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LOCAL SOLUTIONS WITH POLYNOMIAL DECAY IN THE VELOCITY VARIABLES TO THE BOLTZMANN EQUATION FOR SOFT POTENTIALS

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1. INTRODUCTION

In the present note we consider the Cauchy problem for the spatially inhomogeneous Boltzmann equation,

$$\partial_t f + v\cdot\nabla_x f = Q(f,f), \quad f(0,x,v) = f_0(x,v), \tag{1.1}$$

where $f = f(t,x,v)$ is the density distribution function of particles with velocity $v \in \mathbb{R}^3$ at time t and position x. The right hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\,(v-v_*,\sigma)\left\{g(v_*')f(v') - g(v_*)f(v)\right\} d\sigma dv_*\,,$$

where the conservation of momentum and energy implies that for $\sigma \in \mathbb{S}^2$

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \;\; v_*' = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma\,.$$

The non-negative cross section B usually takes the form

$$B = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma\,, \;\; 0 \le \theta \le \frac{\pi}{2}, \tag{1.2}$$

where

$$\begin{aligned}&\Phi(|z|) = \Phi_\gamma(|z|) = |z|^\gamma, \;\; \text{for some } \gamma > -3,\\ &b(\cos\theta)\theta^{2+2s} \;\to K \;\; \text{when} \;\; \theta \to 0+, \text{for } 0 < s < 1 \text{ and } K > 0.\end{aligned}$$

In fact, for the physical model, if the inter-molecule potential satisfies the inverse power law potential $U(\rho) = \rho^{-(q-1)}, q > 2$(, where ρ denotes the distance between two interacting molecules), then s and γ are given by

$$0 < s = 1/(q-1) < 1\;, \quad 1 > \gamma = 1-4s = (q-5)/(q-1) > -3\;.$$

As usual, the hard ($\gamma > 0$) and soft ($\gamma < 0$) potentials correspond to $q > 5$ and $2 < q < 5$, respectively, and the Maxwellian potential ($\gamma = 0$) corresponds to $q = 5$.

The angle θ is the deviation angle, i.e., the angle between post- and pre-collisional velocities (see Figure 1 in the next page). Though the range of θ is originally a full interval $[0,\pi]$, it should be noted that the angle θ in (1.2) is now restricted to $[0,\pi/2]$, as in [1], by replacing $b(\cos\theta)$ by its "symmetrized" version

$$[b(\cos\theta) + b(\cos(\pi-\theta))]\mathbf{1}_{0\le\theta\le\pi/2},$$

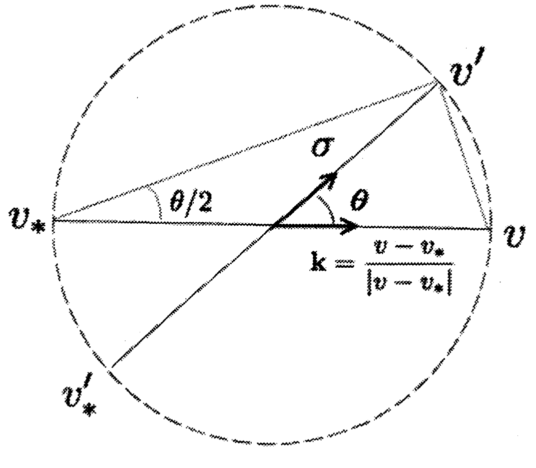

FIGURE 1. post- and pre-collisional velocities

which is possible due to the invariance of the product $f(v')f(v'_*)$ in the collision operator $Q(f,f)$ under the change of variables $\sigma \to -\sigma$. This device enables us to use the *regular* change of variables between post- and pre-collisional velocities (in the proof of the celebrated cancellation lemma in [1]),

$$v \mapsto v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma\,,$$

where the Jacobian is found to be

$$\left|\frac{\partial v}{\partial v'}\right| = \frac{8}{\left|I + \mathbf{k}\otimes\sigma\right|} = \frac{8}{|1 + \mathbf{k}\cdot\sigma|} = \frac{4}{\cos^2(\theta/2)} \le 8, \quad \theta \in [0, \pi/2].$$

In [15, 2], the *singular* change of variables $v_* \to v'$ (, whose Jacobian is computed as

$$\left|\frac{\partial v_*}{\partial v'}\right| = \frac{8}{\left|I - \mathbf{k}\otimes\sigma\right|} = \frac{8}{|1 - \mathbf{k}\cdot\sigma|} = \frac{4}{\sin^2(\theta/2)} \sim \theta^{-2},\ \theta \in [0, \pi/2],)$$

was also introduced to show the existence of solutions to the "linearized" Boltzmann equation, and was used in [3, 4, 9] to prove the uniqueness of solutions with polynomial decay in the velocity variable to the nonlinear Boltzmann equation for Maxwellian and soft potentials. Especially in [9], the uniqueness of solutions was considered in the following function space; for $m \in \mathbb{R}, \ell \ge 0$ and $T > 0$,

$$\begin{aligned}\mathcal{P}^m_\ell([0,T]\times\mathbb{R}^6_{x,v}) &= \Big\{f \in C^0([0,T]; \mathcal{S}'(\mathbb{R}^6_{x,v}));\\ &\qquad s.t.\ f \in L^\infty([0,T]\times\mathbb{R}^3_x;\ H^m_\ell(\mathbb{R}^3_v))\Big\},\end{aligned}$$

$$\|f\|_{H^m_\ell(\mathbb{R}^3_v)} = \left(\int_{\mathbb{R}^3} |\langle v\rangle^\ell (\langle D_v\rangle^m f(v))|^2 dv\right)^{1/2}, \quad \langle v\rangle = (1 + |v|^2)^{1/2}\,.$$

An effective use of the *singular* change of variables gives us

Theorem 1.1 ([9]). *Assume that the cross section B takes the form (1.2) with $0 < s < 1$ and $\max\{-3, -3/2 - 2s\} < \gamma \le 0$. Suppose that the Cauchy problem (1.1) admits two weak solutions $f_1(t), f_2(t) \in \mathcal{P}^{2s}_{\ell_0}([0,T]\times\mathbb{R}^6_{x,v})$ with $0 < T < +\infty$ and $\ell_0 \ge 14$ having the same initial datum $f_0 \in L^\infty(\mathbb{R}^3_x; H^{2s}_{\ell_0}(\mathbb{R}^3_v))$. If one solution is non-negative, then $f_1(t) \equiv f_2(t)$.*

Here the weak solution to the Cauchy problem (1.1) is defined by

$$\begin{aligned}\int_{\mathbb{R}^6} f(t,x,v)\eta(t,x,v)dxdv - \int_{\mathbb{R}^6} f_0(x,v)\eta(0,x,v)dxdv \\ - \int_0^t d\tau \int_{\mathbb{R}^6} f(\tau,x,v)(\partial_\tau + v\cdot\nabla_x)\eta(\tau,x,v)dxdv \\ = \int_0^t d\tau \int_{\mathbb{R}^6} Q(f,f)(\tau,x,v)\eta(\tau,x,v)dxdv,\end{aligned}$$

where $\eta \in C^1(\mathbb{R}; C_0^\infty(\mathbb{R}^6))$ is a test function.

Compared with the uniqueness of polynomial decay solutions in the velocity variables, there are few results concerning the existence of such slowly decay solutions in spatially inhomogeneous case (cf., renormalized solutions by [13, 11], and [12] in the cutoff case). In fact, the existence of classical solutions for the spatially inhomogeneous Boltzmann equation has been usually discussed for solutions with the Maxwellian decay weight in the velocity variables (see [3, 4, 5, 6, 8, 10, 14] in the non-cutoff case). In the next section we state a local existence result concerning polynomial decay solutions in the velocity variable to the full nonlinear Boltzmann equation in a certain soft potential case, by an effective use of the *singular* change of variables between post- and pre-collisional velocities.

2. Local existence for soft potentials

Throughout this section we confine ourselves to the case

$$0 < s < \frac{1}{2}, \quad -\frac{3}{2} < \gamma \le 0, \tag{2.3}$$

because of the technical difficulties, though the uniqueness result,Theorem 1.1, holds under the more general situation $0 < s < 1$ and $\max\{-3, -2s-3/2\} < \gamma \le 0$.

We introduce our working function spaces as follows: Set

$$\partial^\alpha_\beta = \partial^\alpha_x \partial^\beta_v, \qquad \alpha, \beta \in \mathbb{N}^3.$$

and

$$\mathcal{W} = \begin{cases} \langle v\rangle & \text{if } 0 < s \le 1/4, \\ \langle v\rangle^{2s/(1-2s)} & \text{if } 1/4 < s < 1/2, \end{cases} \tag{2.4}$$

which ensures $\langle v\rangle^{2s} \le \mathcal{W}^{1-2s}$ and $\langle v\rangle \le \mathcal{W}$ for the later use. As in [4, 7], we use a kind of cutoff function in both space and velocity variables,

$$\phi(x,v) = \frac{1}{1+|v|^2+|x|^2}, \tag{2.5}$$

which possesses the commutator property $|[v\cdot\nabla_x, \phi]| = 2|v\cdot x|\phi^2 \le \phi$. For $k \in \mathbb{N}$, $\ell \in \mathbb{R}$ with $k < \ell$, we define

$$\begin{aligned}\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6) = \{g \mid \|g\|^2_{\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6)} \\ = \sum_{|\alpha+\beta|\le k} \sup_{a\in\mathbb{R}^3} \int_{\mathbb{R}^6} \left|\phi(x-a,v)\mathcal{W}^{\ell-|\alpha+\beta|}\partial^\alpha_\beta g(x,v)\right|^2 dxdv < +\infty\}.\end{aligned} \tag{2.6}$$

The function space $\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6)$ is a variant of the uniformly local Sobolev space $H^{k,\ell}_{ul}(\mathbb{R}^6)$ employed in [8], which is defined by replacing $\mathcal{W}^{\ell-|\alpha+\beta|}$ and $\phi(x,v)$ by

$\langle v\rangle^\ell$ and a usual smooth cutoff function $\phi_1(x) \in C_0^\infty(\mathbb{R}^3)$, respectively. In [8], bounded classical solutions for the initial data $f_0(x,v)$ satisfying

$$\exists \rho_0 > 0 \ s.t. \ e^{\rho_0\langle v\rangle^2} f_0 \in H_{ul}^{k,0}(\mathbb{R}^6) \tag{2.7}$$

are constructed in the whole space without specifying any limit behaviors at the spatial infinity and without the smallness condition on initial data, under the assumptions on the cross section B with

$$0 < s < 1/2, \quad -3/2 < \gamma, \quad \gamma + 2s < 1.$$

From the point view of the local existence of polynomial decay solution in the velocity variable, we have the following improvement of Theorem 1.1 of [8] for the soft potential case;

Theorem 2.1. *Assume that the cross section B takes the form* (1.2) *with* (2.3), *that is,* $0 < s < 1/2$, $-3/2 < \gamma \leq 0$. *If the initial data f_0 is non-negative and belongs to $\mathcal{H}_{ul}^{k,\ell}(\mathbb{R}^6)$ for $k \geq 6, \ell \geq k+7$, then, there exists a $T_* > 0$ such that the Cauchy problem* (1.1) *admits a non-negative unique solution in the function space* $C^0([0,T_*];\mathcal{H}_{ul}^{k,\ell}(\mathbb{R}^6))$.

Remark 2.2. *The rectangle below expresses the domain of (γ, s) covered by Theorem 2.1. The previous local existence result under* (2.7) *in* [8] *covers an additional triangle region below the line $\gamma = 1 - 2s$, which is contained in the hard potential region $\gamma > 0$. Time global solutions near a global equilibrium,*

$$f = \mu + \sqrt{\mu} g, \ \mu = e^{-|v|^2/2}/(2\pi)^{3/2}.$$

were given in [4, 5, 6], [14], *which cover the full region* $0 < s < 1$, $\gamma > \max\{-3, -3/2 - 2s\}$ *indicated by the figure below.*

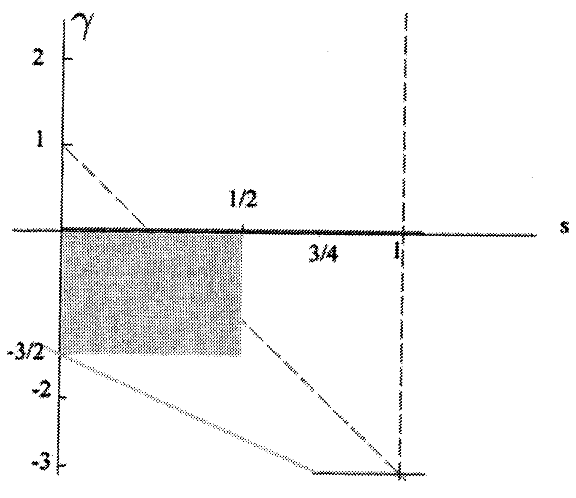

FIGURE 2. dashed line: $\gamma = 1 - 4s$ in case of inverse power law potential

For the proof of Theorem 2.1, we construct the approximate solutions by angular cutoff approximation. That is, for $0 < \varepsilon \ll 1$, we approximate (cutoff) the cross section by

$$b_\epsilon(\cos\theta) = \begin{cases} b(\cos\theta) & (\theta \geq 2\epsilon), \\ 0 & (\theta < 2\epsilon). \end{cases}$$

Theorem 2.3 (Cutoff case). *Assume that* $-3/2 < \gamma \le 0$ *and replace the angular factor of the cross section* b *by* b_ε. *If the initial data* f_0 *is non-negative and belongs to* $\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6)$ *for* $k \ge 5, \ell \ge k+7$, *then, there exists a* $T_\varepsilon > 0$ *such that the Cauchy problem* (1.1) *admits a non-negative unique solution* $f^\varepsilon(t,x,v)$ *in the function space* $C^0([0,T_\varepsilon];\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6))$.

Remark 2.4. *In the cutoff case, the order of derivative* k *can be taken not less than* 5 *instead of* 6 *for the non-cutoff case in our analysis. We might improve the order of derivatives by use of the fractional derivatives employed in* [10], *instead of* ∂_x, ∂_v.

Another key ingredient is to obtain a uniform estimate for solutions in the given function space. Let $T > 0$ and $f(t) \in C^0([0,T];\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6))$ with $k \ge 6$ and $\ell \ge k+7$. If we put

$$\mathcal{E}(t) = \|f(t)\|^2_{\mathcal{H}^{k,\ell}_{ul}},$$

then there exists a $C > 0$ depending only on s, γ, k, ℓ and $K > 0$ in the hypothesis of b such that

$$\mathcal{E}(t) \le \mathcal{E}(0) + C\int_0^t \mathcal{E}(\tau)(1+\mathcal{E}(\tau))d\tau\,, \quad t \in [0,T]\,, \tag{2.8}$$

where we refer [16] to the detail derivation of this estimate, by means of both *regular* and *singular* changes of variables between post- and pre-collisional velocities. It follows from (2.8) that we have

$$\mathcal{E}(t) \le \frac{\mathcal{E}(0)e^{Ct}}{1-(e^{Ct}-1)\mathcal{E}(0)}\,,$$

by exactly the same calculation as the one after (4.3.11) of [3]. If we choose $T_* > 0$ small enough such that

$$T_* = \frac{1}{C}\log\left(1+\frac{3}{1+4\|f_0\|_{\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6)}}\right)$$

then we obtain a uniform estimate

$$\|f(t)\|_{\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6)} \le 2\|f_0\|_{\mathcal{H}^{k,\ell}_{ul}(\mathbb{R}^6)} \text{ for } t \in [0,T_*]\,. \tag{2.9}$$

The proof of Theorem 2.1 can be completed in the almost same way as in the proof of Theorem 4.11 of [3] and the subsequent paragraph there, taking into account the uniform estimate (2.9) and Theorem 2.3.

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