On equivariant homeomorphisms of boundaries of
CAT(0) groups and Coxeter groups

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1. INTRODUCTION

In this note, we introduce on equivariant homeomorphisms of boundaries of
CAT(0) groups (and Coxeter groups) and (boundary-)rigidity in [17].

A geometric action on a CAT(0) space is an action by isometries which is proper
and cocompact. We note that every CAT(0) space \( X \) on which some group \( G \)
acts geometrically is a proper space and we can consider its ideal boundary \( \partial X \)
(cf. [4], [11]). A group \( G \) is called a \textit{CAT(0) group}, if \( G \) acts geometrically on
some CAT(0) space \( X \).

It is well-known that if a Gromov hyperbolic group \( G \) acts geometrically on two
negatively curved spaces \( X \) and \( Y \), then the natural quasi-isometry \( \phi : Gx_0 \to Gy_0 \) extends
continuously to a \( G \)-equivariant homeomorphism \( \bar{\phi} : \partial X \to \partial Y \) of the boundaries of \( X \) and \( Y \) (cf. [4], [5], [11], [12], [13]).

M. Gromov [13] asked whether the boundaries of two CAT(0) spaces \( X \) and
\( Y \) are \( G \)-equivariant homeomorphic whenever a CAT(0) group \( G \) acts geometrically
on the two CAT(0) spaces \( X \) and \( Y \). P. L. Bowers and K. Ruane [3] have
constructed an example that the natural quasi-isometry \( Gx_0 \to Gy_0 \) \( (gx_0 \mapsto gy_0) \) does not extend continuously to any map between the boundaries \( \partial X \) and \( \partial Y \) of
\( X \) and \( Y \). Also, C. Croke and B. Kleiner [6] have constructed a CAT(0) group \( G \)
which acts geometrically on two CAT(0) spaces \( X \) and \( Y \) whose boundaries are
not homeomorphic, and J. Wilson [26] has proved that this CAT(0) group has
uncountably many boundaries.

In this note, we suppose that a CAT(0) group \( G \) acts geometrically on two
CAT(0) spaces \( X \) and \( Y \). Let \( x_0 \in X \) and \( y_0 \in Y \).

Then we consider the following question.
**Question.** When does the quasi-isometry \( \phi : Gx_0 \to Gy_0 \) \((gx_0 \mapsto gy_0)\) continuously extend to a \( G \)-equivariant homeomorphism \( \overline{\phi} : \partial X \to \partial Y \) of the boundaries?

\[
\begin{array}{c}
\sim \quad X \supset Gx_0 \leftrightarrow \partial X \\
G \quad \phi \downarrow \quad \downarrow \overline{\phi} ? \\
\sim \quad Y \supset Gy_0 \leftrightarrow \partial Y
\end{array}
\]

2. **Main theorems**

The following condition (*) comes from observing the Bowers-Ruane’s example.

(*) There exist constants \( N > 0 \) and \( M > 0 \) such that \( GB(x_0, N) = X \), \( GB(y_0, M) = Y \) and for any \( g, a \in G \), if \([x_0, gx_0] \cap B(ax_0, N) \neq \emptyset \) in \( X \) then \([y_0, gy_0] \cap B(ay_0, M) \neq \emptyset \) in \( Y \).

Then we obtain the following theorem.

**Theorem 1** ([17]). If the condition (*) holds, then the quasi-isometry \( \phi : Gx_0 \to Gy_0 \) \((gx_0 \mapsto gy_0)\) continuously extends to a \( G \)-equivariant homeomorphism \( \overline{\phi} : \partial X \to \partial Y \) of the boundaries.

We also consider the following condition (**).

(**) For any sequence \( \{g_i | i \in \mathbb{N}\} \subset G \), the sequence \( \{g_ix_0 | i \in \mathbb{N}\} \) is a Cauchy sequence in \( X \cup \partial X \) if and only if the sequence \( \{g_iy_0 | i \in \mathbb{N}\} \) is a Cauchy sequence in \( Y \cup \partial Y \).

Then we also obtain the following theorem.

**Theorem 2** ([17]). The condition (**) holds if and only if the quasi-isometry \( \phi : Gx_0 \to Gy_0 \) \((gx_0 \mapsto gy_0)\) continuously extends to a \( G \)-equivariant homeomorphism \( \overline{\phi} : \partial X \to \partial Y \) of the boundaries.
3. RIGIDITY OF BOUNDARIES

In this note, a CAT(0) group $G$ is said to be *(boundary-)*rigid, if $G$ determines its ideal boundary up to homeomorphisms, i.e., all boundaries of CAT(0) spaces on which $G$ acts geometrically are homeomorphic.

Also a CAT(0) group $G$ is said to be *equivariant (boundary) rigid*, if $G$ determines its ideal boundary by the equivariant homeomorphisms as above (i.e., if for any two CAT(0) spaces $X$ and $Y$ on which $G$ acts geometrically the quasi-isometry $\phi : Gx_0 \to Gy_0$ continuously extends to a $G$-equivariant homeomorphism $\overline{\phi} : \partial X \to \partial Y$ of the boundaries).

As an application of Theorem 1, we can obtain examples of equivariant rigid CAT(0) groups.

**Example ([17]).** Any group of the form

\[ \mathbb{Z}^{n_1} \ast \cdots \ast \mathbb{Z}^{n_k} \ast A_1 \ast \cdots \ast A_l \]

where $n_i \in \mathbb{N}$ and each $A_j$ is a finite group is an equivariant rigid CAT(0) group.

As an application of Theorem 2, we can also obtain examples of non equivariant rigid CAT(0) groups.

**Example ([17]).** Let $G = F_2 \times \mathbb{Z}$, where $F_2$ is the rank 2 free group generated by $\{a, b\}$. Let $T$ and $T'$ be the Cayley graphs of $F_2$ with respect to the generating set $\{a, b\}$ such that

1. in $T$, all edges $[g, ga]$ and $[g, gb]$ ($g \in F_2$) have the unit length, and
2. in $T'$, the length of $[g, ga]$ is 2 and the length of $[g, gb]$ is 1 for any $g \in F_2$.

Here we note that $F_2$ acts naturally and geometrically on $T$ and $T'$. 
Let $X = T \times \mathbb{R}$ and $Y = T' \times \mathbb{R}$.

We consider the natural actions of the group $G = F_2 \times \mathbb{Z}$ on the CAT(0) spaces $X$ and $Y$. Then the group $G$ acts geometrically on the two CAT(0) spaces $X$ and $Y$, and the quasi-isometry $gx_0 \mapsto gy_0$ (where $x_0 = (1, 0) \in X$ and $y_0 = (1, 0) \in Y$) does not extend continuously to any map from $\partial X$ to $\partial Y$.

Indeed, we can consider the sequence $\{g_n \mid n \in \mathbb{N}\} \subset F_2$ such that $g_1 = ab$ and

$$g_n = \begin{cases} g_{n-1}a^{2^{n-1}} & \text{if } n \text{ is even} \\ g_{n-1}b^{2^{n-1}} & \text{if } n \text{ is odd} \end{cases}$$

for $n \geq 2$. Here we note that the length of the words of $g_n$ in $F_2$ is $2^n$.

Let $\overline{g}_n = (g_n, 2^n) \in F_2 \times \mathbb{Z}$ for $n \in \mathbb{N}$. Then $\{\overline{g}_n x_0\}$ is a Cauchy sequence in $X \cup \partial X$. On the other hand, $\{\overline{g}_n y_0\}$ is not a Cauchy sequence in $Y \cup \partial Y$ (see Figure 1).

Hence, the quasi-isometry $\phi : GX_0 \to GY_0$ ($gx_0 \mapsto gy_0$) does not continuously extend to any map $\overline{\phi} : \partial X \to \partial Y$ of the boundaries.

**Remark** ([17]).

- $G = F_2 \times \mathbb{Z}$ is a non equivariant rigid CAT(0) group.
- $G = F_2 \times \mathbb{Z}$ is a rigid CAT(0) group whose boundary is the suspension of the Cantor set.
• By the same idea, every CAT(0) group of the form $G = F \times H$ where $F$ is a free group of rank $n \geq 2$ and $H$ is an infinite CAT(0) group, is non equivariant rigid.

4. Coxeter groups acting CAT(0) spaces as reflection groups

A Coxeter group $W$ is said to be equivariant rigid as a reflection group, if for any two CAT(0) spaces $X$ and $Y$ on which $W$ acts geometrically as reflection groups, the quasi-isometry $\phi : Wx_0 \to Wy_0$ ($wx_0 \mapsto wy_0$) where $x_0 \in X$ and $y_0 \in Y$ continuously extends to a $W$-equivariant homeomorphism $\overline{\phi} : \partial X \to \partial Y$ of the boundaries.

Theorem 3 ([17]). The following statements hold.

(i) If Coxeter groups $W_1$ and $W_2$ are equivariant rigid as reflection groups, then so is $W_1 \ast W_2$.

(ii) For a Coxeter group $W = W_A \ast_{W_{A\cap B}} W_B$ where $W_{A\cap B}$ is finite, if $W$ determines its Coxeter system up to isomorphism, and if $W_A$ and $W_B$ are equivariant rigid as reflection groups then so is $W$, where $W_T$ is the parabolic subgroup of $W$ generated by $T$.

Corollary 4 ([17]). Any group of the form

$$W = W_1 \ast \cdots \ast W_n$$

where each $W_i$ is a Gromov hyperbolic Coxeter group, an affine Coxeter group or a finite Coxeter group, is an equivariant rigid as a reflection group.

Corollary 5 ([17]). Any Coxeter group of the form

$$W = (\cdots (W_{A_1} \ast_{W_{B_1}} W_{A_2}) \ast_{W_{B_2}} W_{A_3}) \ast \cdots ) \ast_{W_{B_{n-1}}} W_{A_n}$$
where each $W_{A_{i}}$ is a Gromov hyperbolic Coxeter group, an affine Coxeter group or a finite Coxeter group, each $W_{B_{i}}$ is finite and $W$ determines its Coxeter system up to isomorphism, is an equivariant rigid as a reflection group.

**Example.** The Coxeter groups defined by the following diagrams are equivariant rigid as reflection groups.

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5. **Conjecture**

Now we introduce a conjecture.

**Conjecture ([17]).** The group $G = (F_{2} \times \mathbb{Z}) * \mathbb{Z}_{2}$ will be a non-rigid CAT(0) group with uncountably many boundaries.

For $p \geq q \geq 1$, let $T_{p,q}$ be the Cayley graph of the free group $F_{2}$ with the generating set $\{a,b\}$ such that

- the length of $[g,ga]$ is $p$ and the length of $[g,gb]$ is $q$ for any $g \in F$.

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$T_{p,q}$
Then $F_2 \times \mathbb{Z}$ acts naturally on $T_{p,q} \times \mathbb{R}$. We can construct a cuboidal cell complex $\Sigma_{p,q}$ on which $G = (F_2 \times \mathbb{Z}) \ast \mathbb{Z}_2$ acts geometrically, where the 1-skeleton of $\Sigma_{p,q}$ is the Cayley graph of $G$ and $T_{p,q} \subset \Sigma_{p,q}^{(1)}$.

Then, the author thinks that if $\frac{p}{q} \neq \frac{p'}{q'}$ then the boundaries $\partial \Sigma_{p,q}$ and $\partial \Sigma_{p',q'}$ will be not homeomorphic.

6. ON RIGIDITY

Finally, we introduce problems of rigidity in group actions.

Let $G$ and $H$ be groups acting geometrically (i.e. properly and cocompactly by isometries) on metric spaces $(X, d_X)$ and $(Y, d_Y)$ respectively. We consider orbits $G x_0 \subset X$ and $H y_0 \subset Y$ where $x_0 \in X$ and $y_0 \in Y$.

Let $\phi : G \to H$ be a map and let $\phi' : G x_0 \to H y_0$ ($g x_0 \mapsto \phi(g) y_0$).

Here if $X$ and $Y$ are Gromov hyperbolic spaces, CAT(0) spaces or Busemann spaces, then we can define the boundaries $\partial X$ and $\partial Y$.

Then it is well-known that if $\phi : G \to H$ is an isomorphism then $\phi' : G x_0 \to H y_0$ is a quasi-isometry and moreover if $G$ is Gromov hyperbolic then $\phi'$ induces an equivariant homeomorphism $\tilde{\phi} : \partial X \to \partial Y$.

Theorem 2 implies that if $\phi : G \to H$ is an isomorphism and the map $\phi' : G x_0 \to H y_0$ satisfies the condition (**) then $\phi'$ induces an equivariant homeomorphism $\tilde{\phi} : \partial X \to \partial Y$. 
Then there are problems of rigidity.

(I) If $\phi : G \to H$ is an isomorphism then when does there exist an homeomorphism $\overline{\phi} : \partial X \to \partial Y$?

(II) If $\phi : G \to H$ is an isomorphism then when does $\phi'$ induce an equivariant homeomorphism $\overline{\phi} : \partial X \to \partial Y$?

(III) If $X = Y$ and $Gx_0 = Hx_0$ then when are groups $G$ and $H$ virtually isomorphic (i.e. there exist finite-index subgroups $G'$ and $H'$ of $G$ and $H$ respectively such that $G'$ and $H'$ are isomorphic)?

(IV) If $X = Y$ and $Gx_0 = Hx_0$ then when do there exist finite-index subgroups $G'$ and $H'$ of $G$ and $H$ respectively such that $G'$ and $H'$ are conjugate in the isometry group $\text{Isom}(X)$ of $X$?

(V) If there is an isomorphism $\phi : G \to H$ then when does there exist a homeomorphism (or homotopy equivalence) $\psi : X/G \to Y/H$?

Here it seems that (III)–(V) are relate to [1], [8], [9], [14], [18], [19], [20], [22] and [23].

REFERENCES


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