The Stokes semigroup on non-decaying spaces

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Abstract

In this brief note, we review recent results on the analyticity of the Stokes semigroup in spaces of bounded functions. The Stokes equations are well understood on $L^p$ space, $p \in (1, \infty)$, for various kinds of domains such as bounded or exterior domains with smooth boundaries. However, the situation is very different on $L^\infty$ since in this case the Helmholtz projection does not act as a bounded operator on $L^\infty$ anymore. The purpose of this note is to review an approach to prove the analyticity of the semigroup on $L^\infty$, especially, on $L^\infty_\sigma$ and $BUC_\sigma$ for exterior domains and perturbed half spaces. Note that for merely bounded initial data, even existence of solutions are non-trivial. We approximate merely bounded initial data on $L^\infty_\sigma$ and prove the unique existence of solutions together with the analyticity of the semigroup. This note is based on joint works with Y. Giga [2], [3] and the thesis [1].

1 Introduction

We consider the initial-boundary problem for the Stokes equations in the domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$:

\begin{align*}
v_t - \Delta v + \nabla q &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
\text{div} v &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
v &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
v &= v_0 \quad \text{on} \quad \Omega \times \{t = 0\}.
\end{align*}

(1.1) (1.2) (1.3) (1.4)

It is well known that the solution operator (called the Stokes semigroup)

\[ S(t) : v_0 \mapsto v(\cdot, t), \quad t \geq 0, \]

forms an analytic semigroup on the solenoidal $L^p$ space, $L^p_\sigma(\Omega)$, $p \in (1, \infty)$, for various kind of domains $\Omega$, such as bounded and exterior domains with smooth boundaries [25], [13]. However, it had been a long-standing open problem whether or not the Stokes semigroup $\{S(t)\}_{t \geq 0}$ is analytic on $L^\infty$-type spaces even if $\Omega$ is bounded. When $\Omega$ is a half space, it is known that the Stokes semigroup $\{S(t)\}_{t \geq 0}$ is analytic on $L^\infty$-type spaces since explicit solution formulas are available [6], [19], [26].

In [2], Y. Giga and the author gave an affirmative answer to this open problem at least when $\Omega$ is bounded as a typical example. Later, this approach was extended to exterior domains [3] and perturbed half spaces ($n \geq 3$) [1]. The propose of this note is to review an approach to
prove the existence of solutions for merely bounded initial data as well as the analyticity of the semigroup on $L^\infty$-type spaces.

We begin with a typical statement for bounded domains. Let $C_{0,\sigma}(\Omega)$ denote the $L^\infty$-closure of $C^\infty_c(\Omega)$, the space of all smooth solenoidal vector fields with compact support in $\Omega$. When $\Omega$ is bounded, $C_{0,\sigma}(\Omega)$ agrees with the space of all solenoidal vector fields continuous in $\overline{\Omega}$ vanishing on $\partial\Omega$ [18]. A typical result proved in [2, Theorem 1.1] is the following:

**Theorem 1.1** (Analyticity on $C_{0,\sigma}$). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^3$-boundary. Then, the solution operator (the Stokes semigroup) $S(t) : v_0 \mapsto v(\cdot, t)$ is a $C_0$-analytic semigroup on $C_{0,\sigma}(\Omega)$.

The approach to prove Theorem 1.1 was to establish an a priori estimate for

$$N(v, q)(x, t) = |v(x, t)| + t^2|\nabla v(x, t)| + t|\nabla^2 v(x, t)| + t|\partial_t v(x, t)| + t|\nabla q(x, t)|$$

(1.5)
of the form

$$\sup_{0 < r < T_0} \|N(v, q)\|_\infty(t) \leq C\|v_0\|_\infty$$

(1.6)
for some $T_0 > 0$ and $C$ depending only on the domain $\Omega$, where $\|v_0\|_\infty = \|v_0\|_{L^\infty(\Omega)}$ denotes the sup-norm of $|v_0|$ in $\Omega$. The a priori estimate (1.6) was proved by an indirect argument called a blow-up argument which is often used in the study of non-linear elliptic and parabolic equations [12], [14], [21], [20] (see also [17], [16] for the Navier–Stokes equations). Later, a direct approach to prove Theorem 1.1 was found in [4]. The approach in the paper is to derive $L^\infty$-estimates for solutions of the resolvent problem corresponding to (1.1)–(1.4) based on the Masuda–Stewart technique for elliptic operators.

In both approaches, a key is to estimate pressure gradient in terms of velocity, i.e.,

$$\sup_{x \in \Omega} d_\Omega(x)|\nabla q(x, \cdot)| \leq C\|w\|_{L^\infty(\partial\Omega)},$$

(1.7)
where

$$w(v) = -(\nabla v - \nabla^T v)n_\Omega.$$ 

(1.8)
Here, $d_\Omega$ denotes the distance from $x \in \Omega$ to $\partial\Omega$, i.e., $d_\Omega(x) = \inf_{y \in \partial\Omega} |x - y|$ and $n_\Omega$ denotes the unit outward normal vector field on $\partial\Omega$. For $n = 3$, $w(v)$ is nothing but a tangential component of vorticity, i.e., $-\text{curl } v \times n_\Omega$. For $n = 2$, $w(v)$ agrees with $-\text{curl } v n^1_\Omega$, where $n^1_\Omega = (n^2_\Omega, -n^3_\Omega)$.

The estimate (1.7) plays an important role for estimating pressure gradient $\nabla q = (I - P)\Delta v$ by the velocity $v$ on $L^\infty$ since the Helmholtz projection $P$ does not act as a bounded operator on $L^\infty$. Actually, the estimate (1.7) is a special case of the estimate for the homogeneous Neumann problem of the form

$$\Delta q = 0 \quad \text{in } \Omega, \quad \frac{\partial q}{\partial n_\Omega} = \text{div}_{\partial\Omega} w \quad \text{on } \partial\Omega, $$

(1.9)
where $\text{div}_{\partial\Omega}$ denotes the surface divergence on $\partial\Omega$. Since the divergence-free condition for velocity implies

$$\Delta v \cdot n_\Omega = \text{div}_{\partial\Omega} w(v) \quad \text{on } \partial\Omega,$$
the pressure $q$ solves the Neumann problem (1.9) for $w = w(v)$. The estimate (1.7) is valid for various domains, but it may not be true for general domains so we call $\Omega$ strictly admissible if the a priori estimate (1.7) holds for all solutions of the Neumann problem (1.9). Of course, a
half space is strictly admissible. Moreover, it was proved that bounded domains [2, Theorem 2.5] and exterior domains [3, Theorem 3.1] of class $C^3$ are strictly admissible. However, layer domains are not strictly admissible. In fact, in a layer domain, $\Omega = \{x = (x', x_n) \in \mathbb{R}^n | 0 < x_n < 1\}$, $P = x_1$ does not satisfy the estimate (1.9) for $w = 0$. We conjecture that quasi-cylindrical domains, i.e., $\lim_{|x| \to \infty} d_{\Omega}(x) < \infty$, are not strictly admissible.

Actually, it is possible to extend Theorem 1.1 for general strictly admissible, uniformly $C^3$-domains [2, Theorem 1.3] by using the $L^p$-theory developed in [8], [9], [10] since the space $L^p_{0,\sigma}$ is the $L^\infty$-closure of $C^\infty_{0,\sigma}$. Once we have the a priori estimate (1.6) for $v_0 \in C^\infty_{0,\sigma}$, it is extendable for $v_0 \in L^p_{0,\sigma}$. Note that the $L^p$-theory is also available for uniformly $C^3$-domains for which the Helmholtz projection is bounded on $L^p$ [11] so we are able to extend Theorem 1.1 through the $L^p$-theory for domains such as exterior domains or perturbed half spaces.

2 Non-decaying solenoidal spaces

It is natural to extend Theorem 1.1 for the larger space than $C_{0,\sigma}$,

$$L^\infty_{\sigma}(\Omega) = \left\{ f \in L^\infty(\Omega) \mid \int_{\Omega} f \cdot \nabla \varphi dx = 0 \quad \text{for all} \quad \varphi \in \hat{W}^{1,1}(\Omega) \right\},$$

where $\hat{W}^{1,1}(\Omega)$ denotes the homogeneous Sobolev space $\hat{W}^{1,1}(\Omega) = \{ \varphi \in L^1_{loc}(\Omega) \mid \nabla \varphi \in L^1(\Omega) \}$. Since the space $L^\infty_{\sigma}$ includes discontinuous functions, we approximate $v_0 \in L^\infty_{\sigma}(\Omega)$ by elements of $C^\infty_{c,\sigma}$ by the pointwise convergence in $\Omega$. We extend the Stokes semigroup $S(t)$ to $L^\infty_{\sigma}$ by the following approximation [2, Lemma 6.3].

**Lemma 2.1** (Approximation). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. There exists a constant $C = C_{\Omega}$ such that for $v_0 \in L^\infty_{\sigma}(\Omega)$, there exists a sequence $\{v_{0,m}\}_{m=1}^{\infty} \subset C^\infty_{c,\sigma}(\Omega)$ such that

$$\|v_{0,m}\|_{L_{\infty}(\Omega)} \leq C\|v_0\|_{L_{\infty}(\Omega)},$$

$$v_{0,m} \to v_0 \quad \text{a.e. in } \Omega \quad \text{as } m \to \infty. \quad (2.1)$$

If we do not care about the divergence-free condition for the sequence $\{v_{0,m}\}_{m=1}^{\infty}$, it is easy to construct the sequence satisfying (2.1). Lemma 2.1 says that we are able to approximate $v_0 \in L^\infty_{\sigma}$ by solenoidal vector fields $\{v_{0,m}\}_{m=1}^{\infty} \subset C^\infty_{c,\sigma}$ keeping the sup-norm, i.e., $\|v_{0,m}\|_{\infty} \leq C\|v_0\|_{\infty}$. If $\Omega$ is star-shaped, i.e., $\lambda\Omega \subset \Omega$, $\lambda < 1$, it is easy to construct the sequence satisfying (2.1). In fact, for $v_0 \in L^\infty_{\sigma}(\Omega)$, set $v_{0,\lambda}(x) = \alpha(x) v_{0}(\lambda x)$ for $x \in \lambda\Omega$ and $v_{0,\lambda}(x) = 0$ for $x \in \Omega \backslash \lambda\Omega$ so that $v_{0,\lambda}$ is a compactly supported solenoidal vector field in $\Omega$. Then, we get the desired sequence with $C = 1$ in (2.1) by multiplying the mollifier $\eta_\varepsilon$ to $v_{0,\lambda}$, i.e., $v_{0,m} = \eta_{\frac{1}{m}} \ast v_{0,\lambda_m}$. For general bounded domains, we are able to prove Lemma 2.1 by decomposing $\Omega$ into star-shaped domains.

By the above approximation, we are able to prove that the Stokes semigroup $S(t)$ is a (non-$C_0$)-analytic semigroup on $L^\infty_{\sigma}(\Omega)$ [2, Theorem 1.5]. Note that the semigroup $S(t)$ is not type $C_0$ since $S(t)v_0$ is smooth for $t > 0$ so $S(t)v_0 \to v_0$ on $L^\infty$ as $t \downarrow 0$ may not hold for general $v_0 \in L^\infty_{\sigma}$. This means that $S(t)$ is a non-$C_0$-analytic semigroup.

Now, we observe the extension of $S(t)$ to $L^\infty_{\sigma}(\Omega)$ for unbounded domains $\Omega$. Note that the space $L^\infty_{\sigma}$ includes non-decaying functions as $|x| \to \infty$ so the existence of solutions for $v_0 \in L^\infty_{\sigma}(\Omega)$ are non-trivial problem. However, if Lemma 2.1 is valid for the unbounded domain
\( \Omega \) (satisfying the strictly admissibility), we are able to prove the existence of solutions for \( v_0 \in L^\infty_\sigma(\Omega) \) satisfying the estimate (1.6) (called \( L^\infty \)-solutions). Although the approximation (2.1) is unknown in general, it is known to hold for exterior domains [3, Lemma 5.1] and perturbed half space [1, Lemma 4.3.10]. Let us sketch the approach to prove the existence of solutions for \( v_0 \in L^\infty_\sigma \) based on [3] and (1) for exterior domains and perturbed half spaces.

Our approach is by the \( L^\infty \)-estimate (1.6) and the approximation (2.1). We find a solution \((v, q)\) for \( v_0 \in L^\infty_\sigma \) by a sequence of \( L^p \)-solutions \([(v_m, q_m)]_{m=1}^\infty \) for \( v_{0,m} \in C^\infty_{c,\sigma} \). By the estimates (1.6) and (2.1), the sequence \((v_m, q_m)\) is uniformly bounded, i.e.,

\[
\sup_{0<\tau<T_0} \left\| [N(v_m, q_m)]_{\infty}^{(t)} \right\| \leq C \|v_0\|_{\infty}.
\]

Since \( v_{0,m} \to v_0 \), it is natural to expect that \((v_m, q_m)\) converges to a solution \((v, q)\) for \( v_0 \in L^\infty_\sigma \). In fact, by (1.6) and (2.1), we are able to estimate the Hölder semi-norms of \( q \) in the interior of \( \Omega \times (0, T] \) both in space and time variables. Thus, from the parabolic regularity theory, \([(v_m, q_m)]_{m=1}^\infty \) (subsequently) converges to a limit \((v, q)\) locally uniformly in \( \Omega \times (0, T] \) up to second orders. Actually, the limit \((v, q)\) is continuous in \( \overline{\Omega} \times (0, T] \) up to second derivatives since we have local Hölder estimates up to the boundary based on the Solonnikov’s Hölder estimate for (1.1)–(1.4) [25], [28], [29] (see [2, Theorem 3.5]). The uniqueness of \( L^\infty \)-solutions follows from the a priori estimate (1.6) for \( v_0 = 0 \) so the limit \((v, q)\) is independent of a choice of the sequence \([v_{0,m}]_{m=1}^\infty \subset C^\infty_{c,\sigma} \).

To state a result, let us define solutions of (1.1)–(1.4) for \( v_0 \in L^\infty_\sigma(\Omega) \) [3, Definition 2.7].

**Definition 2.2 (\( L^\infty \)-solutions).** Let \( \Omega \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \), with \( \partial \Omega \neq \emptyset \). Let \((v, \nabla q) \in C^{2,1}(\overline{\Omega} \times (0, T)) \times C(\overline{\Omega} \times (0, T)) \) satisfy (1.1)–(1.3) and (1.4) for \( v_0 \in L^\infty_\sigma(\Omega) \) in the sense that \( v(\cdot, t) \to v_0 \) weakly-* on \( L^\infty(\Omega) \) as \( t \downarrow 0 \). We call \((v, q)\) an \( L^\infty \)-solution if (1.5) and

\[
t^{1/2} d_\Omega(x) |\nabla q(x, t)|
\]

are bounded in \( \Omega \times (0, T) \).

Once we know the unique existence of \( L^\infty \)-solutions, we are able to extend the Stokes semigroup \( S(t) : v_0 \mapsto v(\cdot, t), t \geq 0, \) for \( v_0 \in L^\infty_\sigma \) together with the estimate (1.6). The following statement was proved in [3, Theorem 3.2] for exterior domains and [1, Theorem 4.1.2] for perturbed half spaces.

**Theorem 2.3.** Let \( \Omega \) be an exterior domain in \( \mathbb{R}^n \), \( n \geq 2 \), or a perturbed half space in \( \mathbb{R}^n \), \( n \geq 3 \), with \( C^3 \)-boundary.

(i) (Unique existence of \( L^\infty \)-solutions)

For \( v_0 \in L^\infty_\sigma(\Omega) \), there exists a unique \( L^\infty \)-solution \((v, \nabla q)\) satisfying (1.6) for any fixed \( T_0 \) with some constant \( C \) depending only on \( T_0 \) and \( \Omega \).

(ii) (Analyticity on \( L^\infty_\sigma \))

The Stokes semigroup \( S(t) \) is uniquely extendable to a (non-\( C_0 \))analytic semigroup on \( L^\infty_\sigma(\Omega) \).

**Remark 2.4** (Continuity at time zero). It is natural to restrict \( S(t) \) to the space of uniformly continuous functions \( BUC_{c,\sigma}(\Omega) \) so that \( S(t) \) is a \( C_0 \)-analytic semigroup on \( BUC_{c,\sigma}(\Omega) \). Let \( BUC(\Omega) \) be the space of all uniformly continuous functions in \( \Omega \). Define the space \( BUC_{c,\sigma}(\Omega) \) by

\[
BUC_{c,\sigma}(\Omega) = \{ f \in BUC(\Omega) \mid \text{div} f = 0 \text{ in } \Omega, f = 0 \text{ on } \partial \Omega \}.
\]

Then, \( S(t) \) is a \( C_0 \)-(analytic) semigroup on \( BUC_{c,\sigma}(\Omega) \) at least when \( \Omega \) is an exterior domain. Note that \( C_{0,\sigma}(\Omega) \subset BUC_{c,\sigma}(\Omega) \subset L^\infty_\sigma(\Omega) \). When \( \Omega \) is bounded, the space \( BUC_{c,\sigma}(\Omega) \) agrees with \( C_{0,\sigma}(\Omega) \) [18], [2, Lemma 6.3].
References


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