then (1.1) may be interpreted as describing the spreading of u over a one dimensional environment. The initial function $u_0(x)$ stands for the population of the invading species at a very early stage, which occupies an initial region $[0,h_0]$. It is assumed that the species can only invade further into the environment from the right end of the initial region, and the spreading front expands at a speed that is proportional to the population gradient at the front, which gives rise to the Stefan condition $h'(t)=-\mu u_x(t,h(t))$. A deduction of this condition based on some ecological assumptions can be found in [3].

If the right hand side of the differential equation in (1.1) is replaced by 0, this problem becomes the well-known one-phase Stefan problem (in one space dimension), which

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describes the meting of ice in contact with water, where the free boundary represents the ice-water interphase, and u stands for the water temperature. A nonlinear Stefan problem may arise if water is replaced by a chemically reactive and heat diffusive liquid. Therefore the research on nonlinear Stefan problems may also be useful in such situations.

It was shown in [8] that (1.1) has a unique solution $(u(t,x), h(t))$ defined for all $t > 0$, with $u(t,x) > 0$ and $h'(t) > 0$. The long-time dynamical behavior of (1.1) is characterized by a **spreading-vanishing dichotomy**: as time $t \to \infty$, either

(i) (spreading) $h(t) \to \infty$ and $u(t,x) \to a/b$, or

(ii) (vanishing) $h(t) \to h_\infty \le \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and $u(t,x) \to 0$.

Furthermore, when spreading occurs, for large time, the spreading speed approaches a positive constant $k_0 \in (0, 2\sqrt{ad})$, i.e., $h(t) = [k_0 + o(1)]t$ as $t \to \infty$. It is shown that vanishing indeed happens when $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and $\mu > 0$ is less than a certain positive threshold value μ^*, depending on u_0.

If the left boundary $x = 0$ in (1.1) is replaced by a free boundary $x = g(t)$ governed by $g'(t) = -\mu u_x(t, g(t))$, it was proved in [8] that a similar spreading-vanishing dichotomy holds, and in the case of spreading, both the left front $x = g(t)$ and the right front $x = h(t)$ go to infinity at the same asymptotic speed k_0.

This spreading-vanishing phenomenon of (1.1) is strikingly different from the result obtained via the usual approach to describe the front propagation, where the Cauchy problem of the following diffusive logistic equation over the entire space $\mathbb{R}^1$ is used:

$$u_t - d\Delta u = u(a - bu),\ t > 0,\ x \in \mathbb{R}^1. \tag{1.3}$$

In the pioneering works of Fisher [13] and Kolmogorov et al [19], traveling wave solutions have been found for (1.3): For any $c \ge c^* := 2\sqrt{ad}$, there exists a solution $u(t,x) := W(x - ct)$ with the property that

$$W'(y) < 0 \text{ for } y \in \mathbb{R}^1,\ W(-\infty) = a/b,\ \ W(+\infty) = 0;$$

no such solution exists if $c < c^*$. The number c^* is called the minimal speed of the traveling waves. Fisher [13] claims that c^* is the spreading speed for the advantageous genes in his research, and used a probabilistic argument to support his claim. Skellam [23] was able to use a linear model (i.e., (1.3) with $b = 0$) and a similar probabilistic argument to show that c^* should be the speed of spreading. A precise description and rigorous proof of this fact were given by Aronson and Weinberger (see Section 4 in [1]), who showed that for a new population $u(t,x)$ (governed by the above logistic equation) with initial distribution $u(0,x)$ confined to a compact set of x (i.e., $u(0,x) = 0$ outside a compact set), one has

$$\lim_{t\to\infty,\ |x| \le (c^*-\epsilon)t} u(t,x) = a/b, \qquad \lim_{t\to\infty,\ |x| \ge (c^*+\epsilon)t} u(t,x) = 0$$

for any small $\epsilon > 0$. (This result is also true in higher dimensions; see [2].)

The above result indicates that in the long run, the invading species u determined by the Cauchy problem of (1.3) always establishes itself in the new environment, with an asymptotic spreading speed c^*. In other words, spreading always happens.

In the sections below, we report several extensions of the results of [8]. In section 2, we discuss the case of high space dimension with radial symmetry and spatial-temporal inhomogeneity. In section 3 we consider the situation of more general nonlinearities. In section 4, we focus on the case of high space dimension with no radial symmetry, where the regularity of the free boundary becomes an important issue. Finally, in section 5 we briefly mention some recent works on multi-species systems with free boundary and some possible directions of future research.

2. High space dimension with radial symmetry and heterogeneity

2.1. The spatially heterogeneous case. In [4], the following generalization of (1.1) was considered:

$$\begin{cases} u_t - d\Delta u = u(\alpha(r) - \beta(r)u), & t > 0,\ 0 < r < h(t), \\ u_r(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0, u(0,r) = u_0(r), & 0 \le r \le h_0, \end{cases} \tag{2.1}$$

where $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, $r = h(t)$ is the moving boundary, h_0, μ and d are given positive constants, $\alpha, \beta \in C^{\nu_0}([0,\infty))$ for some $\nu_0 \in (0,1)$, and there are positive constants $\kappa_1 \le \kappa_2$ such that

$$\kappa_1 \le \alpha(r) \le \kappa_2, \quad \kappa_1 \le \beta(r) \le \kappa_2 \quad \text{for } r \in [0,\infty). \tag{2.2}$$

The initial function $u_0(r)$ satisfies

$$u_0 \in C^2([0,h_0]),\ u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \text{ in } [0,h_0). \tag{2.3}$$

This describes the situation that the solution u is radially symmetric ($u = u(t,r)$, $r = |x|$, $x \in \mathbb{R}^N$, $N \ge 2$) and the environment may vary in space (but radially symmetrically). The main features of (1.1) are retained in this case.

Theorem 2.1 (Existence and uniqueness).[4] *Problem* (2.1) *has a unique solution* $(u(t,r),h(t))$, *which is defined for all* $t > 0$. *Moreover,* $u(t,r) > 0$, $h'(t) > 0$ *for* $t > 0$ *and* $0 \le r < h(t)$, *and* $h \in C^1([0,\infty))$, $u \in C^{1,2}(D)$, *with* $D = \{(t,r) : t > 0, 0 \le r \le h(t)\}$.

It follows that $r = h(t)$ is monotonic increasing and therefore there exists $h_\infty \in (0,+\infty]$ such that $\lim_{t\to+\infty} h(t) = h_\infty$.

Let $\lambda_1(d,\alpha,R)$ be the principal eigenvalue of the problem

$$\begin{cases} -d\Delta\phi = \lambda\alpha(|x|)\phi & \text{in } B_R \\ \phi = 0 & \text{on } \partial B_R. \end{cases} \tag{2.4}$$

It is well-known that $\lambda(d,\alpha,\cdot)$ is a strictly decreasing continuous function and

$$\lim_{R\to0^+} \lambda_1(d,\alpha,R) = +\infty, \quad \lim_{R\to+\infty} \lambda_1(d,\alpha,R) = 0.$$

Therefore, for fixed $d > 0$ and $\alpha \in C^{\nu_0}([0,\infty))$, there is a unique $R^* := R^*(d,\alpha)$ such that

$$\lambda_1(d,\alpha,R^*) = 1 \tag{2.5}$$

and

$$1 > \lambda_1(d,\alpha,R) \text{ for } R > R^*; \quad 1 < \lambda_1(d,\alpha,R) \text{ for } R < R^*.$$

Theorem 2.2 (Spreading-vanishing dichotomy).[4] *Let* $(u(t,r),h(t))$ *be the solution of the free boundary problem* (2.1). *Then the following alternative holds:*

Either

(i) Spreading: $h_\infty = +\infty$ *and*

$$\lim_{t\to+\infty} u(t,r) = \hat{U}(r) \text{ locally uniformly for } r \in [0,\infty),$$

where $\hat{U}$ *is the unique positive solution of*

$$-d\Delta\hat{U} = \hat{U}(\alpha(r) - \beta(r)\hat{U}) \text{ for } r \in (0,\infty),\ \hat{U}'(0) = 0,$$

or

(ii) Vanishing: $h_\infty \le R^*$ *and* $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([0,h(t)])} = 0$.

Theorem 2.3 (Spreading-vanishing criteria).[4] *If $h_0 \ge R^*$, then spreading always happens. If $h_0 < R^*$, then there exists $\mu^* > 0$ depending on u_0 such that vanishing occurs if $\mu \le \mu^*$, and spreading happens if $\mu > \mu^*$.*

When spreading happens, the following limits

$$\liminf_{t\to\infty} \frac{h(t)}{t}, \ \limsup_{t\to\infty} \frac{h(t)}{t}$$

can be estimated by making use of the numbers α_∞, α^∞, β_∞ and β^∞, defined below,

$$\alpha_\infty := \liminf_{r\to\infty} \alpha(r), \ \alpha^\infty := \limsup_{r\to\infty} \alpha(r),$$

$$\beta_\infty := \liminf_{r\to\infty} \beta(r), \ \beta^\infty := \limsup_{r\to\infty} \beta(r),$$

and the asymptotic spreading speed of (1.1).

The asymptotic spreading speed of (1.1) is determined by the associated *semi-waves*. Let us recall that for each $c \ge c^* := 2\sqrt{ad}$, the following problem

$$-dw'' - cw' = aw - bw^2, \ w(-\infty) = a/b, \ w(\infty) = 0$$

has a unique solution $w(x)$ (up to translation in x), and moreover, $w'(x) < 0$ for all x. If $c < c^*$, then no such solution exists. Such a solution is called a traveling wave with speed c because $u(t,x) := w(x - ct)$ satisfies

$$u_t - du_{xx} = au - bu^2 \text{ for all } t, \ x \in \mathbb{R}^1, \tag{2.6}$$

and as t increases, the curve $u = u(t,x)$ in the ux-plane resembles a wave which does not change its shape but travels to the right at speed c. It is well known (see [2]) that c^* is the asymptotic spreading speed of the solution to the Cauchy problem of (2.6) with initial function $u(0,x) = u_0(x)$ that is nonnegative, not identically zero, and with compact support.

The semi-waves are determined by the following problem over the half line:

$$-dV'' - kV' = aV - bV^2 \quad \text{in } (-\infty,0), \quad V(-\infty) = a/b, \ V(0) = 0. \tag{2.7}$$

If V is a solution to (2.7), then clearly $v(t,x) := V(x - kt)$ satisfies

$$v_t - dv_{xx} = av - bv^2 \text{ for } t \in \mathbb{R}^1, x < kt; \ v(t,kt) = 0.$$

We will call V a **semi-wave**, since as t increases the graph of the curve $v = v(t,x)$, which is defined on the half line $x < kt$, resembles a wave traveling to the right at speed k, with the wave front at $x = kt$.

Set $U(x) = V(-x)$; then clearly (2.7) is equivalent to

$$-dU'' + kU' = aU - bU^2 \quad \text{in } (0,\infty), \quad U(0) = 0, \ U(\infty) = a/b. \tag{2.8}$$

We have the following result, which is a correction of Proposition 4.1 in [8] (see [3]).

Proposition 2.4. *For any given constants $a > 0$, $b > 0$, $d > 0$ and $k \in [0, 2\sqrt{ad})$, problem (2.8) admits a unique positive solution $U = U_k$, and it satisfies $U_k'(x) > 0$ for $x \ge 0$, $U_{k_1}'(0) > U_{k_2}'(0)$, $U_{k_1}(x) > U_{k_2}(x)$ for $x > 0$ and $0 \le k_1 < k_2 < 2\sqrt{ad}$.*

Moreover, for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu, a, b, d) \in (0, 2\sqrt{ad})$ such that $\mu U_{k_0}'(0) = k_0$.

It was shown in [8] that when spreading happens for (1.1), then

$$\lim_{t\to\infty}\frac{h(t)}{t}=k_0(\mu,a,b,d).$$

This result was sharpened in [12] to

$$\lim_{t\to\infty}[h(t)-k_0t]=\hat{H} \text{ for some } \hat{H}\in\mathbb{R}$$

and

$$\lim_{t\to\infty}\sup_{x\in[0,h(t)]}|u(t,x)-U_{k_0}(h(t)-x)|=0.$$

Coming back to (2.1), we have the following result.

Theorem 2.5.[4] *If spreading happens for* (2.1), *then*

$$\overline{\lim}_{t\to+\infty}\frac{h(t)}{t}\le k_0(\mu,\alpha^\infty,\beta_\infty,d),\ \underline{\lim}_{t\to+\infty}\frac{h(t)}{t}\ge k_0(\mu,\alpha_\infty,\beta^\infty,d). \tag{2.9}$$

2.2. **The spatially asymptotically periodic case and pulsating semi-waves.** If we assume further that there exist positive L-periodic functions a and b in $C^{\nu_0}(\mathbb{R})$ such that

$$\lim_{r\to+\infty}\big(|\alpha(r)-a(r)|+|\beta(r)-b(r)|\big)=0, \tag{2.10}$$

then we can show that $\lim_{t\to\infty}\frac{h(t)}{t}$ exists. This is the main result of [7]. To determine this limit, we need to study the *pulsating semi-waves* of the one dimensional problem

$$\begin{cases} u_t-du_{xx}=u[a(x)-b(x)u], & t\in\mathbb{R},\ -\infty<x<h(t),\\ u(t,h(t))=0,\ h'(t)=-\mu u_x(t,h(t)), & t\in\mathbb{R}. \end{cases} \tag{2.11}$$

We call $(u(t,x),h(t))$ a **pulsating semi-wave** of (2.11) if it solves (2.11) and

(i) $u(t,x)=U(h(t),h(t)-x)>0$ for $t\in R,\ x<h(t)$,

(ii) there exists $T>0$ such that $h'(t)$ is a positive T-periodic function and $h(t+T)-h(t)=L$,

(iii) $U(\tau,\xi)$ is a function in $C^{1,2}(\mathbb{R}\times[0,+\infty))$ that is L-periodic in τ.

Theorem 2.6.[7] *Problem* (2.11) *always has a pulsating semi-wave* $(\tilde{u},\tilde{h})$. *The pulsating semi-wave is unique up to translations in t. Furthermore,* $\lim_{t\to\pm\infty}\tilde{h}(t)/t=L/T$, $\tilde{u}_t(t,x)>0$, *and* $\tilde{u}(t,x)\to\phi(x)$ *as* $t\to+\infty$ *uniformly in any interval of the form* $(-\infty,M]$, $M\in\mathbb{R}$, *where ϕ is the unique positive solution of*

$$-d\phi_{xx}=\phi[a(x)-b(x)\phi],\ x\in\mathbb{R}^1.$$

Theorem 2.7.[7] *Suppose that* (2.10) *holds, and* (u,h) *is the unique solution of* (2.1) *and* $\lim_{t\to\infty}h(t)=\infty$; *then*

$$\lim_{t\to\infty}\frac{h(t)}{t}=L/T.$$

2.3. **Heterogeneity in both space and time.** In [6], the following generalization of (2.1) is considered:

$$\begin{cases} u_t-d\Delta u=u(\alpha(t,r)-\beta(t,r)u), & t>0,\ 0<r<h(t),\\ u_r(t,0)=0,\ u(t,h(t))=0, & t>0,\\ h'(t)=-\mu u_r(t,h(t)), & t>0,\\ h(0)=h_0, u(0,r)=u_0(r), & 0\le r\le h_0, \end{cases} \tag{2.12}$$

where $\Delta u = u_{rr} + \frac{N-1}{r} u_r$ $(N \geq 1)$; $r = h(t)$ is the free boundary; h_0, μ and d are given positive constants; $u_0 \in C^2([0, h_0])$ is positive in $[0, h_0)$ and $u_0'(0) = u_0(h_0) = 0$; the functions $\alpha(t,r)$ and $\beta(t,r)$ satisfy the following conditions:

$$\left\{\begin{array}{ll} \text{(i)} & \alpha, \beta \in C^{\nu_0/2, \nu_0}(\mathbb{R} \times [0,\infty)) \text{ for some } \nu_0 \in (0,1), \\ & \text{and are } T\text{-periodic in } t \text{ for some } T > 0; \\ \text{(ii)} & \text{there are positive constants } \kappa_1, \kappa_2 \text{ such that} \\ & \kappa_1 \leq \alpha(t,r) \leq \kappa_2,\ \kappa_1 \leq \beta(t,r) \leq \kappa_2,\ \forall r \in [0,\infty),\ \forall t \in [0,T]. \end{array}\right. \tag{2.13}$$

This describes the situation that the solution u is radially symmetric ($u = u(t,r)$, $r = |x|$, $x \in \mathbb{R}^N$) and the environment may vary in time and space (but radially symmetric). In this situation, similar results to those proved in the previous sections hold.

Theorem 2.8 (Existence and uniqueness).[6] *Problem* (2.12) *admits a unique solution* $(u(t,r), h(t))$, *which is defined for all* $t > 0$. *Moreover,* $h \in C^1([0,\infty))$, $u \in C^{1,2}(D)$ *with* $D = \{(t,r) : t > 0, 0 \leq r \leq h(t)\}$, *and* $u(t,r) > 0$ *for* $t > 0$ *and* $0 \leq r < h(t)$, $h'(t) > 0$ *for* $t > 0$.

Theorem 2.9 (Spreading-vanishing dichotomy).[6] *Let* $(u(t,r), h(t))$ *be the solution of* (2.12). *Then the following alternative holds:*

Either

(i) Spreading: $\lim_{t\to\infty} h(t) = +\infty$ *and*

$$\lim_{t\to\infty} |u(t,r) - \hat{U}(t,r)| = 0 \ \textit{locally uniformly for } r \in [0,\infty),$$

where $\hat{U}(t,|x|)$ *is the unique positive* T*-periodic solution of*

$$U_t - d\Delta U = U[\alpha(t,|x|) - \beta(t,|x|)U], \quad (t,x) \in \mathbb{R}^1 \times \mathbb{R}^N,$$

or

(ii) Vanishing: $\lim_{t\to\infty} h(t) \leq R^*$ *and* $\lim_{t\to+\infty} ||u(t,\cdot)||_{C([0,h(t)])} = 0$, *where* $R^* > 0$ *is the unique value such that the following linear problem has a positive* T*-periodic solution when* $R = R^*$:

$$\left\{\begin{array}{rcll} \phi_t - d\Delta\phi & = & \alpha(t,|x|)\phi & \textit{for } t \in \mathbb{R}^1 \textit{ and } |x| < R, \\ \phi & = & 0 & \textit{for } t \in \mathbb{R}^1 \textit{ and } |x| = R. \end{array}\right.$$

Theorem 2.10 (Spreading-vanishing criteria).[6]

(a) *If* $h_0 \geq R^*$, *then spreading always occurs.*

(b) *If* $h_0 < R^*$, *then there exists a unique* $\mu^* > 0$ *depending on* u_0 *such that vanishing occurs if* $0 < \mu \leq \mu^*$, *and spreading happens if* $\mu > \mu^*$.

Let us note that, when $h_0 < R^*$, since μ^* varies with u_0, for fixed μ, whether spreading or vanishing happens depends on the size of u_0.

Theorem 2.11 (Spreading speed and profile).[6] *Suppose that*

$$\lim_{r\to\infty} \alpha(t,r) = \alpha_*(t), \ \lim_{r\to\infty} \beta(t,r) = \beta_*(t)$$

uniformly for $t \in [0,T]$. *Then in the case of spreading, there exists a positive* T*-periodic function* $k_0(t)$ *such that*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \overline{k}_0 := \frac{1}{T}\int_0^T k_0(t)dt.$$

Moreover, for any $c \in (0, \overline{k}_0)$, *we have*

$$\lim_{t\to\infty} \max_{0\le r\le ct} |u(t,r) - \hat{U}(t,r)| = 0.$$

We remark that while the proofs of Theorems 2.8, 2.9 and 2.10 are similar to those of the corresponding ones in subsection 2.1, the proof of Theorem 2.11 requires completely new techniques.

Remark 2.12. In [21], similar results are obtained for a logistic model with seasonal successions.

3. More general nonlinearities

In [10], the following problem was considered:

$$\begin{cases} u_t = u_{xx} + f(u), & g(t) < x < h(t),\ t > 0, \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu\, u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu\, u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, \quad u(0,x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{3.1}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t,x)$, μ is a given positive constant, $f : [0,\infty) \to \mathbb{R}$ is a C^1 function satisfying

$$f(0) = 0. \tag{3.2}$$

The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$\begin{aligned} \mathscr{X}(h_0) := \Big\{ \phi \in C^2([-h_0, h_0]) :\ & \phi(-h_0) = \phi(h_0) = 0,\ \phi'(-h_0) > 0, \\ & \phi'(h_0) < 0,\ \phi(x) > 0 \text{ in } (-h_0, h_0) \Big\}. \end{aligned} \tag{3.3}$$

Though [10] contained some results which hold for rather general $f(u)$, the long-time dynamics is better understood for three special types of nonlinearities:

(f_M) monostable case, (f_B) bistable case, (f_C) combustion case.

In the monostable case (f_M), it is assumed that f is C^1 and it satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad (1-u)f(u) > 0 \text{ for } u > 0, u \ne 1. \tag{3.4}$$

Clearly $f(u) = u(1-u)$ belongs to (f_M).

In the bistable case (f_B), it is assumed that f is C^1 and it satisfies

$$f(0) = f(\theta) = f(1) = 0, \quad f(u) \begin{cases} < 0 & \text{in } (0,\theta), \\ > 0 & \text{in } (\theta, 1), \\ < 0 & \text{in } (1, \infty) \end{cases} \tag{3.5}$$

for some $\theta \in (0,1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(s)ds > 0. \tag{3.6}$$

A typical bistable $f(u)$ is $u(u-\theta)(1-u)$ with $\theta \in (0, \frac{1}{2})$.

In the combustion case (f_C), f is C^1 and it satisfies

$$f(u) = 0 \text{ in } [0,\theta], \quad f(u) > 0 \text{ in } (\theta, 1), \quad f'(1) < 0, \quad f(u) < 0 \text{ in } [1, \infty) \tag{3.7}$$

for some $\theta \in (0,1)$, and there exists a small $\delta_0 > 0$ such that

$$f(u) \text{ is nondecreasing in } (\theta, \theta + \delta_0). \tag{3.8}$$

It follows from the general result in [10] that for these cases (3.1) always has a unique solution defined for all $t > 0$.

The next three theorems give a rather complete description of the long-time behavior of the solution, and they also reveal the related but different sharp transition natures between vanishing and spreading for these three types of nonlinearities.

Theorem 3.1 (Monostable case).[10] *Assume that f is of* (f_M) *type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval with length no bigger than* $\pi/\sqrt{f'(0)}$ *and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0.$$

Moreover, if $u_0 = \sigma\phi$ with $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0,\phi) \in [0,\infty]$ such that vanishing happens when $0 < \sigma \le \sigma^*$, and spreading happens when $\sigma > \sigma^*$. In addition,

$$\sigma^* \begin{cases} = 0 & \text{if } h_0 \ge \pi/(2\sqrt{f'(0)}),\\ \in (0,\infty] & \text{if } h_0 < \pi/(2\sqrt{f'(0)}),\\ \in (0,\infty) & \text{if } h_0 < \pi/(2\sqrt{f'(0)}) \text{ and if } f \text{ is globally Lipschitz.} \end{cases}$$

Theorem 3.2 (Bistable case).[10] *Assume that f is of* (f_B) *type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0,$$

or

(iii) Transition: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and there exists a continuous function $\gamma : [0,\infty) \to [-h_0, h_0]$ such that*

$$\lim_{t\to\infty} |u(t,x) - v_\infty(x+\gamma(t))| = 0 \text{ locally uniformly in } \mathbb{R}^1,$$

where v_∞ is the unique positive solution to

$$v'' + f(v) = 0 \; (x \in \mathbb{R}^1),\; v'(0) = 0,\; v(-\infty) = v(+\infty) = 0.$$

Moreover, if $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0,\phi) \in (0,\infty]$ such that vanishing happens when $0 < \sigma < \sigma^*$, spreading happens when $\sigma > \sigma^*$, and transition happens when $\sigma = \sigma^*$. In addition, there exists $Z_B > 0$ such that $\sigma^* < \infty$ if $h_0 \ge Z_B$, or if $h_0 < Z_B$ and f is globally Lipschitz.

Theorem 3.3 (Combustion case).[10] *Assume that f is of* (f_C) *type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0,$$

or

(iii) Transition: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = \theta \text{ locally uniformly in } \mathbb{R}^1.$$

Moreover, if $u_0 = \sigma\phi$ *for some* $\phi \in \mathscr{X}(h_0)$, *then there exists* $\sigma^* = \sigma^*(h_0, \phi) \in (0, \infty]$ *such that vanishing happens when* $0 < \sigma < \sigma^*$, *spreading happens when* $\sigma > \sigma^*$, *and transition happens when* $\sigma = \sigma^*$. *In addition, there exists* $Z_C > 0$ *such that* $\sigma^* < \infty$ *if* $h_0 \ge Z_C$, *or if* $h_0 < Z_C$ *and* f *is globally Lipschitz.*

Remark 3.4. The value of σ^* in the above theorems can be $+\infty$ if we drop the assumption that f is globally Lipschitz when h_0 is small. Indeed, this is the case if $f(u)$ goes to $-\infty$ fast enough as $u \to +\infty$, and examples are given in [10].

Remark 3.5. In Section 2, to determine whether spreading or vanishing happens for the special monostable nonlinearity, a threshold value of μ was established, which was shown to be always finite. Here we use σ in $u_0 = \sigma\phi$ as a varying parameter, which appears more natural especially for the bistable and combustion cases, since in these cases the dynamical behavior of (3.1) is more responsive to the change of the initial function than to the change of μ; for example, when $\|u_0\|_\infty \le \theta$, then vanishing always happens regardless of the value of μ.

When spreading happens, the asymptotic spreading speed is determined by the following problem

$$\begin{cases} q_{zz} - cq_z + f(q) = 0 \quad \text{for } z \in (0,\infty), \\ q(0) = 0, \ \mu q_z(0) = c, \ q(\infty) = 1, \ q(z) > 0 \text{ for } z > 0. \end{cases} \tag{3.9}$$

Proposition 3.6.[10] *Assume that* f *is of* (f_M), *or* (f_B), *or* (f_C) *type. Then for each* $\mu > 0$, (3.9) *has a unique solution* $(c, q) = (c^*, q^*)$.

We note that q^* is a "semi-wave" with speed c^*, since the function $v(t,x) = q^*(c^*t - x)$ satisfies

$$v_t = v_{xx} + f(v) \ (t \in \mathbb{R}^1, \ x < c^*t), \ v(t, c^*t) = 0, \ v(t, -\infty) = 1,$$

and it resembles a wave moving to the right at constant speed c^*, with front at $x = c^*t$. In comparison with the normal traveling wave generated by the solution of

$$q_{zz} - cq_z + f(q) = 0 \text{ for } z \in \mathbb{R}^1, \ q(-\infty) = 0, \ q(+\infty) = 1, \tag{3.10}$$

the generator $q^*(z)$ of $v(t,x)$ here is only defined on the half line $\{z \ge 0\}$.

Making use of the above semi-wave, we have the following result.

Theorem 3.7.[10] *Assume that* f *is of* (f_M), *or* (f_B), *or* (f_C) *type, and spreading happens. Let* c^* *be given by Proposition 3.6. Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{-g(t)}{t} = c^*,$$

and for any small $\varepsilon > 0$, *there exist positive constants* δ, M *and* T_0 *such that*

$$\max_{|x|\le(c^*-\varepsilon)t} |u(t,x) - 1| \le Me^{-\delta t} \text{ for all } t \ge T_0. \tag{3.11}$$

Remark 3.8. The asymptotic spreading speed c^* depends on the parameter μ appearing in the free boundary conditions and in (3.9). Therefore we may denote c^* by c^*_μ to stress this dependence. It is well-known (see, e.g., [1, 2]) that when f is of (f_M), or (f_B), or (f_C) type, the asymptotic spreading speed determined by the corresponding Cauchy problem of (3.1) is given by the speed of certain traveling wave solutions generated by a solution of (3.10). Let us denote this speed by c_0. Then it is shown in [10] that c^*_μ is increasing in μ and

$$\lim_{\mu\to\infty} c^*_\mu = c_0.$$

The conclusion in Theorem 3.7 has been significantly strengthened in [12], where the following result is proved:

Theorem 3.9.[12] *Under the conditions of Theorem 3.7, there exist $\hat H, \hat G \in \mathbb{R}$ such that*

$$\lim_{t\to\infty}(h(t)-c^*t-\hat H)=0,\ \lim_{t\to\infty}h'(t)=c^*,$$
$$\lim_{t\to\infty}(g(t)+c^*t-\hat G)=0,\ \lim_{t\to\infty}g'(t)=-c^*,$$

and

$$\lim_{t\to\infty}\ \sup_{x\in[0,\,h(t)]}\ |u(t,x)-q_{c^*}(h(t)-x)|=0, \tag{3.12}$$

$$\lim_{t\to\infty}\ \sup_{x\in[g(t),\,0]}\ |u(t,x)-q_{c^*}(x-g(t))|=0. \tag{3.13}$$

Remark 3.10. In [18, 17], for monostable and bistable nonlinearities, a variant of (3.1) is considered. Instead of considering the problem over $g(t)<x<h(t)$, with $x=g(t)$ and $x=h(t)$ the free boundaries, they assume that the reaction-diffusion equation is satisfied for $x\in[0,h(t))$, where $x=h(t)$ is a free boundary but $x=0$ is a fixed boundary where the solution u satisfies the Dirichlet boundary condition $u(t,0)=0$ for all $t>0$.

Remark 3.11. In [20], the authors consider the case that the Dirichlet boundary condition at $x=0$ in [18, 17] is replaced by a Robin type boundary condition: $u(t,0)=bu_x(t,0)$ ($b\ge 0$) (which includes the Dirichlet case by taking $b=0$). It is shown in [20] that for monostable and bistable nonlinearities, all the results of [10] can be extended to this new case.

4. High dimension without radial symmetry

In space dimension $N\ge 2$ and without assuming radial symmetry, the free boundary problem was considered in [5] and [11]. In such a case, the regularity of the free boundary is not easy to understand, and the free boundary problem can be formulated in the following form

$$\begin{cases} u_t - d\Delta u = g(u) & \text{for } x\in\Omega(t), t>0,\\ u=0 \text{ and } u_t=\mu|\nabla_x u|^2 & \text{for } x\in\Gamma(t), t>0,\\ u(0,x)=u_0(x) & \text{for } x\in\Omega_0,\end{cases} \tag{4.1}$$

where $\Omega(t)\subset\mathbb{R}^n$ ($n\ge 2$) is bounded by the free boundary $\Gamma(t)$, with $\Omega(0)=\Omega_0$, μ and d are given positive constants. It is assumed that Ω_0 is a bounded domain that agrees with the interior of its closure $\overline{\Omega}_0$, $\partial\Omega_0$ satisfies the interior ball condition, and

$u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0)$ is positive in Ω_0 and vanishes on $\partial\Omega_0$. For the nonlinear function g, the following assumptions are made:

$$\begin{cases} \text{(i) } g(0) = 0 \text{ and } g \in C^{1,\alpha}([0, \delta_0]) \text{ for some } \delta_0 > 0 \text{ and } \alpha \in (0,1), \\ \text{(ii) } g(u) \text{ is locally Lipschitz in } [0,\infty),\ g(u) \le 0 \text{ in } [M,\infty) \text{ for some } M > 0. \end{cases} \tag{4.2}$$

We note that these conditions are satisfied by standard monostable, bistable and combustion type nonlinearities.

It was shown in [5] that (4.1) has a unique weak solution $u(t,x)$ defined for all $t > 0$; the free boundary is understood as $\Gamma(t) = \partial\Omega(t)$, $\Omega(t) = \{x : u(t,x) > 0\}$. Moreover, any classical solution of (4.1) is a weak solution, and any weak solution with regular free boundary $\Gamma(t)$ is a classical solution.

The following result reveals the connection of the free boundary problem with the corresponding Cauchy problem.

Theorem 4.1.[5] *Let u_μ be the unique solution to problem* (4.1) *and $\Omega_\mu(t) = \{x : u(t,x) > 0\}$. Then*

$$\lim_{\mu\to\infty} \Omega_\mu(t) = \mathbb{R}^N \ \ \forall t > 0, \tag{4.3}$$

and

$$u_\mu \to U \ \text{ in } C^{\frac{1+\theta}{2},1+\theta}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N) \text{ as } \mu\to\infty, \tag{4.4}$$

where θ can be any number in $(0,1)$ and $U(t,x)$ is the unique solution of the Cauchy problem

$$\begin{cases} U_t - d\Delta U = g(U) & \text{in } (0,\infty)\times\mathbb{R}^N, \\ U(0,x) = \tilde{u}_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{4.5}$$

Here $\tilde{u}_0$ denotes the zero extension of u_0 into $\mathbb{R}^N$.

The regularity of the weak solution and its long-time dynamical behavior are studied in [11], and the following theorems are the main results.

Theorem 4.2.[11] *For any fixed $t > 0$, $\tilde{\Gamma}(t) := \Gamma(t) \setminus \overline{\mathrm{co}}(\Omega_0)$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^n$, and $\tilde{\Gamma} := \{(t,x) : x \in \tilde{\Gamma}(t),\ t > 0\}$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^{n+1}$. In particular, the free boundary is always $C^{2,\alpha}$ if Ω_0 is convex.*

Here $\overline{\mathrm{co}}(\Omega_0)$ stands for the closed convex hull of Ω_0.

Theorem 4.3.[11] *$\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$. Moreover, $\Omega_\infty := \cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^n$, or it is a bounded set. Furthermore, when $\Omega_\infty = \mathbb{R}^n$, for all large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^n$, and there exists a continuous function $M(t)$ such that*

$$\Gamma(t) \subset \{x : M(t) - \frac{d_0}{2}\pi \le |x| \le M(t)\}; \tag{4.6}$$

and when Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$.

Here d_0 is the diameter of Ω_0.

Theorem 4.4.[11] *If $g(u) = au - bu^2$ with a, b positive constants, then there exists $\mu^* \geq 0$ such that $\Omega_\infty = \mathbb{R}^n$ if $\mu > \mu^*$, and Ω_∞ is bounded if $\mu \in (0, \mu^*]$. Moreover, when $\Omega_\infty = \mathbb{R}^n$, the following holds:*

$$\lim_{t\to\infty} \frac{M(t)}{t} = k_0(\mu), \ \lim_{t\to\infty} \max_{|x|\leq ct} \left| u(t,x) - \frac{a}{b} \right| = 0 \ \forall c \in (0, k_0(\mu)),$$

where $k_0(\mu) = k_0(\mu, a, b, d)$ is given in Proposition 2.4 above, with $\lim_{\mu\to\infty} k_0(\mu) = 2\sqrt{ad}$.

There exists $R^* > 0$ such that $\mu^* > 0$ if $\overline{\Omega}_0$ is contained in a ball with radius R^*, and $\mu^* = 0$ if Ω_0 contains a ball of radius R^* (see Theorem 5.11 in [11]).

5. Discussions

In the previous sections, we reported some recent results on nonlinear Stefan problems involving a single equation. As mentioned already, such a mathematical model may be used to describe the spreading of an invasive species into a new environment. From an ecological point of view, it is more realistic to consider the situation that the invasive species invades into an environment where there are already some native species.

The simplest case is that there is one native species in the environment being invaded. Then a two species model have to be used to describe the invasion process. If the native species is a competitor of the invading species, such a situation is considered in [9]; if the native species is a prey to the invading species, the problem is studied in [25]. In [16] and [26], a competition system is considered under the assumption that both species invade along a common free boundary, and [24] considered such a common free boundary case for a predator-prey model. In all these cases, some kind of spreading-vanishing dichotomy has been established. However, no satisfactory result on the spreading speed has been obtained so far.

Even in the single equation case, there are still many questions remain to be investigated. For example, the case where the available environment is a proper unbounded subset of $\mathbb{R}^N$ has not been studied yet. There are also many interesting cases with heterogeneous environment not considered so far. Furthermore, apart from [12], there is no sharp estimate on the spreading speed and spreading profile of the solution when spreading happens in the various free boundary problems. It is also more realistic to consider situations where simple diffusion is replaced by general diffusion with advection. A first step in this direction is taken in [14, 15], where a problem of the form (3.1) with $f(u) = u(1-u)$ is considered, but with u_{xx} replaced by $du_{xx} + \beta u_x$.

References

[1] D.G. Aronson and H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in *Partial Differential Equations and Related Topics*, Lecture Notes in Math., Vol. **446**, Springer, Berlin, 1975, pp. 5–49.

[2] D.G. Aronson and H.F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, *Adv. Math.* **30** (1978), 33–76.

[3] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: analysis of a free boundary model, *Netw. Heterog. Media* **7** (2012), 583-603.

[4] Y. Du and Z.M. Guo, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary II, *J. Diff. Eqns.* **250** (2011), 4336-4366.

[5] Y. Du and Z.M. Guo, The Stefan problem for the Fisher-KPP equation, *J. Diff. Eqns.* **253** (2012), 996-1035.

[6] Y. Du, Z. M. Guo and R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, preprint, 2011. (http://turing.une.edu.au/~ydu/papers/du-guo-peng-Dec-2011)

[7] Y. Du and Xing Liang, Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model, preprint, 2013. (http://turing.une.edu.au/~ydu/papers/du-liang-2013Jan)

[8] Y. Du and Z.G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.* **42** (2010), 377-405.

[9] Y. Du and Z.G. Lin, The diffusive competition model with a free boundary: invasion of a superior or inferior competitor, preprint, 2013, ArXiv1303.0454 (http://arxiv.org/pdf/1303.0454).

[10] Y. Du and Bendong Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, preprint 2011, ArXiv1301.5373 (http://arxiv.org/pdf/1301.5373).

[11] Y. Du, H. Matano and Kelei Wang, Regularity and asymptotic behavior of nonlinear Stefan problems, preprint, 2013. (http://turing.une.edu.au/~ydu/papers/dmw-2013-Jan)

[12] Y. Du, H. Matsuzawa and Maolin Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, preprint, 2013. (http://turing.une.edu.au/~ydu/papers/dmz-2013-Jan)

[13] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* **7** (1937), 335–369.

[14] Hong Gu, Z.G. Lin and Bendong Lou, A reaction-advection-diffusion equation with free boundaries, preprint.

[15] Hong Gu, Z.G. Lin and Bendong Lou, Different asymptotic spreading speeds induced by advection in a diffusion problem with free boundaries, preprint.

[16] J.-S. Guo and C.-H. Wu, On a free boundary problem for a two-species weak competition system, *J. Dyn. Diff. Equat.* **24** (2012), 873-895.

[17] Y. Kaneko K. Oeda and Y. Yamada, Remarks on spreading and vanishing for free boundary problems of some reaction-diffusion equations, preprint.

[18] Y. Kaneko and Y. Yamada, A free boundary problem for a reaction-diffusion equation appearing in ecology, *Adv. Math. Sci. Appl.* **21** (2011), no. 2, 467-492.

[19] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov, Ètude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. Moscou Sér. Internat.* **A1** (1937), 1–26; English transl. in: Dynamics of Curved Fronts, P. Pelcé (ed.), Academic Press, 1988, 105–130.

[20] Xiaowei Liu and Bendong Lou, Asymptotic behavior of solutions to diffusion problems with Robin and free boundary conditions, preprint.

[21] R. Peng and X.-Q. Zhao, The diffusive logistic equation with a free boundary and seasonal succession, *Descrete Cont. Dyn. Syst. A* **33** (2013), 2007-2031.

[22] N. Shigesada and K. Kawasaki, *Biological Invasions: Theory and Practice*, Oxford Series in Ecology and Evolution, Oxford Univ. Press., Oxford, 1997.

[23] J.G. Skellam, Random dispersal in theoretical populations, *Biometrika* **38** (1951), 196-218.

[24] M.X. Wang, On some free boundary problems of the prey-predator model, preprint.

[25] M.X. Wang and J.F. Zhao, Free boundary problems for a pedator-prey model with double free boundaries, preprint.

[26] M.X. Wang and J.F. Zhao, Free boundary problems for a Lotka-Volterra competition system, preprint.