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On an Application of Nash-Moser Theory to the Vacuum Boundary Problem of Gas Dynamics

Tetu Makino *
Yamaguchi University

1 Introduction

We consider spherically symmetric motions of atmosphere governed by the compressible Euler equations:

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\rho\frac{\partial u}{\partial r}+\frac{2}{r}\rho u=0,\\ &\rho\Big(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\Big)+\frac{\partial P}{\partial r}=-\frac{g_0\rho}{r^2}\quad (R_0\leq r)\end{aligned}\tag{1}$$

and the boundary value condition

$$\rho u|_{r=R_0}=0. \tag{2}$$

Here ρ is the density, u the velocity, P the pressure. R_0 (>0) is the radius of the central solid ball, and $g_0=G_0M_0$, G_0 being the gravitational constant, M_0 the mass of the central ball. We assume that

$$P=A\rho^{\gamma}, \tag{3}$$

where A and γ are positive constants such that $1<\gamma\leq 2$.

*Department of Applied Mathematics, Yamaguchi University, Ube 755-8611, JAPAN, e-mail: makino@yamaguchi-u.ac.jp

Equilibria of the problem are given by

$$\bar{\rho}(r) = \begin{cases} A_1\Big(\dfrac{1}{r} - \dfrac{1}{R}\Big)^{\frac{1}{\gamma-1}} & (R_0 \leq r < R) \\ 0 & (R \leq r), \end{cases}$$

where R is an arbitrary number such that $R > R_0$ and

$$A_1 = \Big(\frac{(\gamma-1)g_0}{\gamma A}\Big)^{\frac{1}{\gamma-1}}.$$

Remark The total mass M of the equilibrium is given by

$$M = 4\pi A_1 \int_{R_0}^{R} \Big(\frac{1}{r} - \frac{1}{R}\Big)^{\frac{1}{\gamma-1}} r^2 dr.$$

If M is given in the interval $(0, M^*(\gamma))$, then R is uniquely determined. Here

$$M^*(\gamma) = \frac{4\pi A_1(\gamma-1)}{4-3\gamma} R_0^{-\frac{4-3\gamma}{\gamma-1}}$$

if $\gamma < 4/3$, and $M^*(\gamma) = +\infty$ if $\gamma \geq 4/3$. □

Let us fix one of these equilibria. We are interested in motions around this equilibrium.

We are interested a solution $(\rho(t,r), u(t,r))$, which is continuous on $0 \leq t \leq T, R_0 \leq r < \infty$ and there should be found a continuous curve $r = R_F(t), 0 \leq t \leq T$, such that $|R_F(t) - R| \ll 1, \rho(t,r) > 0$ for $0 \leq t \leq T, R_0 \leq r < R_F(t)$ and $\rho(t,r) = 0$ for $0 \leq t \leq T, R_F(t) \leq r < \infty$. The curve $r = R_F(t)$ is the free boundary at which the density touches the vacuum. It will be shown that the solution satisfies

$$\rho(t,r) = C(t)(R_F(t) - r)^{\frac{1}{\gamma-1}}(1 + O(R_F(t) - r))$$

as $r \to R_F(t) - 0$. Here $C(t)$ is positive and smooth in t. This situation is "physical vacuum boundary" so-called by [3] and [1].

The major difficulty of the analysis comes from the free boundary touching the vacuum, which can move along time. So it is convenient to introduce the Lagrangian mass coordinate

$$m = 4\pi \int_{R_0}^{r} \rho(t,r')r'^2 dr',$$

to fix the interval of independent variable to consider.

Let us take $\bar{r} = \bar{r}(m)$ as the independent variable instead of m, where $m \mapsto \bar{r} = \bar{r}(m)$ is the inverse function of the function

$$\bar{r} \mapsto m = 4\pi \int_{R_0}^{\bar{r}} \bar{\rho}(r')r'^2 dr'.$$

• *Without loss of generality, we can and shall assume that*

$$R_0 = 1, \quad g_0 = \frac{1}{\gamma - 1}, \quad A = \frac{1}{\gamma}, \quad A_1 = 1.$$

Introducing the unknown variable y for perturbation by

$$r = \bar{r}(1 + y), \tag{4}$$

we can write the equation as

$$\frac{\partial^2 y}{\partial t^2} - \frac{1}{\rho r}(1+y)^2 \frac{\partial}{\partial r}\Big(PG\Big(y, r\frac{\partial y}{\partial r}\Big)\Big) - \frac{1}{\gamma - 1}\frac{1}{r^3}H(y) = 0, \tag{5}$$

where

$$G(y, v) := 1 - (1+y)^{-2\gamma}(1+y+v)^{-\gamma} = \gamma(3y+v) + [y, v]_2,$$
$$H(y) := (1+y)^2 - \frac{1}{(1+y)^2} = 4y + [y]_2$$

and we have used the abbreviations r, ρ, P for $\bar{r}, \bar{\rho}, \bar{P}$.

Notational Remark Here and hereafter $[X]_q$ denotes a convergent power series, or an analytic function given by the series, of the form $\sum_{j\geq q} a_j X^j$, and $[X, Y]_q$ stands for a convergent double power series of the form $\sum_{j+k\geq q} a_{jk} X^j Y^k$. □

We are going to study the equation (5) on $1 < r < R$ with the boundary condition

$$y|_{r=1} = 0.$$

Of course y and $r\dfrac{\partial y}{\partial r}$ will be confined to

$$|y| + \left|r\frac{\partial y}{\partial r}\right| < 1.$$

2 Analysis of the linear part

The linear part of the equation (5) is clearly

$$\frac{\partial^2 y}{\partial t^2} + \mathcal{L}\Big(r, \frac{\partial}{\partial r}\Big) y = 0, \tag{6}$$

where

$$\begin{aligned}\mathcal{L}y = \mathcal{L}\Big(r, \frac{d}{dr}\Big) y &:= -\frac{1}{\rho r}\frac{d}{dr}\Big(P\gamma\Big(3y + r\frac{dy}{dr}\Big)\Big) - \frac{1}{\gamma-1}\frac{1}{r^3}(4y) \\ &= -\Big(\frac{1}{r} - \frac{1}{R}\Big)\frac{d^2 y}{dr^2} + \Big(-\frac{4}{r}\Big(\frac{1}{r} - \frac{1}{R}\Big) + \frac{\gamma}{\gamma-1}\frac{1}{r^2}\Big)\frac{dy}{dr} + \frac{3\gamma-4}{\gamma-1}\frac{y}{r^3}.\end{aligned} \tag{7}$$

Let us introduce the independent variable z by

$$z = \frac{R-r}{R} \tag{8}$$

and the parameter N by

$$\frac{\gamma}{\gamma-1} = \frac{N}{2} \quad \text{or} \quad \gamma = 1 + \frac{2}{N-2}. \tag{9}$$

Then we can write

$$R^3\mathcal{L}y = -\frac{z}{1-z}\frac{d^2 y}{dz^2} - \frac{\frac{N}{2} - 4z}{(1-z)^2}\frac{dy}{dz} + \frac{8-N}{2}\frac{1}{(1-z)^3}y. \tag{10}$$

The variable z runs over the interval $[0, 1-1/R]$, the boundary $z = 0$ corresponds to the free boundary touching the vacuum, and the boundary condition at $z = 1 - 1/R$ is the Dirichlet condition $y = 0$.

Although the boundary $z = 1 - 1/R$ is regular, the boundary $z = 0$ is singular. The eigenvalue problem $\mathcal{L}y = \lambda y$ can be written as

$$-z\frac{d^2 y}{dz^2} - \Big(\frac{N}{2}\frac{1}{1-z} - \frac{4z}{1-z}\Big)\frac{dy}{dz} + \frac{8-N}{2}\frac{y}{(1-z)^2} = \lambda R^3(1-z)y. \tag{11}$$

Remark. The equation (11) as an equation on the Riemann sphere has three singular points: $z = 0, 1$ are regular singular points, and $z = \infty$ is an irregular singular point (Poincaré index = 1) provided $\lambda \neq 0$. This is so called a 'confluent Heun equation', and is related to the equation Painlevé V. Not so many informations are available, according to Professor Y. Ohyama ([6]). □

Using the Liouville transformation, we can claim

Proposition 1 *The operator* $\mathfrak{T}_0, \mathcal{D}(\mathfrak{T}_0) = C_0^\infty(0, 1-\frac{1}{R}), \mathfrak{T}_0 y = \mathcal{L}y$ *in*

$$\mathfrak{X} := L^2((0, 1-\frac{1}{R}), z^{\frac{N-2}{2}}(1-z)^{\frac{10-N}{2}}dz),$$

has the Friedrichs extension $\mathfrak{T}$, *a self-adjoint operator whose spectrum consists of simple eigenvalues* $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty$.

Moreover we have

Proposition 2 *If* $N \leq 8$ *(or* $\gamma \geq 4/3$ *), the least eigenvalue* λ_1 *is positive.*

Let us introduce a new variable

$$\begin{aligned} x &= \frac{R^3}{4}\Big(\sqrt{z(1-z)} + \tan^{-1}\sqrt{\frac{z}{1-z}}\Big)^2 \\ &= \frac{R^3}{4}\Big(\frac{\sqrt{(R-r)r}}{R} + \tan^{-1}\sqrt{\frac{R-r}{r}}\Big)^2. \end{aligned} \tag{12}$$

Then we can write

$$\mathcal{L}y = -\triangle y + L_1(x)x\frac{dy}{dx} + L_0(x)y, \tag{13}$$

where

$$\triangle = x\frac{d^2}{dx^2} + \frac{N}{2}\frac{d}{dx}$$

and $L_1(x)$ and $L_0(x)$ are analytic on $0 \leq x < x_\infty := \pi^2R^3/16$. While r runs over the interval $[1, R]$, x runs over $[0, x_R]$, where

$$x_R := \frac{R^3}{4}\Big(\int_0^{1-\frac{1}{R}}\sqrt{\frac{1-z}{z}}dz\Big)^2 < x_\infty = \frac{\pi^2R^3}{16}.$$

Note that

$$x = R^3z + [z]_2 = R^2(R-r) + [R-r]_2.$$

Let us fix a positive eigenvalue $\lambda = \lambda_n$ and an associated eigenfunction $\Phi(x)$ of $\mathcal{L}$. Then

$$y_1(t,x) = \sin(\sqrt{\lambda}t + \theta_0)\Phi(x) \tag{14}$$

is a time-periodic solution of the linearized problem

$$\frac{\partial^2 y}{\partial t^2} + \mathcal{L}y = 0, \qquad y|_{x=x_R} = 0.$$

Moreover we claim

Proposition 3 *The eigenfunction* $\Phi(x)$ *is an analytic function of* $0 \leq x < x_\infty$.

3 Main result

We rewrite the equation (5) by using the linearized part $\mathcal{L}$ defined by (7) as

$$\frac{\partial^2 y}{\partial t^2} + \Big(1 + G_I\Big(y, r\frac{\partial y}{\partial r}\Big)\Big)\mathcal{L}y + G_{II}\Big(r, y, r\frac{\partial y}{\partial r}\Big) = 0, \tag{15}$$

where

$$\begin{aligned}
G_I(y,v) &= (1+y)^2\Big(1 + \frac{1}{\gamma}\partial_v G_2(y,v)\Big) - 1,\\
G_{II}(r,y,v) &= \frac{P}{\rho r^2}G_{II0}(y,v) + \frac{1}{\gamma-1}\frac{1}{r^3}G_{II1}(y,v),\\
G_{II0}(y,v) &= (1+y)^2(3\partial_v G_2 - \partial_y G_2)v,\\
G_{II1}(y,v) &= (1+y)^2\Big(\frac{1}{\gamma}(\partial_v G_2)((4-3\gamma)y - \gamma v) + G_2\Big) - H + 4y(1+y)^2.
\end{aligned}$$

Here

$$G_2 := G - \gamma(3y+v) = [y,v]_2,$$

$$\partial_v G_2 = \frac{\partial G}{\partial v} - \gamma = [y,v]_1, \quad \partial_y G_2 = \frac{\partial G}{\partial y} - 3\gamma = [y,v]_1.$$

We have fixed a solution y_1 of the linearized equation $y_{tt} + \mathcal{L}y = 0$ (see (14)), and we seek a solution y of (5) or (15) of the form

$$y = \varepsilon y_1 + \varepsilon w,$$

where ε is a small positive parameter.

Remark The following discussion is valid if we take

$$y_1 = \sum_{k=1}^{K} c_k \sin(\sqrt{\lambda_{n_k}}t + \theta_k)\cdot\Phi_k(x), \tag{14$'$}$$

where Φ_k is an eigenfunction of $\mathcal{L}$ associated with the positive eigenvalue λ_{n_k} and c_k and θ_k are constants for $k = 1, \cdots, K$. □

Then the equation which governs w turns out to be

$$\frac{\partial^2 w}{\partial t^2} + \Big(1 + \varepsilon a(t,r,w,r\frac{\partial w}{\partial r},\varepsilon)\Big)\mathcal{L}w + \varepsilon b(t,r,w,r\frac{\partial w}{\partial r},\varepsilon) = \varepsilon c(t,r,\varepsilon), \tag{16}$$

where

$$
\begin{aligned}
a(t,r,w,\Omega,\varepsilon) &= \varepsilon^{-1}G_I(\varepsilon(y_1+w),\varepsilon(v_1+\Omega)),\\
b(t,r,w,\Omega,\varepsilon) &= \varepsilon^{-1}G_I(\varepsilon(y_1+w),\varepsilon(v_1+\Omega))\mathcal{L}y_1 + \varepsilon^{-2}G_{II}(r,\varepsilon(y_1+w),\varepsilon(v_1+\Omega))\\
&\quad - \varepsilon^{-1}G_I(\varepsilon y_1,\varepsilon v_1)\mathcal{L}y_1 - \varepsilon^{-2}G_{II}(r,\varepsilon y_1,\varepsilon v_1),\\
c(t,r,\varepsilon) &= -\varepsilon^{-1}G_I(\varepsilon y_1,\varepsilon v_1)\mathcal{L}y_1 - \varepsilon^{-2}G_{II}(r,\varepsilon y_1,\varepsilon v_1).
\end{aligned}
$$

Here v_1 stands for $r\partial y_1/\partial r$.

The main result of this study can be stated as follows:

Theorem 1 *For any $T > 0$, there is a sufficiently small positive $\varepsilon_0(T)$ such that, for $0 < \varepsilon \le \varepsilon_0(T)$, there is a solution w of (16) such that $w \in C^\infty([0,T]\times[1,R])$ and*

$$\sup_{j+k\le n}\left\|\left(\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial r}\right)^k w\right\|_{L^\infty([0,T]\times[1,R])} \le C_n\varepsilon,$$

or a solution $y \in C^\infty([0,T]\times[1,R])$ of (5) or (15) of the form

$$y(t,r) = \varepsilon y_1(t,r) + O(\varepsilon^2),$$

or a motion which can be expressed by the Lagrangian coordinate as

$$r(t,m) = \bar{r}(m)(1+\varepsilon y_1(t,\bar{r}(m)) + O(\varepsilon^2))$$

for $0 \le t \le T, 0 \le m \le M$.

Our task is to find the inverse image $\mathfrak{P}^{-1}(\varepsilon c)$ of the nonlinear mapping $\mathfrak{P}$ defined by

$$\mathfrak{P}(w) = \frac{\partial^2 w}{\partial t^2} + (1+\varepsilon a)\mathcal{L}w + \varepsilon b. \tag{17}$$

Let us note that $\mathfrak{P}(0) = 0$. This task, which will be done by applying the Nash-Moser theorem, will require a certain property of the derivative of $\mathfrak{P}$:

$$
\begin{aligned}
D\mathfrak{P}(w)h &:= \lim_{\tau\to 0}\frac{1}{\tau}(\mathfrak{P}(w+\tau h) - \mathfrak{P}(w))\\
&= \frac{\partial^2 h}{\partial t^2} + (1+\varepsilon a_1)\mathcal{L}h + \varepsilon a_{21} r\frac{\partial h}{\partial r} + \varepsilon a_{20}h,
\end{aligned} \tag{18}
$$

where

$$a_1 = a(t,r,w,r\frac{\partial w}{\partial r},\varepsilon), \quad a_{20} = \frac{\partial a}{\partial w}\mathcal{L}w + \frac{\partial b}{\partial w}, \quad a_{21} = \frac{\partial a}{\partial \Omega}\mathcal{L}w + \frac{\partial b}{\partial \Omega}$$

are smooth functions of $t, r, w, r\frac{\partial w}{\partial r}, \varepsilon$. Here Ω is the dummy of $r\frac{\partial w}{\partial r}$, that is, a and b are functions of $t, r, w, \Omega = r\frac{\partial w}{\partial r}, \varepsilon$ and $\partial a/\partial\Omega$ $[(\partial b/\partial\Omega)]$ denotes the partial derivative of a $[(b)]$ with respect to $\Omega = r\partial w/\partial r$, respectively. We consider $D\mathfrak{P}(w)$ as a second order linear partial differential operator for each fixed w.

• Hereafter we use the variable x defined by (12) instead of r. A function of $1 \le r \le R$ which is infinitely many times continuously differentiable is also so as a function of $0 \le x \le x_R$.

We can claim

Proposition 4 *There are smooth functions b_1, b_0 of $t, x, w, \partial w/\partial x, \partial^2 w/\partial x^2$ such that*

$$D\mathfrak{P}(w)h = \frac{\partial^2 h}{\partial t^2} - (1+\varepsilon a_1)\triangle h + b_1 x\frac{\partial h}{\partial x} + b_0 h. \tag{19}$$

Remark The factor x in the term $b_1 x\frac{\partial h}{\partial x}$ is important. In fact $B\frac{\partial h}{\partial x}$, B being a non-zero constant, without the factor x cannot be considered as a perturbation term, since it has the same order with the principal part $\triangle h = x\frac{\partial^2 h}{\partial x^2} + \frac{N}{2}\frac{\partial h}{\partial x}$. See the proof of the following Lemma 3. □

Using this representation of $D\mathfrak{P}(w)$, we can prove the following energy estimate:

Lemma 1 *If a solution of $D\mathfrak{P}(w)h = g$ satisfies*

$$h|_{x=x_R} = 0, \quad h|_{t=0} = \frac{\partial h}{\partial t}\Big|_{t=0} = 0,$$

then h enjoys the energy inequality

$$\|\partial_t h\|_{\mathfrak{X}} + \|\dot{D}h\|_{\mathfrak{X}} + \|h\|_{\mathfrak{X}} \le C\int_0^T \|g(t')\|_{\mathfrak{X}} dt',$$

where $\partial_t = \partial/\partial t, \dot{D} = \sqrt{x}\partial/\partial x$ *and* C *depends only on* N, R, T, $A := \|\varepsilon\partial_t a_1\|_{L^\infty} + \sqrt{2}\|\varepsilon\dot{D}a_1 + b_1\|_{L^\infty}$ *and* $B := \|b_0\|_{L^\infty}$, *provided that* $|\varepsilon a_1| \leq 1/2$. *Here we have used the notation*

$$\|y\|_{\mathfrak{X}} := \Big(\int_0^{x_R} |y|^2 x^{\frac{N}{2}-1} dx\Big)^{1/2}.$$

The proof of the main result is done by applying the Nash-Moser theorem formulated by R. Hamilton ([2, p.171, III.1.1.1]):

Nash-Moser Theorem *Let* $\mathfrak{E}_0$ *and* $\mathfrak{E}$ *be tame spaces,* U *an open subset of* $\mathfrak{E}_0$ *and* $\mathfrak{P} : U \to \mathfrak{E}$ *a smooth tame map. Suppose that the equation for the derivative* $D\mathfrak{P}(w)h = g$ *has a unique solution* $h = V\mathfrak{P}(w)g$ *for all* w *in* U *and all* g, *and that the family of inverse* $V\mathfrak{P} : U \times \mathfrak{E} \to \mathfrak{E}_0$ *is a smooth tame map. Then* $\mathfrak{P}$ *is locally invertible.*

In order to apply the Nash-Moser theorem, we consider the spaces of functions of t and x:

$$\mathfrak{E} := \{y \in C^\infty([-2\tau_1, T] \times [0, x_R]) \quad | \quad y(t,x) = 0 \quad \text{for} \quad -2\tau_1 \leq t \leq -\tau_1\},$$
$$\mathfrak{E}_0 := \{w \in \mathfrak{E} \ | \quad w|_{x=x_R} = 0\}.$$

Here τ_1 is a positive number. Let U be the set of all functions w in $\mathfrak{E}_0$ such that $|w| + |\partial w/\partial x| < 1$ and suppose that $|\varepsilon| \leq \varepsilon_1$, ε_1 being a small positive number. Then we can consider that the nonlinear mapping $\mathfrak{P}$ maps $U(\subset \mathfrak{E}_0)$ into $\mathfrak{E}$.

For $y \in \mathfrak{E}$, $n \in \mathbb{N}$, let us define

$$\|y\|_n^{(\infty)} := \sup_{0 \leq j+k \leq n} \Big\|\Big(-\frac{\partial^2}{\partial t^2}\Big)^j (-\triangle)^k y\Big\|_{L^\infty([-2\tau_1,T]\times[0,x_R])}.$$

Then we can claim that $\mathfrak{E}$ turns out to be tame by this grading $(\|\cdot\|_n^{(\infty)})_n$.

On the other hand, let us define

$$\|y\|_n^{(2)} := \Big(\sum_{0\leq j+k\leq n} \int_{-\tau_1}^{T} \|\Big(-\frac{\partial^2}{\partial t^2}\Big)^j (-\triangle)^k y\|_{\mathfrak{X}}^2 dt\Big)^{1/2}.$$

Here $\mathfrak{X} = L^2((0, x_R); x^{\frac{N}{2}-1}dx)$ and

$$\|y\|_{\mathfrak{X}} := \Big(\int_0^{x_R} |y(x)|^2 x^{\frac{N}{2}-1} dx\Big)^{1/2}.$$

By Sobolev imbedding, we see that the grading $(\|\cdot\|_n^{(2)})_n$ is tamely equivalent to the grading $(\|\cdot\|_n^{(\infty)})_n$, that is, we have

$$\frac{1}{C}\|y\|_n^{(2)} \leq \|y\|_n^{(\infty)} \leq C\|y\|_{n+s}^{(2)}$$

with $2s > 1 + N/2$. Hence $\mathfrak{E}$ is tame with respect to $(\|\cdot\|_n^{(2)})_n$, too. The grading $(\|\cdot\|_n^{(2)})_n$ will be suitable for estimates of solutions of the associated linear wave equations.

Note that $\mathfrak{E}_0$ is a closed subspace of $\mathfrak{E}$ endowed with these gradings.

We can claim that

$$\|\mathfrak{P}(w)\|_n^{(\infty)} \leq C(1 + \|w\|_{n+1}^{(\infty)}),$$

provided that $\|w\|_1^{(\infty)} \leq M$. This says that the mapping $\mathfrak{P}$ is tame with respect to the grading $(\|\cdot\|_n^{(\infty)})_n$.

Therefore the problem is concentrated to estimates of the solution and its higher derivatives of the linear equation

$$D\mathfrak{P}(w)h = g,$$

when w is fixed in $\mathfrak{E}_0$ and g is given in $\mathfrak{E}$. Actually a tame estimate

$$\|h\|_n^{(2)} \leq C(1 + \|g\|_n^{(2)} + \|w\|_{n+3+s}^{(2)})$$

with $2s > 1 + N/2$, provided that $\|g\|_1^{(2)} \leq M$ and $\|w\|_{3+s}^{(2)} \leq M$, can be derived. See [4, Section 5]. Here h is the solution of the equation

$$D\mathfrak{P}(w)h \equiv \frac{\partial^2 h}{\partial t^2} - (1 + \varepsilon a_1)\triangle h + b_1 x\frac{\partial h}{\partial x} + b_0 h = g$$

for given $g \in \mathfrak{E}$, provided that $|\varepsilon a_1| \leq 1/2$. This estimate says that the mapping $(w, g) \mapsto h$ is tame with respect to the grading $(\|\cdot\|_n^{(2)})_n$. This completes the proof of the applicability of Nash-Moser theorem.

4 Open problem – initial-boundary value problem

Let us consider the initial-boundary value problem

$$\frac{\partial^2 y}{\partial t^2} + \Big(1 + G_I\Big(y, r\frac{\partial y}{\partial r}\Big)\Big)\mathcal{L}y + G_{II}\Big(r, y, r\frac{\partial y}{\partial r}\Big) = 0, \tag{15}$$

$$y|_{r=1} = 0, \tag{20}$$

$$y|_{t=0} = \psi_0(r), \qquad \frac{\partial y}{\partial t}\Big|_{t=0} = \psi_1(r), \tag{21}$$

where $\psi_0(r), \psi_1(r)$ are given smooth functions of $1 \leq r \leq R$ such that $\psi_0(1) = \psi_1(1) = 0$. We want to establish a local-in-time existence theorem to this initial-boundary value problem. But this task is still not yet done. This may be an interesting open problem, so let us explain the difficulty.

Let us formulate the compatibility condition. For the sake of simplicity, we write the equation (15) as

$$\frac{\partial^2 y}{\partial t^2} = \mathcal{B}(r, y, y', y''),$$

where y', y'' stand for $\partial y/\partial r, \partial^2 y/\partial r^2$. Put

$$\mathcal{B}_0(r, [y_0]) = \mathcal{B}(r, [y_0]),$$

where $[y]$ stands for $(y, y', y'') = (y, \partial y/\partial r, \partial^2 y/\partial r^2)$. For $p \geq 1$ we define $\mathcal{B}_p(r, [y_0], [y_1], \cdots, [y_p])$ by the recurrence formula

$$\mathcal{B}_{p+1} := \sum_{j=0}^{p} \Big[y_{j+1}\partial_{y_j} + y'_{j+1}\partial_{y'_j} + y''_{j+1}\partial_{y''_j} \Big] \mathcal{B}_p.$$

Then we can define the sequence $(\psi_p)_{p\in\mathbb{N}}$ of functions of $1 \leq r \leq R$ by

$$\begin{aligned} &\psi_2(r) := \mathcal{B}(r, [\psi_0]), \\ &\psi_3(r) := \mathcal{B}_1(r, [\psi_0], [\psi_1]), \\ &\psi_{p+2}(r) := \mathcal{B}_p(r, [\psi_0], [\psi_1], \cdots, [\psi_p]) \quad \text{for} \quad p \geq 2. \end{aligned}$$

If $y \in C^\infty([0, T] \times [1, R])$ is a smooth solution of (15)(20)(21), then we have

$$\frac{\partial^p y}{\partial t^p}\Big|_{t=0} = \psi_p(r).$$

Therefore it is necessary that

$$\psi_p(1) = 0 \qquad \text{for} \qquad \forall p \geq 0 \tag{22}$$

in order that there is a smooth solution y to (15)(20)(21). So the condition (22) is the compatibility condition to the nonlinear initial-boundary value problem (15)(20)(21). Since ψ_p are determined by ψ_0, ψ_1 inductively, the condition (22) is a condition of (ψ_0, ψ_1).

Our conjecture is that **if (ψ_0, ψ_1) satisfies the compatibility condition, then there is a solution $y \in C^\infty([0,T] \times [1,R])$ to (15)(20)(21), T being sufficiently small.**

But this is not yet verified. Let us explain where is the difficulty.

Assume that we try to prove this local-in-time existence theorem using the Nash-Moser theorem in the same manner as the preceding section. There we constructed a solution w which vanishes identically on $t \leq -2\tau_1$ to simplify the discussion of compatibility. But the situation is different in the present case. Actually we can introduce w by

$$y = y_1^*(t,r) + w,$$

where, for example,

$$y_1^*(t,r) = \psi_0(r) + t\psi_1(r),$$

and get the problem

$$\begin{aligned}
&\mathfrak{P}(w) := \frac{\partial^2 w}{\partial t^2} - \mathcal{B}(r, [y_1^* + w]) + \mathcal{B}(r, [y_1^*]) = \mathcal{B}(r, [y_1^*]),\\
&w|_{r=1} = 0,\\
&w|_{t=0} = \frac{\partial w}{\partial t}\Big|_{t=0} = 0.
\end{aligned}$$

If we try to apply the Nash-Moser theorem in the same manner as the preceding section, we should suppose the linearized problem

$$D\mathfrak{P}(w)h = g, \quad h|_{r=1} = 0, \quad h|_{t=0} = \partial_t h|_{t=0} = 0$$

enjoys the compatibility condition, fixed an arbitrary smooth function $w = w(t,r)$, which is not yet a solution of $\mathfrak{P}(w) = 0$, such that

$$\frac{\partial^p}{\partial t^p}(y_1^* + w)\Big|_{t=0} = \psi_p(r)$$

for $\forall p \in \mathbb{N}$, that is,

$$\frac{\partial^p w}{\partial t^p}\Big|_{t=0} = \psi_p(r)$$

for $p \geq 2$, ψ_p being the function defined in the compatibility condition for the non-linear problem. Here $g = g(t, r)$ is taken from a functional space to which belongs $\mathcal{B}(r, [y_1^*])$. Probably we should assume that $\left.\frac{\partial^p g}{\partial t^p}\right|_{t=0} = \phi_p :=$ $\left.\frac{\partial^p}{\partial t^p}\mathcal{B}(r, [y_1^*])\right|_{t=0}$.

More precisely speaking, the compatibility condition for the linearized problem is as follows:

Putting

$$\tilde{\mathcal{B}}_0(r, [y_0], [H_0]) := \mathcal{B}_1(r, [y_0], [H_0]),$$

for $p \geq 1$ we define $\tilde{\mathcal{B}}_p(r, [y_0], \cdots, [y_p], [H_0], \cdots, [H_p])$ by the recurrence formula

$$\begin{aligned}\tilde{\mathcal{B}}_{p+1} :=& \sum_{j=0}^{p} \Big[y_{j+1}\partial_{y_j} + y'_{j+1}\partial_{y'_j} + y''_{j+1}\partial_{y''_j} + \\ &+ H_{j+1}\partial_{H_j} + H'_{j+1}\partial_{H'_j} + H''_{j+1}\partial_{H''_j}\Big]\tilde{\mathcal{B}}_p;\end{aligned}$$

Then we define the sequence of functions $(h_p)_p$ by $h_0 = h_1 = 0$ and

$$h_{p+2} := \tilde{\mathcal{B}}_p(r, [\psi_0], \cdots, [\psi_p], [h_0], \cdots, [h_p]) + \phi_p;$$

Here ψ_k are those in the compatibility condition for the non-linear problem; Then the compatibility condition for the linearized problem is:

$$h_p(1) = 0 \qquad \text{for} \qquad \forall p \in \mathbb{N}.$$

But it may be difficult to answer whether, in this sense, the compatibility condition of the non-linear problem implies the compatibility condition of the linearized problem or not. This is the difficult point we meet when we try to apply the Nash-Moser theorem.

5 Illustrating example

Let us illustrate the point by the simple problem

$$\begin{aligned}&\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial^2 y}{\partial x^2}\right)^2 \qquad (0 < x < 1), \\ &y|_{x=0} = y|_{x=1} = 0, \\ &y|_{t=0} = \psi_0(x), \qquad \left.\frac{\partial y}{\partial t}\right|_{t=0} = \psi_1(x).\end{aligned}$$

In this example the the linearized problem is

$$\frac{\partial^2 h}{\partial t^2} = \left(1 + 2\frac{\partial^2 y}{\partial x^2}\right)\frac{\partial^2 h}{\partial t^2} + g,$$
$$h|_{x=0} = h|_{x=1} = 0,$$
$$h|_{t=0} = \frac{\partial h}{\partial t}\Big|_{t=0} = 0,$$

where we assume that

$$\frac{\partial^p y}{\partial t^p}\Big|_{t=0} = \psi_p(x), \qquad \frac{\partial^p g}{\partial t^p}\Big|_{t=0} = \phi_p(x).$$

The functions ψ_p are determined by

$$\psi_{p+2} := \triangle\psi_p + \sum_{k=0}^{p}\binom{p}{k}(\triangle\psi_{p-k})(\triangle\psi_k),$$

where and hereafter $\triangle$ stands for $\partial^2/\partial x^2$. Suppose that non-linear compatibility condition

$$\psi_p(0) = \psi_p(1) = 0 \qquad \forall p \in \mathbb{N} \tag{23}$$

holds, and let us consider the restriction that $|\triangle y| < 1/2$. Then

$$\psi_2(x) = (\triangle\psi_0)(x) + (\triangle\psi_0)(x)^2 = 0 \quad \text{for} \quad x = 0, 1$$

implies $\triangle\psi_0(0) = \triangle\psi_0(1) = 0$. Then inductively we have

$$(\triangle\psi_p)(0) = (\triangle\psi_p)(1) = 0. \tag{24}$$

On the other hand the functions h_p are determined by $h_0 = h_1 = 0$ and

$$h_{p+2} := \triangle h_p + 2\sum_{k=0}^{p}\binom{p}{k}(\triangle\psi_{p-k})(\triangle h_k) + \phi_p.$$

Here $\phi_0 = \psi_2, \phi_1 = \psi_3, \phi_2 = 2(\triangle\psi_1)^2$ and $\phi_p = 0$ for $p \geq 3$. Keeping in mind (23)(24), we see that the compatibility condition for the linearized problem

$$h_p(0) = h_p(1) = 0 \qquad \forall p \in \mathbb{N} \tag{25}$$

is equivalent to the apparently stronger condition

$$(\triangle h_p)(0) = (\triangle h_p)(1) = 0 \qquad \forall p \in \mathbb{N}. \tag{26}$$

But we are not sure that (25) or (26) can be derived from (23)(24).

As for this problem, we note that the loss of regularity can be surmounted without using the Nash-Moser theorem. Actually Professor A. Matsumura suggested the following trick ([5]).

Differentiating the equation by t we get

$$\frac{\partial^2 v}{\partial t^2} = (1 + 2\triangle y)\triangle v,$$

where $v = \partial y/\partial t$. If $|Y| < 1/2, G(Y) = Y + Y^2$ has an inverse function $G^{-1}(V) = (\sqrt{1+4V} - 1)/2 = V + [V]_2$ and if y is a solution then we have

$$\frac{\partial^2 v}{\partial t^2} = \Big(1 + 2G^{-1}\Big(\frac{\partial v}{\partial t}\Big)\Big)\triangle v,$$

which is a semi-linear equation. Once we have a solution $v = v(t,x)$ of this equation under the boundary condition

$$v|_{x=0} = v|_{x=1} = 0$$

and the initial condition

$$v|_{t=0} = \psi_1(x), \qquad \frac{\partial v}{\partial t}\Big|_{t=0} = G(\triangle\psi_0),$$

we take the unique $y(t,\cdot)$ such that

$$\triangle y = G^{-1}\Big(\frac{\partial v}{\partial t}\Big), \qquad y|_{x=0} = y|_{x=1} = 0.$$

TRhen the function y turns out to be a solution of the original problem. (In fact, $G(\triangle y) = v_t$ implies

$$DG(\triangle y)\frac{\partial}{\partial t}\triangle y = DG(\triangle y)\triangle v;$$

Since $DG(Y) \neq 0$ for $|Y| < 1/2$, we get $\triangle y_t = \triangle v$. The boundary conditions guarantee that $y_t = v$ and we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial v}{\partial t} = G(\triangle y)$$

and so on.

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