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Function spaces and isometrical extensions of bounded isometries of separable metric spaces

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1 Introduction

In this note, unless stated otherwise, we assume that all maps are continuous functions. Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ denote the set of integers, the set of natural numbers and the set of real numbers, respectively. Also, let I, Δ and $\mathbb{Q}$ be the unit interval $[0,1]$, a Cantor set and the Hilbert cube I^{∞}, respectively. For any compact metric space Z, $C(Z)$ denotes the function space of all (continuous) maps from Z to $\mathbb{R}$ with the supremum metric $\tilde{d}$, i.e.,

$$\tilde{d}(f,g)=\sup\{|f(z)-g(z)| \mid z\in Z\}$$

for $f,g\in C(Z)$.

A map $i:(X,d_X)\rightarrow(Y,d_Y)$ between separable metric spaces is an *isometrical embedding* from (X,d_X) into (Y,d_Y) if i satisfies the condition $d_Y(i(x),i(x'))=d_X(x,x')$ for each $x,x'\in X$. A map $g:(X,d_X)\rightarrow(Y,d_Y)$ between separable metric spaces is an *isometry* if g is surjective and $d_Y(g(x),g(x'))=d_X(x,x')$ for each $x,x'\in X$. For a separable metric space (X,d), let $Iso(X)$ be the group of all isometries of X equipped with the pointwise convergent topology, i.e.,

$$Iso(X)=\{g:X\rightarrow X \mid g \text{ is an isometry}\}.$$

A well-known theorem of Banach and Mazur is the result that $C(I)$ $(I=[0,1])$ is a universal space of separable metric spaces up to isometry (see [1,3,9]). Also, Urysohn [11] constructed a complete separable metric space $\mathbb{U}$ that is also universal up to isometry. In [12], Uspenskij proved that for any separable metric space X there is a natural isometrical embedding $i:X\rightarrow\mathbb{U}$ such that i induces a natural continuous monomorphism $i^{\star}$: $Iso(X)\rightarrow Iso(\mathbb{U})$ satisfying that $i^{\star}(g)\in Iso(\mathbb{U})$ is an extension of $g\in Iso(X)$ (see [2,3,5,7,12,13] for more detailed properties of $\mathbb{U}$).

In this note, we study the extension property of "bounded" isometries of separable metric spaces in function spaces $C(\mathbb{Q})$ and $C(\Delta)$. Also, we know that $C(I)$ does not have the extension property. Let (X,d) be a separable metric space and $x_0\in X$. A subgroup G of $Iso(X)$ is *bounded* if diam $G(x_0)<\infty$, where $G(x_0)=\{g(x_0)| \ g\in G\}$ $(\subset X)$. The definition of "bounded subgroup" of $Iso(X)$ does not depend on the choice of the point $x_0\in X$. Also, each $g\in Iso(X)$ is *bounded* if $\operatorname{diam}\{g^{n}(x_0)| \ n\in\mathbb{Z}\}<\infty$. Note that if (X,d) is bounded, i.e., $\operatorname{diam}_d X<\infty$, then $Iso(X)$ itself is bounded. In particular, if X is a compact metric space, then $Iso(X)$ is bounded. In [6], Mazur and Ulam proved that if B and B' are Banach spaces, then every isometry $T:B\rightarrow B'$ with $T(0)=0$ is linealy isometric and moreover, Banach and Stone proved that if X and Y are compact Hausdorff spaces, then every isometry $T : C(X) \to C(Y)$ with $T(0) = 0$ is linealy isometric and moreover, T is induced by a homeomorphism $h : Y \to X$ (see [1,10]).

Theorem 1.1. (Banach [1] and Stone [10]) *Let X and Y be compact Hausdorff spaces. Then the followings hold.*
(1) $C(X)$ is isometric to $C(Y)$ if and only if X is homeomorphic to Y.
(2) If $T : C(X) \to C(Y)$ is a linear isometry, then there is a homeomorphism $h : Y \to X$ and a (continuous) map $\alpha : Y \to \mathbb{R}$ with $|\alpha(y)| = 1$ for $y \in Y$ such that

$$(T(f))(y) = \alpha(y) \cdot (f \circ h)(y)$$

for $f \in C(X)$ and $y \in Y$. Moreover, if Y is connected, $T(f) = f \circ h$ or $T(f) = -(f \circ h)$.

For any Banach space B, let

$$LinIso(B) = \{f \in Iso(B) | \ f \text{ is linear } \}.$$

Note that $LinIso(B)$ is bounded, because $LinIso(B)(0) = \{0\}$.

2 Extensions of bounded isometries in function spaces

In this section, we assume that (X, d) is a separable metric space and x_0 is a fixed point of X. In [9], Sierpiński considered the space

$$X' = \{f : X \to \mathbb{R} \mid f(x_0) = 0 \text{ and } |f(x) - f(y)| \leq d(x, y) \text{ for } x, y \in X\}$$

which is a topological space equipped with the pointwise convergent topology (see also [3]) and by use of the spaces X', he proved that $C(I)$ is a universal space of separable metric spaces up to isometry. We modify the Sierpiński's method of [9]. In this paper, for any bounded subgroup G of $Iso(X)$, we consider the following more general space

$$\tilde{X} \ (= \tilde{X}_G) = \{f : X \to \mathbb{R} \mid f(z) \in [-\text{diam}(G(x_0)), \text{diam}(G(x_0))] \ \text{ for } z \in G(x_0) \text{ and }$$
$$|f(x) - f(y)| \leq d(x, y) \text{ for } x, y \in X\}$$

which is a topological space equipped with the pointwise convergent topology. We have the following lemmas.

Lemma 2.1. *$\tilde{X}(= \tilde{X}_G)$ is a compact metric absolute retract (= AR). Moreover, if $g \in G$, then $\tilde{g} : \tilde{X} \to \tilde{X}$ is a homeomorphism, where $\tilde{g}$ is defined by $\tilde{g}(f) = f \circ g$ for $f \in \tilde{X}$.*

Lemma 2.2. *Suppose that $p_G : Z \to \tilde{X}(= \tilde{X}_G)$ is a map from a compact metric space Z onto $\tilde{X}$ such that for each $g \in G$ there is a (lift) homeomorphism $L_g : Z \to Z$ satisfying the following commutative diagram.*

$$\begin{array}{ccc} Z & \xrightarrow{L_g} & Z \\ p_G \downarrow & & \downarrow p_G \\ \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \end{array}$$

Then there is an isometrical embedding $i_G : X \to C(Z)$ such that for each $g \in G$, the following commutative diagram holds.

$$\begin{array}{ccc} X & \overset{g}{\to} & X \\ i_G \downarrow & & \downarrow i_G \\ C(Z) & \overset{\tilde{L_g}}{\to} & C(Z) \end{array}$$

where $\tilde{L_g} : C(Z) \to C(Z)$ is the isometry defined by $\tilde{L_g}(f) = f \circ L_g$ for $f \in C(Z)$. In particular, $\tilde{L_g} \in LinIso(C(Z))$ is an isometrical extension of $g \in G$.

Here we have the following theorem of $C(\mathbb{Q})$ which implies that $C(\mathbb{Q})$ is universal concerning isometrical extensions of bounded isometry groups of separable metric spaces.

Theorem 2.3. *Let (X, d) be a separable metric space and let G be any bounded subgroup of $Iso(X)$. Then there is an isometrical embedding $i_G : X \to C(\mathbb{Q})$ such that i_G induces a continuous monomorphism $i_G^\star : G \to LinIso(C(\mathbb{Q}))$ such that $i_G^\star(g) \in LinIso(C(\mathbb{Q}))$ is an extension of $g \in G$.*

Corollary 2.4. *Suppose that (X, d) is a bounded separable metric space. Then there is an isometrical embedding $i : X \to C(\mathbb{Q})$ such that i induces a continuous monomorphism $i^\star : Iso(X) \to LinIso(C(\mathbb{Q}))$ such that $i^\star(g) \in LinIso(C(\mathbb{Q}))$ is an extension of $g \in Iso(X)$.*

Remark 1. Note that for any Banach space B, $LinIso(B)$ is a bounded group. Hence in this note, we can not omit the condition that G is bounded.

If we observe the proof of Lemma 2.2, we see that some converse assertions of Lemma 2.2 are also true. In fact, we have the following.

Proposition 2.5. *Suppose that $p_G : Z \to \tilde{X}(= \tilde{X}_G)$ is a map from a compact metric space Z onto $\tilde{X}$, $i_G : X \to C(Z)$ is the isometrical embedding as in the proof of Lemma 2.2 and $g \in G$. Let $L_g : Z \to Z$ be a homeomorphism. Then the followings hold.*
(1) The following diagram is commutative:

$$\begin{array}{ccc} Z & \overset{L_g}{\to} & Z \\ p_G \downarrow & & \downarrow p_G \\ \tilde{X} & \overset{\tilde{g}}{\to} & \tilde{X} \end{array}$$

if and only if the following diagram is commutative:

$$\begin{array}{ccc} X & \overset{g}{\to} & X \\ i_G \downarrow & & \downarrow i_G \\ C(Z) & \overset{\tilde{L_g}}{\to} & C(Z) \end{array}$$

(2) The following diagram is commutative:

$$\begin{array}{ccc} Z & \overset{L_g}{\to} & Z \\ p_G \downarrow & & \downarrow p_G \\ \tilde{X} & \overset{-\tilde{g}}{\to} & \tilde{X} \end{array}$$

if and only if the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ i_G \downarrow & & \downarrow i_G \\ C(Z) & \xrightarrow{\tilde{L}_g} & C(Z) \end{array}$$

Example. Let $X = \{x_i |\ i = 0, 1, 2\}$ be the set of three elements and let d be the metric on X defined by $d(x_i, x_j) = r > 0\ (i \neq j)$. Define the isometry $g : X \to X$ by $g(x_0) = x_0, g(x_1) = x_2$ and $g(x_2) = x_1$. Let $G = \{id_X, g\}$. Note that $G(x_0) = \{x_0\}$. Then there is an isometrical embedding $i_G : X \to C(\mathbb{Q})$ such that there is no isometrical extension of g on $C(\mathbb{Q})$. In particular, $C(\mathbb{Q})$ is not equal to the Urysohn universal space $\mathbb{U}$, because that $\mathbb{U}$ has the following strong property: Any isometry between finite subsets of $\mathbb{U}$ can be extended to an isometry of $\mathbb{U}$.

Next we will consider the case of the function space $C(\Delta)$. Let $H(X)$ be the set of all homeomorphisms of a space X.

Proposition 2.6. *Let X be a compact metric space and let G be a countable subset of $H(X)$. Then there is an onto map $p_G : \Delta \to X$ such that for any $g \in G$ there is a (lift) homeomorphism $L_g : \Delta \to \Delta$ of Δ such that the following diagram is commutative.*

$$\begin{array}{ccc} \Delta & \xrightarrow{L_g} & \Delta \\ p_G \downarrow & & \downarrow p_G \\ X & \xrightarrow{g} & X \end{array}$$

Then we have the following theorem of $C(\Delta)$.

Theorem 2.7. *Let (X, d) be any separable metric space and let G be a countable bounded subgroup of $Iso(X)$. Then there is an isometrical embedding $i_G : X \to C(\Delta)$ such that there exist a countable subgroup $G^\star$ of $LinIso(C(\Delta))$ and a continuous epimorphism $r^\star : G^\star \to G$ such that each $g^\star \in G^\star$ is an extension of $r^\star(g^\star) \in G$. In particular, if $g \in G$, then there is an extension $g^\star \in LinIso(C(\Delta))$ of g.*

Remark 2. Note that the space $H(\Delta)$ of all homeomorphisms of Δ is homeomorphic to the space P of irrationals, and hence $H(\Delta)$ is zero-dimensional. If G is any bounded subgroup of $Iso(X)$ with $\dim G \geq 1$, there is no embedding from G to $H(\Delta)$.

Corollary 2.8. *Let (X, d) be any separable metric space. If $g \in Iso(X)$ is periodic i.e., $g^n = \mathrm{id}_X$ for some $n \in \mathbb{N}$, then there is an isometrical embedding $i_g : X \to C(\Delta)$ such that there is an extension $g^\star \in LinIso(C(\Delta))$ of g with $(g^\star)^n = \mathrm{id}_{C(\Delta)}$.*

Finally, we consider the case of $C(I)$. We have the following proposition of $C(I)$.

Proposition 2.9. *Let (X, d) be any separable metric space and let $g \in Iso(X)$ such that g has a periodic point x_0 with period $n \in \mathbb{N}$. If $n \geq 3$, there is no isometrical embedding i from X to $C(I)$ such that g has an extension in $LinIso(C(I))$.*

Now, we have the following problem.

Problem 2.10. *Let (X, d) be any separable metric space. Is it true that there is an isometrical embedding i from X to $C(\mathbb{Q})$ such that each $g \in Iso(X)$ has an extension which is an affine isometry of $C(\mathbb{Q})$?*

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