数理解析研究所講究録
第 1883 巻 2014 年 66-74

On some decay properties of solutions for the Stokes equations with surface tension and gravity in the half space

早稲田大学・基幹・数学応数　斎藤平和
Hirokazu Saito
Department of Pure and Applied Mathematics of Graduate school,
Fundamental Science and Engineering, Waseda University

早稲田大学・数学，理工学術院総合研究所　柴田良弘
Yoshihiro Shibata
Department of Mathematics and
Research Institute of Science and Engineering, Waseda University

Abstract

In the present paper, we consider decay properties of solutions for the Stokes equations with the surface tension and gravity in the half space when $t \to \infty$. The Stokes equations arises in the study of a free boundary problem for the Navier-Stokes equations in unbounded domains. We are interested in the global-time wellposedness of the free boundary problem. When we construct solutions of the free boundary problem, decay properties for the Stokes equations and Banach's fixed point theorem will be combined. That is the reason why we consider decay properties of solutions for the Stokes equations. This paper shows the $L_q - L_r$ estimates with $1 < r < 2 < q < \infty$ of solutions for the Stokes equations, mainly. Our technique is based on the analysis of some resolvent problem which is obtained by the Laplace transform of the Stokes equations. By examining the spectrum of the resolvent problem in detail, we show the decay properties of solutions of the Stokes equations.

1 Introduction

This article is brief survey of the results related to [9], mainly.

In the present paper, we consider decay properties of solutions for the Stokes equations with the surface tension and gravity in the half space $\mathbb{R}_+^N = \{x = (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1},\ x_N > 0\}$ $(N \geq 2)$:

$$\begin{cases} \partial_t u - \operatorname{Div} S(u, \theta) = 0, \quad \operatorname{div} u = 0 & \text{in } \mathbb{R}_+^N, \quad t > 0, \\ \partial_t h + u_N = 0 & \text{on } \mathbb{R}_0^N, \quad t > 0, \\ S(u, \theta)\mathbf{n} + (\gamma_a - \sigma\Delta')h\mathbf{n} = 0 & \text{on } \mathbb{R}_0^N, \quad t > 0, \\ u|_{t=0} = f(x), \quad h|_{t=0} = d(x'). \end{cases} \tag{1.1}$$

Here, $u = u(x,t) = (u_1(x,t), \ldots, u_N(x,t))^{T}$[1)] and $\theta = \theta(x,t)$ are unknown N-component velocity vector and scalar pressure at $(x,t) \in \mathbb{R}_+^N \times (0,\infty)$, respectively, and $h = h(x',t)$ is also unknown scalar function at $(x',t) \in \mathbb{R}^{N-1} \times (0,\infty)$ which is explained more precisely below; $f(x)$ and $d(x')$ are given initial data for $u(x,t)$ and $h(x',t)$, respectively; $\mathbb{R}_0^N$ is the boundary of $\mathbb{R}_+^N$ and $\mathbf{n} = (0,\ldots,0,-1)^T$ is the unit outer normal vector on $\mathbb{R}_0^N$; the derivatives $\operatorname{div} u$ and $\Delta' h$ denote

$$\operatorname{div} u = \sum_{j=1}^N D_j u_j \quad (D_j = \frac{\partial}{\partial x_j}), \quad \Delta' h = \sum_{j=1}^{N-1} D_j^2 h;$$

$S(u,\theta) = -\theta I + D(u)$ is the stress tensor for the newtonian fluids, where I is the $N \times N$ identity matrix and $D(u)$ is also $N \times N$ matrix whose (i,j) component $D_{ij}(u)$ is give by $D_{ij}(u) = D_i u_j + D_j u_i$; γ_a is

[1)] M^T denotes the transposed M.

the gravitational acceleration and $\sigma > 0$ is the surface tension coefficient. For the matrix $M = M(x) = (M_{ij}(x))$, $\operatorname{Div} M$ denotes that

$$i\text{th component of } \operatorname{Div} M = \sum_{j=1}^{N} D_j M_{ij}(x),$$

and therefore $\operatorname{Div} S(u, \theta)$ is given by

$$i\text{th component of } \operatorname{Div} S(u,\theta) = -D_i\theta + \Delta u_i + D_i \operatorname{div} u.$$

Now, we introduce the following two nonlinear problem:

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = \operatorname{Div} S(v,\pi) - \gamma_a \nabla x_N, \quad \operatorname{div} v = 0 & \text{in } \Omega(t), \quad t>0, \\ V = v\cdot \mathbf{n}_t & \text{on } \Gamma(t), \quad t>0, \\ S(v,\pi)\mathbf{n}_t = \sigma\kappa\mathbf{n}_t & \text{on } \Gamma(t), \quad t>0, \\ v = 0 & \text{on } \Gamma_b, \quad t>0, \\ v|_{t=0} = v_0 & \text{in } \Omega(0), \end{cases} \tag{1.2}$$

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = \operatorname{Div} S(v,\pi) - \gamma_a \nabla x_N, \quad \operatorname{div} v = 0 & \text{in } \Omega(t), \quad t>0, \\ V = v\cdot \mathbf{n}_t & \text{on } \Gamma(t), \quad t>0, \\ S(v,\pi)\mathbf{n}_t = \sigma\kappa\mathbf{n}_t & \text{on } \Gamma(t), \quad t>0, \\ v|_{t=0} = v_0 & \text{in } \Omega(0). \end{cases} \tag{1.3}$$

These two problems are free boundary problems for the Navier-Stokes equations of incompressible flows for the newtonian fluids. Here, $(v, \pi, \Gamma(t))$ is unknown, where $v = v(x,t) = (v_1(x,t), \dots, v_N(x,t))^T$ is the N-component velocity vector, $\pi = \pi(x,t)$ is the pressure, and for a scalar function $h = h(x',t)$ defined on $\mathbb{R}^N_0 \times (0,\infty)$, $\Gamma(t)$ is given by

$$\Gamma(t) = \{x = (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1},\ x_N = h(x',t)\};$$

$\Gamma_b = \{x = (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1},\ x_N = -b\}$ $(b > 0)$ is the fixed boundary; V is the velocity of the evolution of $\Gamma(t)$ in the normal direction and $\mathbf{n}_t$ is the unit outer normal vector on $\Gamma(t)$; $\kappa = \kappa(x,t)$ is the mean curvature of $\Gamma(t)$ which is negative when $\Omega(t)$ is convex in a neighborhood of $x \in \Gamma(t)$. The domain $\Omega(t)$ is give by

$$\begin{aligned} \Omega(t) &= \{x = (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1},\ -b < x_N < h(x',t)\} \quad \text{for } (1.2), \\ \Omega(t) &= \{x = (x', x_N) \in \mathbb{R}^N \mid x' \in \mathbb{R}^{N-1},\ x_N < h(x',t)\} \quad \text{for } (1.3). \end{aligned}$$

First, we see the history of the problem (1.2). This problem is first studied by Beale [3] mathematically. Beale [3] shows the local-time unique existence theorem in the case that $\sigma = 0$, and show the fact that global-time solutions depending analytically on the initial data (v_0, h_0) can not exist even if (v_0, h_0) is sufficiently small. After that, Beale [4] proves the unique existence theorem globally in time for small initial data by taking into account the surface tension $\sigma\kappa\mathbf{n}_t$ with $\sigma > 0$. Beale and Nishida [5] gives large-time behavior of solutions for Beale [4], but the paper has just outline of proof. We can find the detailed proof in Hataya [7]. Tani and Tanaka [12] shows the global-time unique existence theorem for small initial data in the case with or without surface tension under weaker assumptions of the initial data than Beale's. And also, Hataya and Kawashima [6] gives some decay properties of large-time behavior of solutions for small initial data in the case of $\sigma = 0$. In addition to these results, there are Allain [2] and Tani [13] as long as we know. Note that these all results are in the framework of L_2 in time and L_2 in space. As another approach to (1.2), Abels [1] uses the L_q settings in both time and space, and he obtain the unique existence theorem locally in time in the case of $\sigma = 0$ for $q > N$.

Next, we see the history of the problem (1.3). We have seen a lot of results of (1.2) above, on the other hand results of (1.3) are not so many. Shibata and Shimizu [10, 11] shows the maximal $L_p - L_q$ regularity theorem for the linearized problem of (1.3) and resolvent estimates for the resolvent problem obtained by the Laplace transform of the linearized problem. And Prüss and Simonett [8] considers the two-phase problem corresponding to (1.3), and shows the unique existence of solutions locally in time for sufficiently small initial data and its instability when upper fluid is heavier than lower one. But, we think that these all results Shibata and Shimizu [10, 11], and Prüss and Simonett [8] can not be applied directly to show the global-time wellposedness of the problem (1.3).

Our final goal is to show the global-time wellposedness of (1.3) in the scaling critical spaces L_p in time and L_q in space with $(2/p) + (N/q) = 1$. As the first step, we consider decay properties of (1.1), noting that the equations (1.1) is homogeneous part of the linearized equations of (1.3). Especially, we obtain the $L_q - L_r$ estimates of the solution of (1.1) for $1 < r < 2 < q < \infty$, and also obtain estimates of L_∞ norm of the lower order terms: u, ∇u, $D_j h$ and $D_j D_k h$ for $j, k = 1, \dots, N-1$. The restriction $r < 2 < q$ arises from a kind of hyperbolic effect of the term $h(x', t)$.

Here, we introduce a interesting fact concerning the difference between the problem (1.2) and (1.3). The remarkable one appears in the analysis of Lopatinski determinant of the linearized problem. Beale and Nishida [5], and Hataya [7] show the following expansion of spectrum λ:

$$\lambda = -\frac{\gamma_a b}{3}|\xi'|^2 + O(|\xi'|^3) \quad \text{as} \quad |\xi'| \to 0,$$

where $\xi' = (\xi_1, \dots, \xi_{N-1})$ is the tangential variable in the Fourier space. On the other hand, our case has the spectrum $\lambda_\pm$:

$$\lambda_\pm = \pm i\gamma_a^{1/2}|\xi'|^{1/2} - 2|\xi'|^2 + O(|\xi'|^{5/2}) \quad \text{as} \quad |\xi'| \to 0+.$$

From this viewpoint, our case is more complex than the case of Beale and Nishida [5], and Hataya [7], because we have not only $-|\xi'|^2$ but also $\pm i\gamma_a^{1/2}|\xi'|^{1/2}$ which yields oscillations in time.

This paper is organized as follows. In Section 2, we will state the main results of [9]. In Section 3, the strategy of their approach is explained.

2 Main results

First, we introduce some symbols and functional spaces in order to state our main results precisely. Let Ω be any domain and Γ be its boundary. $L_q(\Omega)$ and $W_q^m(\Omega)$ are usual Lebesgue and Sobolev spaces, respectively, for $1 \le q \le \infty$ and $m \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of all natural numbers. And, we use the convention $W_q^0(\Omega) = L_q(\Omega)$. For $1 \le q < \infty$ and $s > 0$ that s is not integer, the Slobodeckij space is defined by

$$W_q^s(\Omega) = \{u \in W_q^{[s]}(\Omega) \mid \|u\|_{W_q^s(\Omega)} < \infty\},$$

$$\|u\|_{W_q^s(\Omega)} = \|u\|_{W_q^{[s]}(\Omega)} + \sum_{|\alpha|=[s]} \left(\int_\Omega \int_\Omega \frac{|D_x^\alpha u(x) - D_y u(y)|^q}{|x-y|^{N+(s-[s])q}}\, dxdy\right)^{1/q},$$

where $[s] = \max\{n \mid n < s,\ n \in \mathbf{N} \cup \{0\}\}$ and $D_x^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N})$ for any multi-index $\alpha \in \mathbf{N}_0^N = (\mathbf{N} \cup \{0\})^N$. We use the following functional space for the pressure:

$$\widehat{W}_q^1(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_q(\Omega)^N\} \quad (1 < q < \infty).$$

Moreover, for $1 < q < \infty$ we set $W_{q,\Gamma}^1(\Omega) = \{\theta \in W_q^1(\Omega) \mid \theta|_\Gamma = 0\}$, and

$$\widehat{W}_{q,\Gamma}^1(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_q(\Omega)^N,\ \theta|_\Gamma = 0,$$
$$\text{there exists } \{\theta_j\}_{j=1}^\infty \subset W_{q,\Gamma}^1(\Omega) \text{ such that } \lim_{j\to\infty} \|\nabla(\theta_j - \theta)\|_{L_q(\Omega)} = 0\}.$$

For the simplicity, we use the abbreviations: $W_{q,0}^1(\mathbb{R}_+^N)$ and $\widehat{W}_{q,0}^1(\mathbb{R}_+^N)$ when $\Omega = \mathbb{R}_+^N$ and $\Gamma = \mathbb{R}_0^N$ in the above definitions. The second solenoidal space is defined as follows:

$$J_q(\Omega) = \{u \in L_q(\Omega)^N \mid (u, \nabla\varphi)_\Omega = 0 \text{ for any } \varphi \in \widehat{W}_{q',\Gamma}^1(\Omega)\} \quad (1 < q < \infty),$$

where $(1/q) + (1/q') = 1$ and $(f, g)_\Omega = \int_\Omega f(x) \cdot g(x)\, dx = \sum_{j=1}^N \int_\Omega f_j(x) g_j(x)\, dx$ for any N-component vector functions $f(x)$ and $g(x)$. Let $C^m(I, X)$ be the set of all X-valued C^m functions defined on the interval I for any Banach space X and any $m \in \mathbf{N}_0$. The letter C denotes a generic constant, and the constant C may change from line to line. Then, we have the following unique existence theorem for (1.1).

Theorem 2.1. *Let $1 < q < \infty$, and let $f \in J_q(\mathbb{R}^N_+)$ and $d \in W_q^{2-(1/q)}(\mathbb{R}^{N-1})$. Then, (1.1) admits a unique solution (u, θ, h) in the spaces:*

$$
\begin{aligned}
&u \in C^1((0,\infty), J_q(\mathbb{R}^N_+)) \cap C^0([0,\infty), J_q(\mathbb{R}^N_+)) \cap C^0((0,\infty), W_q^2(\mathbb{R}^N_+)^N), \quad \theta \in C^0((0,\infty), \widehat{W}_q^1(\mathbb{R}^N_+)),\\
&h \in C^1((0,\infty), W_q^{2-(1/q)}(\mathbb{R}^{N-1})) \cap C^0([0,\infty), W_q^{2-(1/q)}(\mathbb{R}^{N-1})) \cap C^0((0,\infty), W_q^{3-(1/q)}(\mathbb{R}^{N-1})).
\end{aligned}
$$

Next, we introduce large-time behaviors of the solutions obtained in Theorem 2.1. For the purpose, we extend $h(x', t)$ to a function $H(x, t)$ defined in $\mathbb{R}^N_+ \times (0, \infty)$ through the equations:

$$
\begin{cases}
\Delta H = 0 \quad \text{in} \quad \mathbb{R}^N_+, \quad t > 0,\\
\quad H = h \quad \text{on} \quad \mathbb{R}^N_0, \quad t > 0,
\end{cases} \tag{2.1}
$$

and we set

$$
\begin{aligned}
X_{q,r} &= (L_q(\mathbb{R}^N_+) \cap L_r(\mathbb{R}^N_+))^N \times (W_q^{2-(1/q)}(\mathbb{R}^{N-1}) \cap L_r(\mathbb{R}^{N-1})),\\
\ell(q, r) &= \min\left\{\frac{1}{2}\left(\frac{1}{r} - \frac{1}{q}\right), \frac{1}{8}\left(2 - \frac{1}{q}\right)\right\}.
\end{aligned}
$$

Then, there holds the following theorem.

Theorem 2.2. *Let $1 < r < 2 < q < \infty$ and $F = (f(x), d(x'))$ for*

$$
f \in J_q(\mathbb{R}^N_+) \cap L_r(\mathbb{R}^N_+)^N, \quad d \in W_q^{2-(1/q)}(\mathbb{R}^{N-1}) \cap L_r(\mathbb{R}^{N-1}).
$$

Let (u, θ, h) be the solution in Theorem 2.1 and H be the extension in (2.1). Then, there exists a positive constant C such that for any $t \geq 1$ there hold

$$
\begin{aligned}
\|\partial_t u(t)\|_{L_q(\mathbb{R}^N_+)} + \|\nabla\theta(t)\|_{L_q(\mathbb{R}^N_+)} &\leq C \max\left\{t^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{4}},\ t^{-1}\right\} \|F\|_{X_{q,r}},\\
\|\nabla^2 u(t)\|_{L_q(\mathbb{R}^N_+)} &\leq C \max\left\{t^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\ell(q,r)-\frac{1}{4}},\ t^{-1}\right\} \|F\|_{X_{q,r}},\\
\|u(t)\|_{L_q(\mathbb{R}^N_+)} &\leq C t^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})} \|F\|_{X_{q,r}},\\
\|\nabla u(t)\|_{L_q(\mathbb{R}^N_+)} &\leq C t^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\ell(q,r)-\frac{1}{8}} \|F\|_{X_{q,r}},\\
\|D_x^\alpha \partial_t H(t)\|_{L_q(\mathbb{R}^N_+)} &\leq C t^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{|\alpha|}{2}} \|F\|_{X_{q,r}} \quad (|\alpha| \leq 2),\\
\|D_x^\alpha \nabla H(t)\|_{L_q(\mathbb{R}^N_+)} &\leq C t^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{4}-\frac{|\alpha|}{2}} \|F\|_{X_{q,r}} \quad (|\alpha| \leq 2).
\end{aligned}
$$

Moreover, if we assume that $q > N$, then there exists a positive constant C such that for any $t \geq 1$ there hold

$$
\begin{aligned}
&\|u(t)\|_{L_\infty(\mathbb{R}^N_+)} \leq C t^{-(\frac{N-1}{2r}+\frac{1}{4})} \|F\|_{X_{q,r}}, \qquad \|\nabla u(t)\|_{L_\infty(\mathbb{R}^N_+)} \leq C t^{-(\frac{N-1}{2r}+\frac{3}{8})} \|F\|_{X_{q,r}},\\
&\|\nabla H(t)\|_{L_\infty(\mathbb{R}^N_+)} \leq C t^{-(\frac{N-1}{2r}+\frac{1}{2})} \|F\|_{X_{q,r}} \qquad \|\nabla^2 H(t)\|_{L_\infty(\mathbb{R}^N_+)} \leq C t^{-(\frac{N-1}{2r}+1)} \|F\|_{X_{q,r}}.
\end{aligned}
$$

3 Outline of the proof

We show the outline of the proof of Theorem 2.2. By applying the Laplace transform to (1.1), we obtain the resolvent problem independent of time variable t:

$$
\begin{cases}
\lambda v - \Delta v + \nabla\pi = f(x), \quad \operatorname{div} v = 0 & \text{in} \quad \mathbb{R}^N_+,\\
\lambda\eta + v_N = d(x') & \text{on} \quad \mathbb{R}^N_0,\\
S(v, \pi)\mathbf{n} + (\gamma_a - \sigma\Delta')\eta\mathbf{n} = 0 & \text{on} \quad \mathbb{R}^N_0,
\end{cases} \tag{3.1}
$$

where (v, π, η) is the Laplace transform of (u, θ, h) and the resolvent parameter λ is in Σ_ε given by

$$
\Sigma_\varepsilon = \{\lambda \in \mathbf{C} \mid |\arg\lambda| < \pi - \varepsilon, \ \lambda \neq 0\} \quad (0 < \varepsilon < \pi/2).
$$

We also use the symbol $\Sigma_{\varepsilon,\lambda_0}$ given by

$$\Sigma_{\varepsilon,\lambda_0} = \{\lambda \in \mathbf{C} \mid |\arg\lambda| \le \pi - \varepsilon,\ |\lambda| \ge \lambda_0\} \quad (0 < \varepsilon < \pi/2,\ \lambda_0 > 0).$$

In order to progress the argument, we define the partial Fourier transform $\mathcal{F}_{x'}$ with respect to tangential variable x' and its inverse formula $\mathcal{F}_{\xi'}^{-1}$ for functions $f(x', x_N)$ and $g(\xi', x_N)$, respectively, as follows:

$$\mathcal{F}_{x'}[f(x', x_N)](\xi') = \widehat{f}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\, dx',$$
$$\mathcal{F}_{\xi'}^{-1}[g(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', x_N)\, d\xi'.$$

We can give the exact solution formulas of the resolvent problem (3.1) as follows. First, we apply the partial Fourier transform with respect to the tangential variable x' to (3.1), and after that we solve the obtained ordinary differential equations with respect to x_N in the Fourier space by seeing $\xi' \in \mathbb{R}^{N-1}$ as a parameter. Then, we have the exact formulas of $\widehat{v}(\xi', x_N) = (\widehat{v}_1(\xi', x_N), \ldots, \widehat{v}_N(\xi', x_N))^T$, $\widehat{\pi}(\xi', x_N)$ and $\widehat{\eta}(\xi')$ in the Fourier space. Finally, the inverse partial Fourier transforms of $\widehat{v}(\xi', x_N)$, $\widehat{\pi}(\xi', x_N)$ and $\widehat{\eta}(\xi')$ yield the exact solution formulas of (3.1). Here, we concentrate on the term:

$$\mathcal{F}_{\xi'}^{-1}\left[\frac{D(A,B)}{(B+A)L(A,B)}\widehat{d}(\xi')\right](x'), \tag{3.2}$$

which is a part of the solution $\eta = \eta(x', \lambda)$. The symbols in (3.2) are given by

$$\begin{aligned} &A = |\xi'|, \quad B = \sqrt{\lambda + |\xi'|^2}\ (\mathrm{Re}\, B \ge 0), \quad D(A,B) = B^3 + AB^2 + 3A^2B - A^3, \\ &L(A,B) = (B-A)D(A,B) + A(\gamma_a + \sigma A^2). \end{aligned} \tag{3.3}$$

The following lemma is proved in [11, Lemma 7.2].

Lemma 3.1. *Let $0 < \varepsilon < \pi/2$ and $\alpha' \in \mathbf{N}_0^{N-1}$. There exist a positive number $\lambda_0 = \lambda_0(\varepsilon, \gamma_a, \sigma) \ge 2$, depending only on ε, γ_a and σ, and a positive constant C, depending only on α', ε and λ_0, such that for any $(\xi', \lambda) \in (\mathbb{R}^{N-1} \setminus \{0\}) \times \Sigma_{\varepsilon,\lambda_0}$ there holds*

$$|D_{\xi'}^{\alpha'} L(A,B)^{-1}| \le C\left\{|\lambda|(|\lambda|^{\frac{1}{2}} + A)^2 + A(\gamma_a + \sigma A^2)\right\}^{-1} A^{-|\alpha'|}.$$

We set

$$I(x,t) = \frac{1}{2\pi i}\mathcal{F}_{\xi'}^{-1}\left[\int_\Gamma e^{\lambda t}\frac{D(A,B)}{(B+A)L(A,B)}\, d\lambda\, e^{-Ax_N}\widehat{d}(\xi')\right](x'),$$

where $\Gamma = \Gamma_+ \cup \Gamma_-$ is give by

$$\Gamma + = \{\lambda \in \mathbf{C} \mid \lambda = 2\lambda_0 + se^{i(3/4)\pi},\ s: 0 \to \infty\},$$
$$\Gamma_- = \{\lambda \in \mathbf{C} \mid \lambda = 2\lambda_0 + se^{-i(3/4)\pi},\ s: \infty \to 0\}$$

for $\lambda_0 = \lambda_0(\pi/4, \gamma_a, \sigma)$ in Lemma 3.1. Note that $I(x,t)$ is a part of the solution of (2.1). In the present proof, we only show decay properties of $I(x,t)$, but the other terms in $u(x,t)$, $\theta(x,t)$ and $H(x,t)$ are calculated by techniques similar to the present case. In order to derive decay properties from $I(x,t)$, we divide $I(x,t)$ into

$$\begin{aligned} I(x,t) &= I_0(x,t) + I_\infty(x,t), \\ I_a(x,t) &= \frac{1}{2\pi i}\mathcal{F}_{\xi'}^{-1}\left[\int_\Gamma e^{\lambda t}\frac{\varphi_a(\xi')D(A,B)}{(B+A)L(A,B)}e^{-Ax_N}\widehat{d}(\xi')\, d\lambda\right](x') \quad (a \in \{0, \infty\}), \end{aligned}$$

where $\varphi_0(\xi')$ and $\varphi_\infty(\xi')$ are cut-off functions such that $\varphi_0(\xi') = \varphi(\xi'/A_0)$, $\varphi_\infty(\xi') = 1 - \varphi_0(\xi')$ and $\varphi(\xi') \in C_0^\infty(\mathbb{R}^{N-1})$ satisfies

$$\varphi(\xi') = \begin{cases} 1 & (|\xi'| \le \frac{1}{3}), \\ 0 & (|\xi'| \le \frac{2}{3}). \end{cases}$$

Note that the positive number $0 < A_0 \le 1$ in $\varphi_0(\xi')$ can be chosen sufficiently small when we need to do so.

3.1 Analysisi of $I_0(x,t)$

$L(A,B)$ has the following four roots $B_j^\pm$ $(j=1,2)$ as a function of B

$$B_j^\pm = e^{\pm\frac{i(2j-1)\pi}{4}}\gamma_a^{1/4}A^{1/4} - \frac{A^{7/4}}{e^{\pm\frac{i(2j-1)\pi}{4}}\gamma_a^{1/4}} - \frac{\sigma A^{9/4}}{e^{\pm\frac{i(2j-1)3\pi}{4}}\gamma_a^{3/4}} + O(A^{10/4}) \quad (A\to 0+).$$

Moreover, setting $\lambda_\pm = (B_1^\pm)^2 - A^2$, we obtain

$$\lambda_\pm = \pm i\gamma_a^{1/2}A^{1/2} - 2A^2 \pm \frac{2\sigma}{i\gamma_a^{1/2}}A^{5/2} + O(A^{11/4}) \quad (A\to 0+).$$

Note that $\lambda_\pm$ appear only in our brunch since we use the brunch such that $\mathrm{Re}\,B\geq 0$ in (3.3). Then, by using Cauchy's integral theorem, we change the integral path Γ to the paths:

$$\begin{aligned}
\Gamma_0^\pm &= \{\lambda\in\mathbf{C}\,|\,\lambda=\lambda_\pm+(\gamma_a^{1/2}/4)A^{1/2}e^{\pm iu},\ u:0\to 2\pi\},\\
\Gamma_1^\pm &= \{\lambda\in\mathbf{C}\,|\,\lambda=-A^2+(A^2/4)e^{\pm iu},\ u:0\to\pi/2\},\\
\Gamma_2^\pm &= \{\lambda\in\mathbf{C}\,|\,\lambda=-(A^2(1-u)+\gamma_0u)\pm i((A^2/4)(1-u)+\gamma_0u),\ u:0\to 1\},\\
\Gamma_3^\pm &= \{\lambda\in\mathbf{C}\,|\,\lambda=-(\gamma_0\pm i\gamma_0)+ue^{\pm i(\pi-\varepsilon_0)},\ u:0\to\infty\},
\end{aligned}$$

where $\varepsilon_0=\tan^{-1}\{(A^2/8)/A^2\}=\tan^{-1}(1/8)$ and γ_0 is the same number as $\lambda_0=\lambda_0(\varepsilon_0,\gamma_a,\sigma)$ in Lemma 3.1, noting that λ_0 determined by ε_0 especially. Then, $I_0(x,t)$ is divided into

$$I_0(x,t)=\sum_{n=0}^{3}I_0^{\pm,n}(x,t),\quad I_0^{\pm,n}(x,t)=\frac{1}{2\pi i}\mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_n^\pm}e^{\lambda t}\frac{\varphi_0(\xi')D(A,B)}{(B+A)L(A,B)}\,d\lambda\,e^{-Ax_N}\widehat{d}(\xi')\right](x').$$

We have the following theorem for $I_0(x,t)$.

Theorem 3.2. *Let $1<r<2<q\leq\infty$ and $\alpha\in\mathbf{N}_0^N$, and let $d\in L_r(\mathbb{R}^{N-1})$. Then, there exists a positive constant C such that for any $t\geq 1$ there hold*

$$\begin{aligned}
\|D_x^\alpha\nabla I_0(t)\|_{L_q(\mathbb{R}_+^N)} &\leq Ct^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{4}-\frac{|\alpha|}{2}}\|d\|_{L_r(\mathbb{R}^{N-1})},\\
\|D_x^\alpha\partial_t I_0(t)\|_{L_q(\mathbb{R}_+^N)} &\leq Ct^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{|\alpha|}{2}}\|d\|_{L_r(\mathbb{R}^{N-1})}.
\end{aligned}$$

Proof. The bad decay rate arises from the residue parts, that is $I_0^{\pm,0}(x,t)$. We, therefore, consider only $I_0^{\pm,0}(x,t)$ here. See [9] concerning the terms $I_0^{\pm,n}(x,t)$ $(n=,1,2,3)$. Since $L(A.B)=(B-B_1^+)(B-B_1^-)(B-B_2^+)(B-B_2^-)$, by the residue theorem we have

$$\begin{aligned}
&(D_jI_0^\pm(x,t),D_NI_0^\pm(x,t),\partial_tI_0^\pm(x,t))\\
&=\frac{1}{2\pi i}\mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma_0^\pm}e^{\lambda t}\frac{\varphi_0(\xi')(i\xi_j,-A,\lambda)(B+B_1^\pm)D(A,B)}{(B+A)(\lambda-\lambda_\pm)(B-B_1^\mp)(B-B_2^+)(B-B_2^-)}\,d\lambda\,e^{-Ax_N}\widehat{d}(\xi')\right](x')\\
&=\mathcal{F}_{\xi'}^{-1}\left[e^{\lambda_\pm t}\frac{\varphi_0(\xi')(i\xi_j,-A,\lambda_\pm)(2B_1^\pm)D(A,B_1^\pm)}{(B_1^\pm+A)(B_1^\pm-B_1^\mp)(B_1^\pm-B_2^+)(B_1^\pm-B_2^-)}\,e^{-Ax_N}\widehat{d}(\xi')\right](x')
\end{aligned}$$

for $j=1,\dots,N-1$. Note that $|D(A,B_1^\pm)|\leq CA^{3/4}$,

$$|B_1^\pm+A|\geq CA^{1/4},\quad |B_1^\pm-B_1^\mp|\geq CA^{1/4},\quad |B_1^\pm-B_2^+|\geq CA^{1/4},\quad |B_1^\pm-B_2^-|\geq CA^{1/4}$$

on $\mathrm{supp}\,\varphi_0$ with some positive constant C and

$$e^{\lambda_\pm t}=e^{\pm i\gamma_a^{1/2}A^{1/2}t}e^{(-2A^2+O(A^{5/2}))t},$$

and then we obtain, by using the $N-1$ dimensions heat kernel $\mathcal{F}_{\xi'}^{-1}[e^{-A^2t}](x')$ and Parseval's theorem,

$$\|D_JI_0^{\pm,0}(\cdot,x_N,t)\|_{L_q(\mathbb{R}^{N-1})}\leq Ct^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\left\|\left(e^{-(A^2/3)t}Ae^{-Ax_N}\right)\left(e^{-(A^2/3)t}\widehat{d}(\xi')\right)\right\|_{L_2(\mathbb{R}^{N-1})}$$

$$\leq Ct^{-\frac{N-1}{2}(\frac{1}{2}-\frac{1}{q})}\frac{\|e^{-(A^2/3)t}\widehat{d}(\xi')\|_{L_2(\mathbb{R}^{N-1})}}{t^{1/2}+x_N}$$
$$\leq Ct^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\frac{\|d\|_{L_r(\mathbb{R}^{N-1})}}{t^{1/2}+x_N}$$

for $J=1,\dots,N$. Similarly, we obtain

$$\|\partial_t I_0^{\pm,0}(\cdot,x_N,t)\|_{L_q(\mathbb{R}^{N-1})}\leq Ct^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})}\frac{\|d\|_{L_r(\mathbb{R}^{N-1})}}{t^{1/4}+x_N^{1/2}}.$$

Finally, taking $\|\cdot\|_{L_q(0,\infty)}$ in the above inequalities yields the required inequalities with $\alpha=0$. For the case of $\alpha\neq 0$, we can prove analogously. □

3.2 Analysis of $I_\infty(x,t)$

$L(A,B)$ has the following four roots B_j $(j=1,\dots,4)$ as a function of B:

$$B_j=a_jA+\frac{\sigma}{4(1-a_j-a_j^3)}+\frac{(1+3a_j^2)\sigma^2}{32(1-a_j-a_j^3)^3}A^{-1}+O(A^{-2})\quad(A\to\infty),$$

where a_j $(j=1,\dots,4)$ are numbers, satisfying the equation: $x^4+2x^2-4x+1=0$, such that

$$a_1=1,\quad 0<a_2<\frac{1}{2},\quad \operatorname{Re}a_j<0\ (j=3,4).$$

Setting $\lambda_j=(B_j)^2-A^2$ for $j=1,2$ implies that

$$\begin{aligned}\lambda_1&=-(\sigma/2)A-(3/16)\sigma^2+O(A^{-1})\quad(A\to\infty),\\ \lambda_2&=-(1-a_2^2)A^2+\frac{a_2\sigma}{2(1-a_2-a_2^3)}A+O(1)\quad(A\to\infty).\end{aligned}\tag{3.4}$$

The following lemma is the key when we consider $I_\infty(x,t)$.

Lemma 3.3. *Let $\xi'\in\mathbb{R}^{N-1}\setminus\{0\}$. Then, $L(A,B)\neq 0$ provided that $\lambda\in\{z\in\mathbf{C}\,|\,\operatorname{Re}z\geq 0\}$.*

We set
$$L_0=\{\lambda\in\mathbf{C}\,|\,L(A,B)=0,\ \operatorname{Re}B\geq 0,\ A\in\operatorname{supp}\varphi_\infty\},$$
and then we obtain the following lemma by (3.4) and Lemma 3.3.

Lemma 3.4. *There exist positive numbers $0<\varepsilon_\infty<\pi/2$ and $\lambda_\infty>0$ such that*

$$L_0\subset\Sigma_{\varepsilon_\infty}\cap\{z\in\mathbf{C}\,|\,\operatorname{Re}z<-\lambda_\infty\}.$$

By using λ_∞ in Lemma 3.3, we put $\gamma_\infty=\min\{\lambda_\infty,\ 4^{-1}\times(A_0/6)^2\}$, and we change Γ to the paths:

$$\begin{aligned}\Gamma_4^\pm&=\{\lambda\in\mathbf{C}\,|\,\lambda=-\gamma_\infty\pm iu,\ u:0\to\tau_0\},\\ \Gamma_5^\pm&=\{\lambda\in\mathbf{C}\,|\,\lambda=-\gamma_\infty\pm i\tau_0+ue^{\pm i(\pi-\varepsilon_\infty)},\ u:0\to\infty\},\end{aligned}$$

where $\tau_0>0$ is the same number as $\lambda_0=\lambda_0(\varepsilon_\infty,\gamma_a,\sigma)$ in Lemma 3.1. Then, $I_\infty(x,t)$ can be written by

$$I_\infty(x,t)=\sum_{n=4}^{5}I_\infty^{\pm,n}(x,t),\quad I_\infty^{\pm,n}(x,t)=\frac{1}{2\pi i}\int_{\Gamma_n^\pm}e^{\lambda t}\mathcal{F}_{\xi'}^{-1}\left[\frac{\varphi_\infty(\xi')D(A,B)}{(B+A)L(A,B)}e^{-Ax_N}\widehat{d}(\xi')\right](x')\,d\lambda.$$

We have the following theorem for $I_\infty(x,t)$.

Theorem 3.5. *Let $1<q<\infty$ and $d\in W_q^{2-(1/q)}(\mathbb{R}^{N-1})$. Then, there exist a positive number $\delta>0$ and a positive constant C such that for any $t\geq 1$ there holds*

$$\|\partial_tI_\infty(t)\|_{W_q^2(\mathbb{R}_+^N)}+\|\nabla I_\infty(t)\|_{W_q^2(\mathbb{R}_+^N)}\leq Ce^{-\delta t}\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})}.$$

Proof. Set

$$H_\infty(x,\lambda) = \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty(\xi')D(A,B)}{(B+A)L(A,B)}e^{-Ax_N}\widehat{d}(\xi')\right](x') \quad (\lambda \in \Gamma_4^\pm \cup \Gamma_5^\pm). \tag{3.5}$$

First, we write (3.5) by integral. For the purpose, we extend $d \in W_q^{2-(1/q)}(\mathbb{R}^{N-1})$ to d^* defined in $\mathbb{R}^N_+$ satisfying $d^* = d$ on $\mathbb{R}^N_0$ and

$$\|d^*\|_{W_q^2(\mathbb{R}^N_+)} \le C\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})}. \tag{3.6}$$

By using the relation:

$$\begin{aligned}\widehat{d^*}(\xi,0) &= -\int_0^\infty \frac{d}{dy_N}\left(e^{-Ay_N}\widehat{d^*}(\xi,y_N)\right)dy_N\\ &= \int_0^\infty Ae^{-Ay_N}\widehat{d^*}(\xi,y_N)\,dy_N - \int_0^\infty e^{-Ay_N}\widehat{D_N d^*}(\xi,y_N)\,dy_N\end{aligned}$$

and $A^2 = -\sum_{j=1}^{N-1}(i\xi_j)^2$ in (3.5), we have

$$\begin{aligned}H_\infty(x,\lambda) = &-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[\frac{\varphi_\infty(\xi')D(A,B)}{A^2(B+A)L(A,B)}Ae^{-A(x_N+y_N)}\widehat{\Delta' d^*}(\xi',y_N)\right](x')\,dy_N\\ &+\sum_{j=1}^{N-1}\int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[(i\xi_jA^{-1})\frac{\varphi_\infty(\xi')D(A,B)}{A^2(B+A)L(A,B)}Ae^{-A(x_N+y_N)}\widehat{D_N D_j d^*}(\xi',y_N)\right](x')\,dy_N.\end{aligned} \tag{3.7}$$

Now, there holds the following lemma.

Lemma 3.6. *Let $\alpha' \in \mathbf{N}_0^{N-1}$. Then, there exists a positive constant C such that for any $\xi' \in (\mathbb{R}^{N-1}\setminus\{0\})$ there hold*

$$\left|D^{\alpha'}_{\xi'}\left(\frac{\varphi_\infty(\xi')D(A,B)}{A^2(B+A)L(A,B)}\right)\right| \le CA^{-3-|\alpha'|} \quad (\lambda \in \Gamma_4^\pm),$$

$$\left|D^{\alpha'}_{\xi'}\left(\frac{\varphi_\infty(\xi')D(A,B)}{A^2(B+A)L(A,B)}\right)\right| \le C\frac{(|\lambda|^{1/2}+A)^2}{A^2\{|\lambda|(|\lambda|^{1/2}+A)^2 + A(\gamma_a+\sigma A^2)\}}A^{-|\alpha'|} \quad (\lambda \in \Gamma_5^\pm),$$

where C is independent of λ.

Proof. See [9]. □

By Lemma 3.6, (3.6), (3.7) and [11, Lemma 5.4], we have the resolvent estimates:

$$\|\lambda H_\infty\|_{W_q^2(\mathbb{R}^N_+)} + \|\nabla H_\infty\|_{W_q^2(\mathbb{R}^N_+)} \le C\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})}$$

for any $\lambda \in \Gamma_4^\pm \cup \Gamma_5^\pm$ with some positive constant C independent of λ. We can easily show the required estimate for $I_\infty(x,t)$ by combining the above resolvent estimates with the exact formula of $I_\infty(x,t)$. This completes the proof. □

By Theorem 3.2 and Theorem 3.5, we have

$$\begin{aligned}\|D_x^\alpha \nabla I(t)\|_{L_q(\mathbb{R}^N_+)} &\le \|D_x^\alpha \nabla I_0(t)\|_{L_q(\mathbb{R}^N_+)} + \|D_x^\alpha \nabla I_\infty(t)\|_{L_q(\mathbb{R}^N_+)}\\ &\le Ct^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{1}{4}-\frac{|\alpha|}{2}}\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})\cap L_r(\mathbb{R}^{N-1})}\end{aligned}$$

for any multi-index $\alpha \in \mathbf{N}_0^N$ with $|\alpha| \le 2$ and $1 < r < 2 < q < \infty$. Similarly, we obtain

$$\|D_x^\alpha \partial_t I(t)\|_{L_q(\mathbb{R}^N_+)} \le Ct^{-\frac{N-1}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})-\frac{|\alpha|}{2}}\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})\cap L_r(\mathbb{R}^{N-1})}.$$

Finally, we consider the L_∞ norms. Let $q > N$. Then, by Theorem 3.5 and Sobolev's inequality there holds

$$\|\nabla I_\infty(t)\|_{W^1_\infty(\mathbb{R}^N_+)} \le Ce^{-\delta t}\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})}.$$

Combining the above inequality and Theorem 3.2 yield that

$$\|\nabla I(t)\|_{L_\infty(\mathbb{R}^N_+)} \leq Ct^{-\left(\frac{N-1}{2r}+\frac{1}{2}\right)}\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})\cap L_r(\mathbb{R}^{N-1})},$$
$$\|\nabla^2 I(t)\|_{L_\infty(\mathbb{R}^N_+)} \leq Ct^{-\left(\frac{N-1}{2r}+1\right)}\|d\|_{W_q^{2-(1/q)}(\mathbb{R}^{N-1})\cap L_r(\mathbb{R}^{N-1})}.$$

Note that these estimates of $I(x,t)$ are corresponding to the estimates of $H(x,t)$ in Theorem 2.2 since $I(x,t)$ is a part of the solution $H(x,t)$.

References

[1] H. Abels, *The initial-value problem for the Navier-Stokes equations with a free surface in L_q-Sobolev spaces*, Adv. Differential Equations 10 (2005), no. 1, 45-64.

[2] G. Allain, *Small-time existence for the Navier-Stokes equations with a free surface and surface tension*, Free boundary problems: application and theory, Vol. IV (Maubuisson, 1984), 355-364, Res. Notes in Math., 121, Pitman, Boston, MA, 1985.

[3] J. T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure Appl. Math. 34 (1981), no. 3, 359-392.

[4] J. T. Beale, *Large-time regularity of viscous surface waves*, Arch. Rational Mech. Anal. 84 (1983/84), no. 4, 307-352.

[5] J. T. Beale and T. Nishida, *Large-time behavior of viscous surface waves*, Recent topics in nonlinear PDE, II (Sendai, 1984), 1-14, North-Holland Math. Stud., 128, North-Holland, Amsterdam, 1985.

[6] Y. Hataya and S. Kawashima, *Decaying solution of the Navier-Stokes flow of infinite volume without surface tension*, Nonlinear Anal. 71 (2009), no. 12.

[7] Y. Hataya, *A remark on Beale-Nishida's paper*, Bull. Inst. Math. Acad. Sin. (N.S.) 6 (2011), no. 3, 293-303.

[8] J. Prüss and G. Simonett, *Analytic solutions for the two-phase Navier-Stokes equations with surface tension and gravity*, Parabolic problems, 507-540, Progr. Nonlinear Differential Equations Appl., 80, Birkhäuser/Springer Basel AG, Basel, 2011.

[9] H. Saito and Y. Shibata, *On the Stokes equations with surface tension under the uniform gravity field in the half space*, preprint.

[10] Y. Shibata and S. Shimizu, *On a resolvent estimate of the Stokes system in a half space arising from a free boundary problem for the Navier-Stokes equations*, Math. Nachr. 282 (2009), no. 3, 482-499.

[11] Y. Shibata and S. Shimizu, *On the L_p-L_q Maximal Regularity of the Stokes Problem with First Order Boundary Condition; Model Problems* J. Math. Soc. Japan, 64 (2012), no.2, 561-626.

[12] A. Tani and N. Tanaka, *Large-time existence of surface waves in incompressible viscous fluids with or without surface tension*, Arch. Rational Mech. Anal. 130 (1995), no. 4, 303-314.

[13] A. Tani, *Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface*, Arch. Rational Mech. Anal. 133 (1996), no. 4, 299-331.