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Function spaces and isometrical extensions of bounded isometries of separable metric spaces

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1 Introduction

In this note, unless stated otherwise, we assume that all maps are continuous functions. Let \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{R} \) denote the set of integers, the set of natural numbers and the set of real numbers, respectively. Also, let \( I, \Delta \) and \( \mathbb{Q} \) be the unit interval \([0, 1]\), a Cantor set and the Hilbert cube \( I^\infty \), respectively. For any compact metric space \( Z \), \( C(Z) \) denotes the function space of all (continuous) maps from \( Z \) to \( \mathbb{R} \) with the supremum metric \( \tilde{d} \), i.e.,

\[
\tilde{d}(f, g) = \sup \{|f(z) - g(z)| \mid z \in Z\}
\]

for \( f, g \in C(Z) \).

A map \( i : (X, d_X) \to (Y, d_Y) \) between separable metric spaces is an isometrical embedding from \( (X, d_X) \) into \( (Y, d_Y) \) if \( i \) satisfies the condition \( d_Y(i(x), i(x')) = d_X(x, x') \) for each \( x, x' \in X \). A map \( g : (X, d_X) \to (Y, d_Y) \) between separable metric spaces is an isometry if \( g \) is surjective and \( d_Y(g(x), g(x')) = d_X(x, x') \) for each \( x, x' \in X \). For a separable metric space \( (X, d) \), let \( \text{Iso}(X) \) be the group of all isometries of \( X \) equipped with the pointwise convergent topology, i.e.,

\[
\text{Iso}(X) = \{ g : X \to X \mid g \text{ is an isometry} \}.
\]

A well-known theorem of Banach and Mazur is the result that \( C(I) \ (I = [0, 1]) \) is a universal space of separable metric spaces up to isometry (see \([1,3,9]\)). Also, Urysohn \([11]\) constructed a complete separable metric space \( U \) that is also universal up to isometry. In \([12]\), Uspenskij proved that for any separable metric space \( X \) there is a natural isometrical embedding \( i : X \to U \) such that \( i \) induces a natural continuous monomorphism \( i^* : \text{Iso}(X) \to \text{Iso}(U) \) satisfying that \( i^*(g) \in \text{Iso}(U) \) is an extension of \( g \in \text{Iso}(X) \) (see \([2,3,5,7,12,13]\) for more detailed properties of \( U \)).

In this note, we study the extension property of "bounded" isometries of separable metric spaces in function spaces \( C(\mathbb{Q}) \) and \( C(\Delta) \). Also, we know that \( C(I) \) does not have the extension property. Let \( (X, d) \) be a separable metric space and \( x_0 \in X \). A subgroup \( G \) of \( \text{Iso}(X) \) is bounded if \( \text{diam} \ G(x_0) < \infty \), where \( G(x_0) = \{ g(x_0) \mid g \in G \} \subset X \). The definition of "bounded subgroup" of \( \text{Iso}(X) \) does not depend on the choice of the point \( x_0 \in X \). Also, each \( g \in \text{Iso}(X) \) is bounded if \( \text{diam} \{ g^n(x_0) \mid n \in \mathbb{Z} \} < \infty \). Note that if \( (X, d) \) is bounded, i.e., \( \text{diam}_d X < \infty \), then \( \text{Iso}(X) \) itself is bounded. In particular, if \( X \) is a compact metric space, then \( \text{Iso}(X) \) is bounded. In \([6]\), Mazur and Ulam proved that if \( B \) and \( B' \) are Banach spaces, then every isometry \( T : B \to B' \) with \( T(0) = 0 \) is linealy
isometric and moreover, Banach and Stone proved that if $X$ and $Y$ are compact Hausdorff spaces, then every isometry $T : C(X) \to C(Y)$ with $T(0) = 0$ is linearly isometric and moreover, $T$ is induced by a homeomorphism $h : Y \to X$ (see [1,10]).

**Theorem 1.1.** (Banach [1] and Stone [10]) Let $X$ and $Y$ be compact Hausdorff spaces. Then the followings hold.

1. $C(X)$ is isometric to $C(Y)$ if and only if $X$ is homeomorphic to $Y$.
2. If $T : C(X) \to C(Y)$ is a linear isometry, then there is a homeomorphism $h : Y \to X$ and a (continuous) map $\alpha : Y \to \mathbb{R}$ with $|\alpha(y)| = 1$ for $y \in Y$ such that

   \[(T(f))(y) = \alpha(y) \cdot (f \circ h)(y)\]

   for $f \in C(X)$ and $y \in Y$. Moreover, if $Y$ is connected, $T(f) = f \circ h$ or $T(f) = - (f \circ h)$.

For any Banach space $B$, let

\[\text{LinIso}(B) = \{f \in \text{Iso}(B) | f \text{ is linear}\}.\]

Note that LinIso($B$) is bounded, because LinIso$(B)(0) = \{0\}$.

## 2 Extensions of bounded isometries in function spaces

In this section, we assume that $(X, d)$ is a separable metric space and $x_0$ is a fixed point of $X$. In [9], Sierpiński considered the space

\[X' = \{f : X \to \mathbb{R} | f(x_0) = 0 \text{ and } |f(x) - f(y)| \leq d(x, y) \text{ for } x, y \in X\}\]

which is a topological space equipped with the pointwise convergent topology (see also [3]) and by use of the spaces $X'$, he proved that $C(I)$ is a universal space of separable metric spaces up to isometry. We modify the Sierpiński's method of [9]. In this paper, for any bounded subgroup $G$ of Iso($X$), we consider the following more general space

\[\tilde{X} (= \tilde{X}_G) = \{f : X \to \mathbb{R} | f(z) \in [-\text{diam}(G(x_0)), \text{diam}(G(x_0))] \text{ for } z \in G(x_0) \text{ and } |f(x) - f(y)| \leq d(x, y) \text{ for } x, y \in X\}\]

which is a topological space equipped with the pointwise convergent topology. We have the following lemmas.

**Lemma 2.1.** $\tilde{X} (= \tilde{X}_G)$ is a compact metric absolute retract (= AR). Moreover, if $g \in G$, then $\tilde{g} : \tilde{X} \to \tilde{X}$ is a homeomorphism, where $\tilde{g}$ is defined by $\tilde{g}(f) = f \circ g$ for $f \in \tilde{X}$.

**Lemma 2.2.** Suppose that $p_G : Z \to \tilde{X} (= \tilde{X}_G)$ is a map from a compact metric space $Z$ onto $\tilde{X}$ such that for each $g \in G$ there is a (lift) homeomorphism $L_g : Z \to Z$ satisfying the following commutative diagram.

\[
\begin{array}{ccc}
Z & \xrightarrow{\text{L}} & Z \\
\downarrow p_G & & \downarrow p_G \\
\tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X}
\end{array}
\]
Then there is an isometrical embedding $i_G : X \to C(Z)$ such that for each $g \in G$, the following commutative diagram holds:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
i_G \downarrow & & \downarrow i_G \\
C(Z) & \xrightarrow{L_g} & C(Z)
\end{array}
\]

where $L_g : C(Z) \to C(Z)$ is the isometry defined by $L_g(f) = f \circ L_g$ for $f \in C(Z)$. In particular, $L_g \in \text{LinIso}(C(Z))$ is an isometrical extension of $g \in G$.

Here we have the following theorem of $C(\mathbb{Q})$ which implies that $C(\mathbb{Q})$ is universal concerning isometrical extensions of bounded isometry groups of separable metric spaces.

**Theorem 2.3.** Let $(X, d)$ be a separable metric space and let $G$ be any bounded subgroup of $\text{Iso}(X)$. Then there is an isometrical embedding $i_G : X \to C(\mathbb{Q})$ such that $i_G$ induces a continuous monomorphism $i_G^* : G \to \text{LinIso}(C(\mathbb{Q}))$ such that $i_G^*(g) \in \text{LinIso}(C(\mathbb{Q}))$ is an extension of $g \in G$.

**Corollary 2.4.** Suppose that $(X, d)$ is a bounded separable metric space. Then there is an isometrical embedding $i : X \to C(\mathbb{Q})$ such that $i$ induces a continuous monomorphism $i^* : \text{Iso}(X) \to \text{LinIso}(C(\mathbb{Q}))$ such that $i^*(g) \in \text{LinIso}(C(\mathbb{Q}))$ is an extension of $g \in \text{Iso}(X)$.

Remark 1. Note that for any Banach space $B$, $\text{LinIso}(B)$ is a bounded group. Hence in this note, we can not omit the condition that $G$ is bounded.

If we observe the proof of Lemma 2.2, we see that some converse assertions of Lemma 2.2 are also true. In fact, we have the following.

**Proposition 2.5.** Suppose that $p_G : Z \to \tilde{X}(= \tilde{X}_G)$ is a map from a compact metric space $Z$ onto $\tilde{X}$, $i_G : X \to C(Z)$ is the isometrical embedding as in the proof of Lemma 2.2 and $g \in G$. Let $L_g : Z \to Z$ be a homeomorphism. Then the followings hold.

(1) The following diagram is commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{L_g} & Z \\
p_G \downarrow & & \downarrow p_G \\
\tilde{X} & \xrightarrow{g} & \tilde{X}
\end{array}
\]

if and only if the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
i_G \downarrow & & \downarrow i_G \\
C(Z) & \xrightarrow{L_g} & C(Z)
\end{array}
\]

(2) The following diagram is commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{L_g} & Z \\
p_G \downarrow & & \downarrow p_G \\
\tilde{X} & \xrightarrow{g} & \tilde{X}
\end{array}
\]
if and only if the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
i_G \downarrow & & \downarrow i_G \\
C(Z) & \xrightarrow{L} & C(Z)
\end{array}
\]

Example. Let \( X = \{x_i : i = 0, 1, 2\} \) be the set of three elements and let \( d \) be the metric on \( X \) defined by \( d(x_i, x_j) = r > 0 (i \neq j) \). Define the isometry \( g : X \to X \) by \( g(x_0) = x_0, g(x_1) = x_2 \) and \( g(x_2) = x_1 \). Let \( G = \{id_X, g\} \). Note that \( G(x_0) = \{x_0\} \).

Then there is an isometrical embedding \( i_G : X \to \mathbb{Q} \) such that there is no isometrical extension of \( g \) on \( \mathbb{Q} \). In particular, \( \mathbb{Q} \) is not equal to the Urysohn universal space \( U \), because that \( U \) has the following strong property: Any isometry between finite subsets of \( U \) can be extended to an isometry of \( U \).

Next we will consider the case of the function space \( C(\Delta) \). Let \( H(X) \) be the set of all homeomorphisms of a space \( X \).

**Proposition 2.6.** Let \( X \) be a compact metric space and let \( G \) be a countable subset of \( H(X) \). Then there is an onto map \( p_G : \Delta \to X \) such that for any \( g \in G \) there is a (lift) homeomorphism \( L_g : \Delta \to \Delta \) of \( \Delta \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
\Delta & \xrightarrow{L_g} & \Delta \\
p_G \downarrow & & \downarrow p_G \\
X & \xrightarrow{g} & X
\end{array}
\]

Then we have the following theorem of \( C(\Delta) \).

**Theorem 2.7.** Let \( (X, d) \) be any separable metric space and let \( G \) be a countable bounded subgroup of \( Iso(X) \). Then there is an isometrical embedding \( i_G : X \to C(\Delta) \) such that there exist a countable subgroup \( G^* \) of \( LinIso(C(\Delta)) \) and a continuous epimorphism \( r^* : G^* \to G \) such that each \( g^* \in G^* \) is an extension of \( r^*(g^*) \in G \). In particular, if \( g \in G \), then there is an extension \( g^* \in LinIso(C(\Delta)) \) of \( g \).

**Remark 2.** Note that the space \( H(\Delta) \) of all homeomorphisms of \( \Delta \) is homeomorphic to the space \( P \) of irrationals, and hence \( H(\Delta) \) is zero-dimensional. If \( G \) is any bounded subgroup of \( Iso(X) \) with \( \dim G \geq 1 \), there is no embedding from \( G \) to \( H(\Delta) \).

**Corollary 2.8.** Let \( (X, d) \) be any separable metric space. If \( g \in Iso(X) \) is periodic i.e., \( g^n = id_X \) for some \( n \in \mathbb{N} \), then there is an isometrical embedding \( i_g : X \to C(\Delta) \) such that there is an extension \( g^* \in LinIso(C(\Delta)) \) of \( g \) with \( (g^*)^n = id_{C(\Delta)} \).

Finally, we consider the case of \( C(I) \). We have the following proposition of \( C(I) \).

**Proposition 2.9.** Let \( (X, d) \) be any separable metric space and let \( g \in Iso(X) \) such that \( g \) has a periodic point \( x_0 \) with period \( n \in \mathbb{N} \). If \( n \geq 3 \), there is no isometrical embedding \( i \) from \( X \) to \( C(I) \) such that \( g \) has an extension in \( LinIso(C(I)) \).

Now, we have the following problem.

**Problem 2.10.** Let \( (X, d) \) be any separable metric space. Is it true that there is an isometrical embedding \( i \) from \( X \) to \( C(\mathbb{Q}) \) such that each \( g \in Iso(X) \) has an extension which is an affine isometry of \( C(\mathbb{Q}) \)?
References


