数理解析研究所講究録
第 1881 巻 2014 年 1-17

A Note on Mountain Pass Solutions for a Class of Allen-Cahn Models

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March 16, 2013

Abstract

For an Allen-Cahn model problem having 0 and 1 as solutions, the authors recently established the existence of a large family of solutions between 0 and 1 [13]. These solutions were obtained as constrained minimizers of a functional associated with the model or as limits of such minimizers. In this note, it is shown that corresponding to each minimizer, U, there is a second solution, V, of the problem which is of mountain pass type with $0 < V < U < 1$.

1 Introduction

In a recent paper [13], the authors studied the following Allen-Cahn model equation:

$$-\Delta u + A_{\varepsilon}(x)G'(u) = 0, \quad x \in \mathbb{R}^n \tag{1.1}$$

where $G(u) = u^2(1-u)^2$ is a double well potential, $\varepsilon > 0$, and $A_\varepsilon(x) = 1 + A(x)/\varepsilon$ with $0 \le A \in C^1(\mathbb{R}^n)$, $1-$periodic in $x_1, \cdots, x_n$, Ω is the support of $A|_{[0,1]^n}$ and has a smooth boundary, and $\overline{\Omega} \subset (0,1)^n$. Then $u \equiv 0$ and $u \equiv 1$ are solutions of (1.1), the so-called pure states, and the question of interest is whether there are other solutions of (1.1) having $0 < u < 1$, i.e. so called mixed states. There have been many papers written on this topic, both for Allen-Cahn models: [1]-[4], [15], [18], [20]-[21], as well as in a more general context motivated by Aubry-Mather Theory and initiated by Moser: [17], [8]-[12], [22], [24]. In particular, it was shown in [13] that there is an infinitude of such mixed states that shadow 0 and 1 in any prescribed way on a spatially periodic array of sets. To make this statement more precise, let $T \subset \mathbb{Z}^n$, and set

$$A^T = \cup_{i \in T}(i + \Omega); \qquad B^T = \cup_{i \in \{\mathbb{Z}^n \setminus T\}}(i + \Omega).$$

Then one of the main results in [13] contains

Theorem 1.2. *For G and A as above, there is an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ and each $T \subset \mathbb{Z}^n$, there is a solution $U_{\varepsilon,T}$ of (1.1) with $0 < U_{\varepsilon,T} < 1$. Moreover as $\varepsilon \to 0$, $U_{\varepsilon,T} \to 1$ uniformly on A^T and $U_{\varepsilon,T} \to 0$ uniformly on B^T.*

A more detailed statement can be found in [13]. For finite T, $U_{\varepsilon,T}$ is obtained by a minimization argument where pointwise constraints are imposed on the class of admissible functions. Then, with the aid of further a priori bounds, the case of arbitrary $T \subset Z^n$ is obtained from the finite case by passing to a limit.

The solutions for finite T are local minima of a functional corresponding to (1.1) while 0 and 1 are global minima. Possessing a local minimum such as $U_{\varepsilon,T}$ which is not a global minimum is essentially the geometric hypothesis of the Mountain Pass Theorem. For the current setting, it suggests the possibility of finding mountain pass solutions for (1.1). The main goal of this note is to show that corresponding to each local minimizing solution, $U_{\varepsilon,T}$ of (1.1), there is a second solution, $V_{\varepsilon,T}$, of mountain pass type with $0 < V_{\varepsilon,T} < U_{\varepsilon,T}$. In §2, we will show how to prove this for the case of T finite. Then in §3, we present a slightly refined result.

There is another way to view the geometric situation just described that may also be fruitful. A well known result of Matano, Theorem 4.4 of [16], states that for equations like

$$-\Delta u + f(x,u) = 0 \tag{1.3}$$

on bounded domains in $\mathbb{R}^n$, if there is an ordered pair of solutions, $v < w$ of (1.3) and v and w are stable solutions in an appropriate sense for the corresponding parabolic equation

$$u_t - \Delta u + f(x,u) = 0, \tag{1.4}$$

then there exists a solution, u of (1.3) with $v < u < w$. See also Amann [5] -[7] and Sattinger [23]. We will comment further on this point of view in §2.

2 The proof for finite T

The goal of this section is to show that if T is finite and $\varepsilon \in (0, \varepsilon_0)$ is sufficiently small, there is a mountain pass solution, $V_{\varepsilon,T}$ of (PDE) such that $0 < V_{\varepsilon,T} < U_{\varepsilon,T}$. To do this, first the minimization characterization of $U_{\varepsilon,T}$ in [13] will be recalled. Let $\mathcal{W}$ the closure of $C_0^\infty(\mathbb{R}^n)$ under

$$\|u\| \equiv (\int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \int_{[-1,1]^n} u^2 \, dx)^{\frac{1}{2}}.$$

For $u \in \mathcal{W}$, let $L_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + A_\varepsilon G(u)$ and $J_\varepsilon(u) = \int_{\mathbb{R}^n} L_\varepsilon(u) \, dx$.

Let $d^* = \frac{1}{2}|\partial\Omega - \partial[0,1]^n|$ so $d^* > 0$. Choose any $d \in (0, d^*)$ such that if

$$\Omega_d \equiv \{x \in \Omega \mid |x - \partial\Omega| > d\},$$

then $\partial\Omega_d$ is diffeomorphic to $\partial\Omega$. Set $A_T = \mathbb{Z}^n + \Omega_d$ and $B_T = (\mathbb{Z}^n \setminus T) + \Omega_d$. Choose constants a, b such that $0 < b < \frac{1}{2} < a < 1$ and set

$$\Gamma(T) = \{u \in \mathcal{W} \mid u \geq a \ \ A_T \ \ \textit{and} \ \ u \leq b \ \ \textit{on} \ \ B_T\}.$$

Define

$$c_\varepsilon(T) = \inf_{u \in \Gamma(T)} J_\varepsilon(u).$$

Let χ_S denote the characteristic function of the set S. Then from [13], we have:

Theorem 2.1. *Let A_ε and G be as above. Then there exists an $\varepsilon_0 > 0$ and independent of T with the property that for each $\varepsilon \in (0, \varepsilon_0)$ and each finite $T \subset \mathbb{Z}^n$,*

1^o $\mathcal{M}_\varepsilon(T) \equiv \{u \in \Gamma(T) \mid J_\varepsilon(u) = c_\varepsilon(T)\} \neq \emptyset.$

2^o *Any* $U \in \mathcal{M}_\varepsilon(T)$ *satisfies* $0 < U < 1$ *and is a classical solution of* (PDE).

3^o $\mathcal{M}_\varepsilon(T)$ *is an ordered set:* $U, V \in \mathcal{M}_\varepsilon(T)$ *implies* $U < V$, $U > V$, *or* $U \equiv V$.

4^o *If* $T \subset S \subset \mathbb{Z}^n$, $U_{\varepsilon,T} \le U_{\varepsilon,S}$ *with strict inequality if* $T \ne S$.

5^o *There exist constants* $C, c > 0$, *independent of* T *and of* $\varepsilon \in (0, \varepsilon_0)$, *satisfying*

$$|U_{\varepsilon,T}(x) - \chi_{T+[0,1]^n}(x)| \le C \exp(-cd(x,T)), \quad x \in \mathbb{R}^n$$

where $d(x,T)$ *denotes the distance from* x *to* T.

Let $T \subset \mathbb{Z}^n$ be finite, $\varepsilon \in (0, \varepsilon_0)$, and $U_{\varepsilon,T} \in \mathcal{M}_\varepsilon(T)$. Then, it is straightforward to show that

$$\limsup_{\varepsilon \to 0} c_\varepsilon(T) = \limsup_{\varepsilon \to 0} J_\varepsilon(U_{\varepsilon,T}) < \infty. \tag{2.2}$$

To obtain a mountain pass solution of (1.1) corresponding to T, set

$$\mathcal{G}_{\varepsilon,T} \equiv \{g \in C([0,1], W^{1,2}(\mathbb{R}^n) \mid 0 \le g(\theta) \le U_{\varepsilon,T},\ g(0) = 0,\ g(1) = U_{\varepsilon,T}\}$$

and define

$$b_\varepsilon(T) = \inf_{g \in \mathcal{G}_{\varepsilon,T}} \max_{\theta \in [0,1]} J_\varepsilon(g(\theta)). \tag{2.3}$$

We will show:

Proposition 2.4. $\liminf_{\varepsilon \to 0} \sqrt{\varepsilon} b_\varepsilon(T) > 0$.

Proof: Arguing indirectly, if the result is false, there exists a sequence $\{\varepsilon_m\}_{m=1}^\infty \subset (0,1)$ such that

$$\lim_{m \to \infty} \varepsilon_m = \lim_{m \to \infty} \sqrt{\varepsilon_m} b_{\varepsilon_m}(T) = 0.$$

For notational convenience, we assume that

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} b_\varepsilon(T) = 0. \tag{2.5}$$

The definition of $b_\varepsilon(T)$ implies there is a $g \in \mathcal{G}_{\varepsilon,T}$ such that

(2.6) $$b_\varepsilon(T) \le \max_{\theta\in[0,1]} J_\varepsilon(g(\theta)) \le b_\varepsilon(T) + 1.$$

By (2.6) for each $\theta \in [0,1]$,

(2.7) $$\int_{\mathbb{R}^n} A_\varepsilon G(g(\theta))\, dx \le b_\varepsilon(T) + 1.$$

Let $B_r(x)$ denote an open ball of radius r about $x \in \mathbb{R}^n$. Note that $b_\varepsilon(T) = b_\varepsilon(T+p)$ for any $p \in \mathbb{Z}^n$. By shifting T by such a p if need be, we can assume $0 \in T$. Suppose $B_r(z) \subset \Omega_d$. Since there is a constant, $\omega_d > 0$ such that for $x \in \Omega_d$, $A(x) \ge \omega_d$, it follows that $A_\varepsilon(x) \ge \omega_d/\varepsilon$, and by (2.7),

(2.8) $$\int_{B_r(z)} G(g(\theta))\, dx \le \int_{\Omega_d} G(g(\theta))\, dx \le \int_{\Omega_d} \frac{\varepsilon}{\omega_d} A_\varepsilon(x) G(g(\theta)) dx$$
$$\le \frac{\varepsilon}{\omega_d}(b_\varepsilon(T) + 1) \equiv M(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Set

$$\mathcal{A}_\varepsilon(\theta) = \{x \in B_r(z) \mid g(\theta) > a\}, \quad \mathcal{B}_\varepsilon(\theta) = \{x \in B_r(z) \mid g(\theta) \le b\},$$

and

$$\mathcal{D}_\varepsilon(\theta) = \{x \in B_r(z) \mid b < g(\theta) \le a\}.$$

The measure of a set, S, will be denoted by $|S|$. Let $\alpha(\theta) = |\mathcal{A}_\varepsilon(\theta)|$, $\beta(\theta) = |\mathcal{B}_\varepsilon(\theta)|$, and $\delta(\theta) = |\mathcal{D}_\varepsilon(\theta)|$. Then

(2.9) $$\alpha(\theta) + \beta(\theta) + \delta(\theta) = |B_r(z)| = |B_r(0)|.$$

There is a constant $\sigma = \sigma(d, a, b) > 0$ such that $G(g(\theta)(x)) \ge \sigma$ for $x \in \mathcal{D}_\varepsilon(\theta) \subset \Omega_d$ for $\theta \in [0,1]$. Hence by (2.8), for all $\theta \in [0,1]$,

$$\sigma\delta(\theta) \le \int_{\mathcal{D}_\varepsilon(t)} G(g(\theta))\, dx \le \int_{B_r(z)} G(g(\theta))\, dx \le M(\varepsilon)$$

or

(2.10) $$\delta(\theta) \le M(\varepsilon)/\sigma.$$

Note that by the choice of $B_r(z)$,

$$\begin{cases} \alpha(0) = 0, & \beta(0) = |B_r(0)| \\ \alpha(1) = |B_r(0)|, & \beta(1) = 0. \end{cases} \tag{2.11}$$

We claim that for all $\varepsilon \in (0, \varepsilon_0)$, there is a $\rho = \rho(\varepsilon) > 0$ such that whenever $|t - s| \leq \rho$,

$$|\alpha(t) - \alpha(s)| \leq 2M(\varepsilon)/\sigma. \tag{2.12}$$

Otherwise for some $\varepsilon \in (0, \varepsilon_0)$ and for each ρ, there is a $t = t(\rho), s = s(\rho)$ with $|t - s| \leq \rho$ and $|\alpha(t) - \alpha(s)| > 2M(\varepsilon)/\sigma$. Without loss of generality, it can be assumed that $\alpha(t) \geq \alpha(s)$. Therefore

$$\alpha(t) > \alpha(s) + 2M(\varepsilon)/\sigma. \tag{2.13}$$

Set

$$X = \{x \in B_r(z) \mid |g(t)(x) - g(s)(x)| \geq a - b\}.$$

Then

$$|X|(a-b)^2 \leq \int_X |g(t)(x) - g(s)(x)|^2 \, dx \leq \|g(t) - g(s)\|^2_{W^{1,2}(\mathbb{R}^n)} \tag{2.14}$$

and the right hand side of (2.14) goes to 0 as $\rho \to 0$. On the other hand, by (2.13),

$$\left|\left(\mathcal{B}_\varepsilon(s) \cup \mathcal{D}_\varepsilon(s)\right) \cap \mathcal{A}_\varepsilon(t)\right| \geq 2M(\varepsilon)/\sigma.$$

Therefore by (2.10), we see

$$\left|\mathcal{B}_\varepsilon(s) \cap \mathcal{A}_\varepsilon(t)\right| \geq M(\varepsilon)/\sigma.$$

and this implies

$$|X| \geq M(\varepsilon)/\sigma. \tag{2.15}$$

For small ρ, (2.15) is contrary to (2.14). Thus (2.12) holds.

In a similar fashion, by making ρ still smaller if necessary, it can be assumed that if $|t - s| \leq \rho(\varepsilon)$,

$$|\beta(t) - \beta(s)| \leq 2M(\varepsilon)/\sigma. \tag{2.16}$$

Further requiring that ε is small compared to $|B_r(0))|$, the inequalities (2.12), (2.16), and (2.11) imply there is a $t^* = t^*(\varepsilon)$ such that $\alpha(t^*)$ and $\beta(t^*)$ belong to $(|B_r(0)|/3,\ 2|B_r(0)|/3)$. By Lemma 3.5 in Chapter 2 of [14], for a constant, C, independent of ε, r, a, b, we have

$$
\begin{aligned}
(2.17)\qquad & \big(\frac{1}{3}|B_r(0)|\big)^{1-\frac{1}{n}}(a-b) \le \alpha(t^*)^{1-\frac{1}{n}}(a-b) \\
&\le \frac{Cr^n}{|B_r(0)\setminus\{x\in B_r(0)\mid g(t^*)(x)>b\}|}\int_{\mathcal{D}_\varepsilon(t^*)}|\nabla g(t^*)|\,dx \\
&\le \frac{Cr^n|\delta(t^*)|^{\frac{1}{2}}}{\beta(t^*)}\Big(\int_{\mathcal{D}_\varepsilon(t^*)}|\nabla g(t^*)|^2\,dx\Big)^{\frac{1}{2}} \\
&\le \frac{3Cr^n(M(\varepsilon)/\sigma)^{\frac{1}{2}}}{|B_r(0)|}(2J_\varepsilon(g(t^*)))^{\frac{1}{2}} \le \frac{3\sqrt{2}C}{\sqrt{\sigma\omega_d}|B_1(0)|}\sqrt{\varepsilon}(b_\varepsilon+1)
\end{aligned}
$$

via (2.10), (2.8) and (2.6). But as $\varepsilon \to 0$, the right hand side of (2.17) approaches 0 while the left hand side is bounded away from 0, a contradiction. This proves Proposition 2.4. □

Next we have the an upper estimate for $b_\varepsilon(T)$.

Proposition 2.18. $\limsup_{\varepsilon\to 0}\sqrt{\varepsilon}b_\varepsilon(T) < \infty$.

Proof: First, let $\psi : [0,1] \to \mathbb{R}$ be a smooth function such that $\lim_{s\to 0}\psi(s) = -\infty$, $\lim_{s\to 1}\psi(1) = \infty$ and ψ is monotone increasing. Second, choose a function $\phi \in C^1(\mathbb{R})$ such that $0 \le \phi(s) \le 1$, $|\phi'(s)| \le 2$ for $s \in \mathbb{R}$, $\phi(s) = 0$ for $s \le 0$ $\phi(s) = 1$ for $s \ge 1$. Now define

$$
g(\theta)(x_1,\cdots,x_n) = U_{\varepsilon,T}(x_1,\cdots,x_n)\phi\big(\frac{x_1+\psi(\theta)}{\sqrt{\varepsilon}}\big),\quad \theta\in(0,1).
$$

Then using 5^o of Theorem 2.1 shows that $\lim_{s\to 0} g(s) = 0$, $\lim_{s\to 1} g(s) = U_{\varepsilon,T}$, the limits being taken in $W^{1,2}(\mathbb{R}^n)$. Hence defining $g(0) = 0$ and $g(1) = U_{\varepsilon,T}$, shows $g \in \mathcal{G}_{\varepsilon,T}$ and

$$
(2.19)\qquad b_\varepsilon(T) \le \max_{\theta\in[0,1]} J_\varepsilon(g(\theta)).
$$

To estimate the right hand side of (2.19), first observe that

$$
\max_{\theta\in[0,1]}\int_{\mathbb{R}^n}|\nabla g(\theta)|^2dx \le 2\int_{\mathbb{R}^n}(|\nabla U_{\varepsilon,T}|^2 + \frac{1}{\varepsilon}U_{\varepsilon,T}^2\,|\varphi'(\frac{x_1+\psi(\theta)}{\sqrt{\varepsilon}})|^2)dx
$$

and

$$\int_{\mathbb{R}^n} U_{\varepsilon,T}^2 \, |\varphi'(\frac{x_1+\psi(\theta)}{\sqrt{\varepsilon}})|^2)dx \le 4 \int_{0\le x_1+\psi(\theta)\le\sqrt{\varepsilon}} U_{\varepsilon,T}^2 dx$$

Since $0 \le U_{\varepsilon,T} \le 1$, again by 5^o of Theorem 2.1, there exists a constant $D_1 > 0$, independent of small $\varepsilon > 0$ such that for each $l \in \mathbb{R}$,

$$\int_{x_1=l} U_{\varepsilon,T}^2 dx_2 \cdots dx_n \le D_1.$$

This inequality implies that

$$\int_{0\le x_1+\psi(\theta)\le\sqrt{\varepsilon}} U_{\varepsilon,T}^2 dx \le D_1\sqrt{\varepsilon}.$$

Hence by 5^o of Theorem 2.1 yet again, there is a constant, D_2 independent of small $\varepsilon > 0$, so that

$$\max_{\theta\in[0,1]} \int_{\mathbb{R}^n} |\nabla g(\theta)|^2 dx \le D_2/\sqrt{\varepsilon}. \tag{2.20}$$

Next noting that

$$\int_{\mathbb{R}^n} A_\varepsilon(x)G(g(\theta))dx = \int_{0\le x_1+\psi(\theta)\le\sqrt{\varepsilon}} A_\varepsilon(x)G(g(\theta))dx$$

$$+ \int_{\sqrt{\varepsilon}<x_1} A_\varepsilon(x)G(U_{\varepsilon,T})dx$$

and estimating as above shows there is a constant $D_3 > 0$, independent of small $\varepsilon > 0$ and $\theta \in [0,1]$ such that

$$\int_{\mathbb{R}^n} A_\varepsilon(x)G(g(\theta))dx \le D_3/\sqrt{\varepsilon}. \tag{2.21}$$

Combining (2.19) - (2.21) then yields the Proposition. □

Now the main result of this section can be stated.

Theorem 2.22. *For each finite $T \subset \mathbb{Z}^n$, there is an $\varepsilon_1(T) > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, there is a solution, $V_{\varepsilon,T}$ of (1.1) with $0 < V_{\varepsilon,T} < U_{\varepsilon,T}$ and $J_\varepsilon(V_{\varepsilon,T}) = b_\varepsilon(T)$.*

Before proving Theorem 2.22, some remarks are in order. First to take advantage of some results from [11], it is convenient to replace G by a new function, G^* defined by $G^*|_{[0,1]} = G|_{[0,1]}$ and $G^*(z)$ is 1−periodic in z. Thus $G^* \in C^2(\mathbb{R})$ and is a bounded function. Let

$$L_\varepsilon^*(u) = \frac{1}{2}|\nabla u|^2 + A_\varepsilon G^*(u) \quad \textit{and} \quad J_\varepsilon^*(u) = \int_{\mathbb{R}^n} L_\varepsilon^*(u)\, dx.$$

When dealing with functions, u, where $0 \le u \le 1$, $G^*(u) = G(u)$ so $J_\varepsilon^*(u) = J_\varepsilon(u)$. In particular any critical point, u, of J_ε^* with $0 \le u \le 1$ will be a critical point of J_ε and a solution of (1.1).

Let

$$\mathcal{S}_T \equiv \{u \in W^{1,2}(\mathbb{R}^n) \mid 0 \le u \le U_{\varepsilon,T}\}.$$

Consider the parabolic initial value problem associated with G^*:

$$u_t - \Delta u + A_\varepsilon(x) G_u^*(u) = 0, \quad t > 0,\ x \in \mathbb{R}^n \tag{2.23}$$

$$u(x,0) = \varphi(x) \in \mathcal{S}_T.$$

Let $\Phi^t(\varphi)(x)$ denote the solution of (2.23). Then we have

Proposition 2.24. 1^o *For each $\varphi \in \mathcal{S}_T$, there is a solution, $\Phi^t(\varphi)$ of (2.23) which exists for all $t > 0$.*

2^o *For each $t > 0$, $\Phi^t : \mathcal{S}_T \to \mathcal{S}_T$.*

3^o *For each $\varphi \in \mathcal{S}_T$, $J_\varepsilon(\Phi^t(\varphi)) \le J_\varepsilon(\Phi^s(\varphi))$ if $t \ge s \ge 0$.*

4^o *For each $\varphi \in \mathcal{S}_T$, there is a solution, $\Psi(\varphi)$, of (1.1), with $\Psi(\varphi) \in \mathcal{S}_T$ and a sequence $(t_k^*) \subset \mathbb{R}$ such that $t_k^* \to \infty$ and $\Phi^{t_k}(\varphi) \to \Psi(\varphi)$ in $W^{1,2}(\mathbb{R}^n)$ as $k \to \infty$.*

Proposition 2.24 was proved in [11] for a more general class of (bounded) nonlinearities than $A_\varepsilon(x)G_u^*$, but with $\mathbb{R}^n$ replaced by an n-torus. However one can use the heat kernel as in [11] (which in turn was based on arguments in [19]) to convert (2.23) to an operator equation in $W^{1,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and then continue to argue as in [11] to get $1^o - 4^o$ of the Proposition.

Next we have:

Proposition 2.25. *On $\mathcal{S}_T$, $J_\varepsilon = J_\varepsilon^*$ satisfies the Palais-Smale condition, i.e. any sequence $(u_k) \subset \mathcal{S}_T$, with $J_\varepsilon(u_k)$ bounded and $J_\varepsilon'(u_k) \to 0$, has a convergent subsequence.*

Proof: Suppose $(\varphi_k) \subset \mathcal{S}_T$ is a Palais-Smale sequence, i.e. $J_\varepsilon(\varphi_k) \leq M$ and $J'_\varepsilon(\varphi_k) \to 0$. Due to the form of J_ε and of $U_{\varepsilon,T}$ (recalling 5^o of Theorem 2.1), (φ_k) is bounded in $W^{1,2}(\mathbb{R}^n)$. Hence there is a $\varphi \in \mathcal{S}_T$ such that along a subsequence, $\varphi_k \to \varphi$ weakly in $W^{1,2}(\mathbb{R}^n)$ and in $L^2_{loc}(\mathbb{R}^n)$ as $k \to \infty$. Since $0 \leq \varphi \leq U_{\varepsilon,T}$, the tails of φ are exponentially small so in fact along our subsequence,

$$\|\varphi_k - \varphi\|_{L^2(\mathbb{R}^n)} \to 0, \quad k \to \infty. \tag{2.26}$$

By (2.26), for any $\psi \in W^{1,2}(\mathbb{R}^n)$, as $k \to \infty$,

$$\begin{aligned} (2.27)\qquad & \left|\int_{\mathbb{R}^n} A_\varepsilon(x)(G_u(x,\varphi_k) - G_u(x,\varphi))\psi\,dx\right| \\ & \qquad \leq K\|\varphi_k - \varphi\|_{L^2(\mathbb{R}^n)}\|\psi\|_{L^2(\mathbb{R}^n)} \to 0 \end{aligned}$$

where $K = \|A_\varepsilon\|_{\mathrm{L}^\infty}\|G_{uu}\|_{\mathrm{L}^\infty[0,1]}$. Hence $J'_\varepsilon(\varphi) = 0$ and

$$\begin{aligned} (2.28)\quad & J'_\varepsilon(\varphi_k)(\varphi_k - \varphi) - J'_\varepsilon(\varphi)(\varphi_k - \varphi) \\ & \quad = \int_{\mathbb{R}^n} |\nabla(\varphi_k - \varphi)|^2 + A_\varepsilon(G_u(x,\varphi_k) - G_u(x,\varphi)(\varphi_k - \varphi)\,dx \to 0. \end{aligned}$$

as $k \to \infty$. Thus by (2.27)–(2.28),

$$\int_{\mathbb{R}^n} |\nabla(\varphi_k - \varphi)|^2\,dx \to 0$$

as $k \to \infty$. Consequently $\varphi_k \to \varphi$ in $W^{1,2}(\mathbb{R}^n)$. □

Now we can give the

Proof of Theorem 2.22: First we note from Proposition 2.4 and (2.2) that for small $\varepsilon > 0$, $b_\varepsilon(T) > J_\varepsilon(U_{\varepsilon,T})$.

Now we follow an argument from [12]. Let $\delta > 0$ and choose $g \in \Gamma(T)$ such that

$$\max_{\theta\in[0,1]} J_\varepsilon(g(\theta)) \leq b_\varepsilon(T) + \delta. \tag{2.29}$$

By $1^o - 2^o$ of Proposition 2.24, for each $\tau > 0$ and $\theta \in [0,1]$, $\Phi^\tau(g(\theta)) \in \mathcal{S}_T$. Therefore there is a $\theta_\tau \in [0,1]$ such that

$$J_\varepsilon(\Phi^\tau(g(\theta_\tau)) \geq b_\varepsilon(T). \tag{2.30}$$

Hence a subsequence of the points, $\{\theta_\tau \mid \tau > 0\}$, converges to some $\theta_\infty \in [0,1]$. Denote this subsequence by $\{\theta_{\tau_p} \equiv \mu_p \mid p \in \mathbb{N}\}$. We claim for all $\tau > 0$,

$$J_\varepsilon(\Phi^\tau(g(\theta_\infty)) \geq b_\varepsilon(T). \tag{2.31}$$

Otherwise for some $\tau > 0$,

$$J_\varepsilon(\Phi^\tau(g(\theta_\infty)) < b_\varepsilon(T). \tag{2.32}$$

Since g is continuous, by (2.32), for large p,

$$J_\varepsilon(\Phi^\tau(g(\mu_p)) < b_\varepsilon(T). \tag{2.33}$$

Thus by 3^o of Proposition 2.24, for large p,

$$J_\varepsilon(\Phi^{\mu_p}(g(\mu_p)) < b_\varepsilon(T), \tag{2.34}$$

contrary to (2.30). Consequently (2.31) holds.

Next observe that by 4^o of Proposition 2.24, there is a sequence $(q_k) \subset \mathbb{R}$ and a solution, v_δ of (1.1) in $\mathcal{S}_T$ such that $\Phi^{q_k}(g(\theta_\infty)) \to v_\delta$ in $W^{1,2}(\mathbb{R}^n)$. Moreover by (2.31) and (2.29),

$$b_\varepsilon(T) \leq J_\varepsilon(v_\delta) \leq b_\varepsilon(T) + \delta. \tag{2.35}$$

Letting $\delta \to 0$, Proposition 2.25 shows there is a solution of (1.1), $V_{\varepsilon,T} \in \mathcal{S}_T$ such that along a subsequence, $v_\delta \to V_{\varepsilon,T}$ in $W^{1,2}(\mathbb{R}^n)$ and $J_\varepsilon(V_{\varepsilon,T}) = b_\varepsilon(T) > c_\varepsilon(T)$. This implies that $0 < V_{\varepsilon,T} < U_{\varepsilon,T}$ and completes the proof. □

To conclude this section, recall that from 3^o of Theorem 2.1, $\mathcal{M}_\varepsilon(T)$ is an ordered set. Suppose that $U, U^* \in \mathcal{M}_\varepsilon(T)$ and there are no members of $\mathcal{M}_\varepsilon(T)$ lying between U and U^*. Then U, U^* is called a gap pair of solutions of (1.1) in $\mathcal{M}_\varepsilon(T)$ and we have:

Theorem 2.36. *If U, U^* is a gap pair of solutions of (1.1) in $\mathcal{M}_\varepsilon(T)$, there exists a solution, V of (1.1) of mountain pass type with $U < V < U^*$. Moreover $J_\varepsilon(V) = d_\varepsilon(T)$ where*

$$d_\varepsilon(T) = \inf_{h \in \mathcal{H}_{\varepsilon,T}} \max_{\theta \in [0,1]} J_\varepsilon(h(\theta))$$

and

$$\mathcal{H}_{\varepsilon,T} \equiv \{h \in C([0,1], W^{1,2}(\mathbb{R}^n)) \mid U \leq h(\theta) \leq U^*,\ h(0) = U, h(1) = U^*\}.$$

Proof: Once it is shown that

$$d_\varepsilon(T) > c_\varepsilon(T), \tag{2.37}$$

the proof is essentially the same as that of Theorem 2.22. Therefore we merely verify (2.37). Modifying an argument from [11], let $\rho \in (0, \|U - U^*\|_{L^2(\mathbb{R}^n)})$ and set

$$P_\rho \equiv \{u \in W^{1,2}(\mathbb{R}^n) \mid U \le u \le U^* \textit{ and } \|U - u\|_{L^2(\mathbb{R}^n)} = \rho\}.$$

We claim

$$\inf_{u \in P_\rho} J_\varepsilon(u) \equiv f_\rho(T) > c_\varepsilon(T). \tag{2.38}$$

Indeed, suppose that (2.38) has been established. Fix $\rho \in (0, \|U - U^*\|_{L^2(\mathbb{R}^n)})$. Since for any $h \in \mathcal{H}_{\varepsilon,T}$, there is a corresponding $\theta_h \in (0,1)$ such that $h(\theta_h) \in P_\rho$, by (2.38),

$$d_\varepsilon(T) \ge f_\rho(T) > c_\varepsilon(T)$$

and (2.37) is established.

Finally to verify (2.38), if it is false, there is a sequence $(u_k) \subset P_\rho$ such that $J_\varepsilon(u_k) \to c_\varepsilon(T)$ as $k \to \infty$. The sequence (u_k) is bounded in $W^{1,2}(\mathbb{R}^n)$. Therefore there is a $v \in W^{1,2}(\mathbb{R}^n)$ with $U \le v \le U^*$ such that, along a subsequence, $u_k \to v$ weakly in $W^{1,2}(\mathbb{R}^n)$ and in $L^2_{loc}(\mathbb{R}^n)$ as $k \to \infty$. Since for each $R > 0$, $\int_{B_R(0)} L_\varepsilon(\cdot)\, dx$ is weakly lower semicontinuous,

$$\int_{B_R(0)} L_\varepsilon(v)\, dx \le \liminf_{k\to\infty} \int_{B_R(0)} L_\varepsilon(u_k)\, dx \le \liminf_{k\to\infty} J_\varepsilon(u_k) = c_\varepsilon(T).$$

Since this is true for all $R > 0$,

$$J_\varepsilon(v) \le c_\varepsilon(T). \tag{2.39}$$

But U, U^* is a gap pair so (2.39) implies $v = U$ or $v = U^*$. This is impossible since

$$\|U - v\|_{L^2(\mathbb{R}^n)} = \rho. \tag{2.40}$$

Indeed, to verify (2.40), observe that

$$\rho^2 = \|U - u_k\|^2_{L^2(\mathbb{R}^n)} = \int_{B_R(0)} |U - u_k|^2\, dx + \int_{\mathbb{R}^n \setminus B_R(0)} |U - u_k|^2\, dx \tag{2.41}$$

As $k \to \infty$ along our subsequence, the first term on the right in (2.41) converges to

$$\int_{B_R(0)} |U - v|^2 \, dx$$

while for each $k \in \mathbb{N}$,

$$\int_{\mathbb{R}^n \setminus B_R(0)} |U - u_k|^2 \, dx \leq \int_{\mathbb{R}^n \setminus B_R(0)} |U - U^*|^2 \, dx.$$

Letting $R \to \infty$ and noting that, by 5^o of Theorem 2.1, U and U^* each decay exponentially to 0 as $|x| \to \infty$, (2.40) follows. □

Remark 2.42. Theorem 1.2 provides the existence of a solution, $U_{\varepsilon,T}$, of (1.1) for any $T \in \mathbb{Z}^n$ and any $\varepsilon \in (0, \varepsilon_0)$ with ε_0 independent of T. If T is finite, Theorem 2.22 gives another solution, $V_{\varepsilon,T}$, for a more restricted set of values of ε : $0 < \varepsilon < \varepsilon_1(T)$ where $\varepsilon_1(T)$ is possibly very small. The size restriction on $\varepsilon_1(T)$ is required to get $b_\varepsilon(T) > c_\varepsilon(T)$. It is natural to ask whether the range of values for the existence of a second solution can be broadened, perhaps even to ε_0. One possible approach to this question is to use the relationship between Theorem 2.22 and the Theorem of Matano [16] mentioned in the Introduction. The latter result is for bounded domains while the current setting is $\mathbb{R}^n$. Ignoring this fact, the Matano result requires stability behavior relative to (1.4) for the solutions $u = 0$ and $u = U_{\varepsilon,T}$ of (1.1). The linearized equation of (1.1) about $u = 0$ is

$$-\Delta v + A_\varepsilon(x) v = 0 \tag{2.43}$$

where $A_\varepsilon \geq 1$, showing that 0 will be a stable solution of (1.4). What happens for initial data for (1.4) near $U_{\varepsilon,T}$ is less clear. However, at the formal level, a calculation shows $\min(U_{\varepsilon,T}, a)$ (where a is as in $\Gamma(T)$) is a weak subsolution of (1.1), at least for small ε, and that may suffice for the stability near $U_{\varepsilon,T}$. Whether these heuristics can be used to obtain an existence theorem remains an open question.

3 A generalization of Theorem 2.22 and some remarks

In this final section we will prove a version of Theorem 2.22 that contains more qualitative information about solutions of mountain pass type. Let

$S \subset T \subset \mathbb{Z}^n$ with T finite and $S \neq T$. Define

$$\begin{aligned} G_\varepsilon(S,T) \;\equiv\; & \{g \in C([0,1], W^{1,2}(\mathbb{R}^n)) \mid U_{\varepsilon,S} \leq g(\theta) \leq U_{\varepsilon,T} \\ & \textit{and} \;\; g(0) = U_{\varepsilon,S},\; g(1) = U_{\varepsilon,T}\} \end{aligned}$$

and set

$$b_\varepsilon(S,T) = \inf_{g \in G_\varepsilon(S,T)} \max_{\theta \in [0,1]} J_\varepsilon(g(\theta)).$$

Then corresponding to Proposition 2.4, we have

Proposition 3.1. $\liminf_{\varepsilon \to 0} \sqrt{\varepsilon} b_\varepsilon(S,T) > 0$.

Proof: The proof is quite close to that of the earlier case so we will be brief. Choose $q \in T \backslash S$ and r, z so that $B_r(z) \subset q + \Omega_d$. Then following the previous proof yields the Proposition. $\square$

Next corresponding to Proposition 2.18, we also have

Proposition 3.2. $\limsup_{\varepsilon \to 0} \sqrt{\varepsilon} b_\varepsilon(S,T) < \infty$.

To continue, replace the set $\mathcal{S}_T$ of §2 by

$$\Psi(S,T) = \{u \in W^{1,2}(\mathbb{R}^n) \mid U_{\varepsilon,S} \leq u \leq U_{\varepsilon,T}\}.$$

Then there are analogues of Proposition 2.24 and Proposition 2.25 corresponding to this replacement. In particular, $\Psi(S,T)$ is invariant under Φ^t. As earlier, these tools and the proof of Theorem 2.22 yield:

Theorem 3.3. *Let S and T be as above. Then there is an $\varepsilon_2 = \varepsilon_2(S,T) > 0$ such that for any $\varepsilon \in (0, \varepsilon_2)$, there is a solution, $W_{\varepsilon,S,T}$ of (1.1) with $U_{\varepsilon,S} < W_{\varepsilon,S,T} < U_{\varepsilon,T}$ and $J_\varepsilon(W_{\varepsilon,S,T}) = b_\varepsilon(S,T)$.*

We conclude with two remarks and some open questions.

Remark 3.4. Consider a finite set $S \in \mathbb{Z}^n$ and an infinite set $T \subset \mathbb{Z}^n$ with $S \subset T$. For each large integer $m > 0$, $S \subset T_m \equiv T \cap [-m,m]^n$. Then by Theorem 2.1, there are corresponding solutions $U_{\varepsilon,S} \in \Gamma(S)$ and $U_{\varepsilon,T_m} \in \Gamma(T_m)$. Theorem 3.3 provides an $\varepsilon_m = \varepsilon_m(S,T_m)$ such that for any $\varepsilon \in (0, \varepsilon_m)$, there is a solution, W_{ε,S,T_m} of (1.1) with $U_{\varepsilon,S} < W_{\varepsilon,S,T_m} < U_{\varepsilon,T_m}$ and $J_\varepsilon(W_{\varepsilon,S,T_m}) = b_\varepsilon(S,T_m)$. In [13], we showed that there is an $\varepsilon_0 > 0$, independent of $m > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and large integer $m > 0$, U_{ε,T_m} is a solution of (1.1), and that as $m \to \infty$, U_{ε,T_m} converges in $C^2_{loc}(\mathbb{R}^n)$ to a solution $U_{\varepsilon,T}$ of (1.1) with the property that $\lim_{\varepsilon\to 0} U_{\varepsilon,T}(x) = 1$ uniformly on $T+\Omega$ and $\lim_{\varepsilon\to 0} U_{\varepsilon,T}(x) = 0$ uniformly on $(\mathbb{Z}^n \setminus T)+\Omega$. It is an interesting open question as to whether $\liminf_{m\to\infty} \varepsilon_m > 0$. If this is the case, for small $\varepsilon > 0$, W_{ε,S,T_m} converges to a solution $W_{\varepsilon,S,T}$ of (1.1) with $U_{\varepsilon,S} < W_{\varepsilon,S,T} \leq U_{\varepsilon,T}$. Hence, it is then also an interesting open question to see whether $W_{\varepsilon,S,T} < U_{\varepsilon,T}$ or $W_{\varepsilon,S,T} = U_{\varepsilon,T}$.

Acknowledgments This research of the first author was supported by the Mid-career Researcher Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (MEST) (No. 2010-0014135).

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