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ISOMETRIES ON THE SYMMETRIC PRODUCTS OF THE EUCLIDEAN SPACES WITH USUAL METRICS (The present situation of set-theoretic and geometric topology and its prospects)

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ISOMETRIES ON THE SYMMETRIC PRODUCTS OF THE EUCLIDEAN SPACES WITH USUAL METRICS

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1. INTRODUCTION

As an interesting construction in topology, Borsuk and Ulam [3] introduced the \( n \)-th symmetric product of a metric space \((X, d)\), denoted by \( F_n(X) \). Namely \( F_n(X) \) is the space of non-empty finite subsets of \( X \) with at most \( n \) elements endowed with the Hausdorff metric \( d_H \), i.e., \( F_n(X) = \{ A \subset X \mid 1 \leq |A| \leq n \} \) and \( d_H(A, B) = \inf \{ \epsilon \mid A \subset B_{d}(B, \epsilon) \text{ and } B \subset B_{d}(A, \epsilon) \} = \max \{ d(a, B), d(b, A) \mid a \in A, b \in B \} \) for any \( A, B \in F_n(X) \) (see [10, p.6]).

For the symmetric products of \( \mathbb{R} \), it is known that \( F_2(\mathbb{R}) \approx \mathbb{R} \times [0, \infty) \) and \( F_3(\mathbb{R}) \approx \mathbb{R}^3 \) (see Section 3). It was proved in [3] that \( F_n(\mathbb{I}) \) is homeomorphic to \( \mathbb{I}^n \) (written \( F_n(\mathbb{I}) \approx \mathbb{I}^n \)) if and only if \( 1 \leq n \leq 3 \), and that for \( n \geq 4 \), \( F_n(\mathbb{I}) \) cannot be embedded into \( \mathbb{R}^n \), where \( \mathbb{I} = [0, 1] \) has the usual metric. Thus, for \( n \geq 4 \), \( F_n(\mathbb{R}) \not\approx \mathbb{R}^n \). Molski [12] showed that \( F_2(\mathbb{I}^2) \approx \mathbb{I}^4 \), and that for \( n \geq 3 \) neither \( F_n(\mathbb{I}^2) \) nor \( F_2(\mathbb{I}^n) \) can be embedded into \( \mathbb{R}^{2n} \). Thus, for \( n \geq 3 \), \( F_n(\mathbb{R}^2) \not\approx \mathbb{R}^{2n} \) and \( F_2(\mathbb{R}^n) \not\approx \mathbb{R}^{2n} \).

Turning toward the symmetric product \( F_n(S^1) \) of the circle \( S^1 \), Chinen and Koyama [9] prove that for \( n \in \mathbb{N} \), both \( F_{2n-1}(S^1) \) and \( F_{2n}(S^1) \) have the same homotopy type of the \((2n-1)\)-sphere \( S^{2n-1} \). In [7] Bott corrected Borsuk’s statement [4] and showed that \( F_3(S^1) \approx S^3 \). In [9], another proof of it is given.

For a metric space \((X, d)\), we denote by Isom\(_d\)(\(X\)) (Isom\((X)\) for short) the group of all isometries from \( X \) into itself, i.e., \( \phi : X \to X \in \text{Isom}_d(X) \) if \( \phi \) is a bijection satisfying that \( d(x, x') = d(\phi(x), \phi(x')) \) for any \( x, x' \in X \). Let \( n \in \mathbb{N} \). Every isometry \( \phi : X \to X \) induces an isometry \( \chi_{(n)}(\phi) : (F_n(X), d_H) \to (F_n(X), d_H) \) defined by \( \chi_{(n)}(\phi)(A) = \phi(A) \) for each \( A \in F_n(X) \). Thus, there exists a natural monomorphism \( \chi_{(n)} : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X)) \). It is clear that \( \chi_{(n)} : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X)) \) is an isomorphism if and only if \( \chi_{(n)} \) is an epimorphism, i.e., for every \( \Phi \in \text{Isom}_{d_H}(F_n(X)) \) there exists \( \phi \in \text{Isom}_d(X) \) such that \( \Phi = \chi_{(n)}(\phi) \).

In this paper, it is of interest to know whether \( \chi_{(n)} : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X)) \) is an isomorphism for a metric space \((X, d)\). Recently, Borovikova and Ibragimov [5] prove that \( (F_3(\mathbb{R}), d_H) \) is bi-Lipschitz equivalent to \( (\mathbb{R}^3, d) \) and that \( \chi_{(3)} : \text{Isom}_d(\mathbb{R}) \to \text{Isom}_{d_H}(F_3(\mathbb{R})) \) is an isomorphism, where \( \mathbb{R} \) has the usual metric \( d \). The following result is a generalization of the result above and the affirmative answer to [6, p.60, Conjecture 2.1].
Theorem 1.1. Let \( l \in \mathbb{N} \) and let \( X = \mathbb{R}^l \) or \( X = S^l \) with the usual metric \( d \). Then \( \chi(n) : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X)) \) is an isomorphism for each \( n \in \mathbb{N} \).

In Section 4, we give the main ideas of proof of Theorem 1.1. In Example 5.2 below, we present a compact metric space \((X, d)\) such that \( \chi(n)(\text{Isom}_d(X)) \neq \text{Isom}_{d_H}(F_n(X)) \) for all \( n \geq 2 \), i.e., \( \chi(n) : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X)) \) is not an isomorphism. And, in Section 3, we provide another proof of [5, Theorem 6]. Its proof is based on the proof of [11, Lemma 2.4].

2. Preliminaries

Notation 2.1. Let denote the set of all natural numbers and real numbers by \( \mathbb{N} \) and \( \mathbb{R} \), respectively. Let \( d \) be the usual metric on \( \mathbb{R}^l \), i.e., \( d(x, y) = \left\{ \sum_{i=1}^{l}(x_i - y_i)^2 \right\}^{1/2} \) for any \( x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in \mathbb{R}^l \). Write \( \mathbb{S}^{l} = \{x = (x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1} | \sum_{i=1}^{l+1}x_i^{2} = 1\} \) with the length metric \( d \). Denote the identity map from \( X \) into itself by \( \text{id}_X \).

Definition 2.2. Let \((X, d)\) be a metric space, let \( x \in X \), let \( Y, Z \) be subsets of \( X \) and let \( \epsilon > 0 \). Set \( d(Y, Z) = \inf \{d(y, x) | y \in Y, z \in Z\} \), and \( B_d(Y, \epsilon) = \{x \in X | d(x, Y) \leq \epsilon\} \). If \( Y = \{y\} \), for simplicity of notation, we write \( B_d(y, \epsilon) = B_d(Y, \epsilon) \) and \( S_d(y, \epsilon) = S_d(Y, \epsilon) \).

For \( n \in \mathbb{N} \), the \( n \)-th symmetric product of \( X \) is defined by
\[
F_n(X) = \{A \subset X \mid 1 \leq |A| \leq n\},
\]
where \( |A| \) is the cardinality of \( A \). Write \( F_{(m)}(X) = \{A \in 2^X \mid |A| = m\} \) for each \( m \in \mathbb{N} \). Let \( \text{Isom}(X, Y) = \{\phi \in \text{Isom}(X) \mid \phi(y) = y \text{ for each } y \in Y\} \) for \( Y \subset X \). Set \( r(A) = \min\{\{1\} \cup \{d(a, a') \mid a, a' \in A, a \neq a'\}\} \) for each \( A \in F_n(X) \).

3. A metric space is bi-Lipschitz equivalent to the symmetric product of \( \mathbb{R} \)

In this section, we give another proof of [5, Theorem 6] which is based on the proof of [11, Lemma 2.4].

Definition 3.1. Let \( n \in \mathbb{N} \). Set \( F^*_n(\mathbb{I}) = \{A \in F_n(\mathbb{I}) \mid 0, 1 \in A\} \). It is known that \( F^*_2(\mathbb{I}) = \{\{0, 1\}\}, F^*_3(\mathbb{I}) = \{\{0, t, 1\} \mid 0 \leq t \leq 1\} \approx S^1 \), and, \( F^*_4(\mathbb{I}) = \{\{0, s, t, 1\} \mid 0 \leq s \leq t \leq 1\} \) is homeomorphic to the dance hat (see [16]). In general, \( F^*_2(\mathbb{I}) \) is contractible but not collapsible, and \( F^*_2(\mathbb{I}) \) has the same homotopy type of \( S^{2n+1} \). In [1], it is called the spaces \( F^*_2(\mathbb{I}), n \geq 2 \), higher dimensional dunce hats (see [1]).
Definition 3.2 ([11]). Let $(X, d)$ be a metric space with $\text{diam } X \leq 2$. Set $\text{Cone}^o(X) = X \times [0, \infty)/(X \times \{0\})$, is said to be the open cone over $X$, with the metric $d_C([(x_1, t_1)], [(x_2, t_2)]) = |t_1 - t_2| + \min\{t_1, t_2\} \cdot d(x_1, x_2)$.

Definition 3.3. Let $f : (X, d) \to (Y, d')$ be a map. The map $f$ is said to be Lipschitz (bi-Lipschitz, respectively) if there exists $L > 0$ such that
\[
    d'(f(x_1), f(x_2)) \leq L d(x_1, x_2)
\]
and
\[
    (L^{-1} d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq L d(x_1, x_2), \text{ respectively})
\]
for any $x_1, x_2 \in X$. $(X, d)$ is said to be bi-Lipschitz equivalent to $(Y, d')$ if there exists a surjective bi-Lipschitz map from $(X, d)$ to $(Y, d')$.

Theorem 3.4 ([11]). Let $n \in \mathbb{N}$ with $n \geq 2$. Then $(F_n(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times \text{Cone}^o(F_n^*(\mathbb{I})), \rho)$, where $\rho = \sqrt{d^2 + (d_{H})_{C}^2}$.

Sketch of Proof. Let $Z = \{A \in F_n(\mathbb{R}) \mid \min A = 0\}$. For every $A \in Z$ there exists the unique $E \in F_n^*(\mathbb{I})$ such that $A = tE$, where $t = \max A$.

Step1: $(F_n(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times Z, \rho_1)$, where $\rho_1 = \sqrt{\rho^2 + (d_{H})_{C}^2}$. In fact, we can show the following.

Step1.1: A map $f : F_n(\mathbb{R}) \to \mathbb{R} \times Z : A \mapsto (\min A, A - \min A)$ is $\sqrt{5}$-Lipschitz.

Step1.2: A map $f^{-1} : \mathbb{R} \times Z \to F_n(\mathbb{R}) : (b, A) \mapsto A + b$ is 2-Lipschitz.

Step2: $(Z, d_H)$ is bi-Lipschitz equivalent to $(\text{Cone}^o(F_n^*(\mathbb{I})), (d_{H})_{C})$. In fact, we can show the following.

Step2.1: A map $g : Z \to \text{Cone}^o(F_n^*(\mathbb{I})) : tE \mapsto [(E, t)]$ is 1-Lipschitz.

Step2.2: A map $g^{-1} : \text{Cone}^o(F_n^*(\mathbb{I})) \to Z : [(E, t)] \mapsto tE$ is 3-Lipschitz.

By the above, $(\text{id}_\mathbb{R} \times g) \circ f : F_n(\mathbb{R}) \to \mathbb{R} \times Z \to \mathbb{R} \times \text{Cone}^o(F_n^*(\mathbb{I}))$ is a bi-Lipschitz equivalence.

Corollary 3.5. $(F_2(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times [0, \infty), d)$.

Proof. By Definition 3.1, $F_2(\mathbb{I})$ is one point, thus $(\text{Cone}^o(F_2^*(\mathbb{I})), (d_{H})_{C})$ is corresponding to $([0, \infty), d)$. By Theorem 3.4, $(F_2(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R} \times [0, \infty), d)$.

The following result is first proved in [5, Theorem 6]. We give another proof by use of Theorem 3.4.

Corollary 3.6 ([5]). $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$.

Sketch of Proof. We note $F_3^*(\mathbb{I}) = \{\{0, t, 1\} \mid 0 \leq t \leq 1\} \approx \mathbb{S}^1$.

Step1: We can show that $(\text{Cone}^o(F_3^*(\mathbb{I})), (d_{H})_{C})$ is bi-Lipschitz equivalent to $(\text{Cone}^o(\mathbb{S}^1), (d_{|\mathbb{S}^1|})_{C})$.

Step2: We can show that $(\mathbb{R}^2, d)$ is bi-Lipschitz equivalent to $(\text{Cone}^o(\mathbb{S}^1), (d_{|\mathbb{S}^1|})_{C})$.

By Theorem 3.4, $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$. □
Remark 3.7. We note that $F_2(\mathbb{R}^2) \approx \mathbb{R}^4$. Indeed, we can define a homeomorphism $h: F_2(\mathbb{R}^2) \to \mathbb{R}^2 \times \text{Cone}^o(S^1/ x \sim -x) (\approx \mathbb{R}^4)$ by

$$h(A) = \begin{cases} (m(A), \left[\frac{2(A-m(A))}{\text{diam} A}, \text{diam} A\right]) & \text{if diam } A \neq 0, \\ (m(A), \text{the cone point}) & \text{if diam } A = 0, \end{cases}$$

where $m(A) = a$ if $A = \{a\}$ and $m(A) = (a + a')/2$ if $A = \{a, a'\}$. In general, we see that $F_2(\mathbb{R}^l) \approx \mathbb{R}^l \times \text{Cone}^o(S^{l-1}/ x \sim -x)$ for each $l \in \mathbb{N}$.

4. ISOMETRIES

Lemma 4.1. Let $n \in \mathbb{N}$ and let $(X, d)$ be a metric space such that

1. $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$ for each $\Phi \in \text{Isom}(F_n(X))$, and that
2. $\text{Isom}(F_n(X), F_1(X)) = \{\text{id}_{F_n(X)}\}$.

Then, $\chi(n): \text{Isom}(X) \to \text{Isom}(F_n(X))$ is an isomorphism.

Proof. Let $\Phi \in \text{Isom}(F_n(X))$ and let $A_x = \{x\} \in F_1(X)$ for each $x \in X$. By assumption, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. Denote $\Phi(A_x) \in F_1(X)$ by $\{\phi(x)\}$ for each $x \in X$. Then, $\phi: X \to X : x \mapsto \phi(x)$ is an isometry. Set $\Phi' = \chi(n)(\phi^{-1}) \circ \Phi \in \text{Isom}(F_n(X))$. We claim that $\Phi'|_{F_1(X)} = \text{id}|_{F_1(X)}$. Indeed, $\Phi|_{F_1(X)} = \chi(n)(\phi)|_{F_1(X)}$ and $\chi(n)(\phi^{-1}) = (\chi(n)(\phi))^{-1}$. By assumption, we have that $\Phi' = \text{id}_{F_n(X)}$, therefore, $\Phi = \chi(n)(\phi)$, which completes the proof.  

Definition 4.2. Let $(X, d)$ be a metric space, let $n \in \mathbb{N}$, let $\epsilon > 0$ and let $A \in F_n(X)$. Define

$$D_n(A, \epsilon) = \sup\{k \in \mathbb{N} | A_1, \ldots, A_k \in S_{d_H}(A, \epsilon), d_H(A_i, A_j) = 2\epsilon (i \neq j)\} \in \mathbb{N} \cup \{\infty\}.$$ 

Lemma 4.3. Let $l, n \in \mathbb{N}$, let $X = \mathbb{R}^l$ or $X = S^l$ and let $\Phi \in \text{Isom}(F_n(X))$. Then, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$.

Sketch of Proof. Let $n \in \mathbb{N}$ with $n \geq 2$.

Step1: Let $A = \{a_1\} \in F_1(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)$. We can show that $D_n(A, \epsilon) = 3$.

Step2: Let $m \in \mathbb{N}$ with $m \geq 2$, let $A = \{a_1, \ldots, a_m\} \in F_m(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)/5$. We can show that $D_n(A, \epsilon) > 3$.

Let $\Phi \in \text{Isom}(F_n(X))$ and let $A \in F_n(X)$. From the definition of $D_n(A, \epsilon)$, we obtain $D_n(A, \epsilon) = D_n(\Phi(A), \epsilon)$ for each $0 < \epsilon < \min\{r(A), r(\Phi(A))\}$. By the above, we see that $A \in F_1(X)$ if and only if $\Phi(A) \in F_1(X)$. Therefore, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. 

Lemma 4.4. Let $l, n \in \mathbb{N}$. Then, $\text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) = \{\text{id}_{F_n(\mathbb{R}^l)}\}$. 

Sketch of Proof.
Step1: Let $l,n \in \mathbb{N}$ and let $\Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$. Then, $\Phi|_{F_2(\mathbb{R}^l)} = \text{id}_{F_2(\mathbb{R}^l)}$.
Step2: Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$ and let $A \in F_{(m)}(\mathbb{R}^l)$. We can show that $\Phi(A) \subset A$. If similar arguments apply to $\Phi^{-1}(A) \subset \Phi^{-1}(A)$, we obtain $A = \Phi^{-1}(\Phi(A)) \subset \Phi(A)$, therefore, $A = \Phi(A)$.

Lemma 4.5. Let $l,n \in \mathbb{N}$. Then $\text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) = \{\text{id}_{F_n(\mathbb{S}^l)}\}$.

Proof. Let $\Phi \in \text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l))$, $m \in \mathbb{N}$ with $2 \leq m \leq n$ and let $A \in F_{(m)}(\mathbb{S}^l)$. We show that $A = \Phi(A)$. Let $a \in A$ and let $a' \in \mathbb{S}^l$ be the anti-point of $a$. Since $d_{H}(\{a'\}, \Phi(A)) = d_{H}(\Phi(\{a'\}), \Phi(A)) = d_{H}(\{a'\}, A) = \pi$, we have $a \in \Phi(A)$, therefore, $A \subset \Phi(A)$. If similar arguments apply to $\Phi(A)$ and $\Phi^{-1}$, we obtain $\Phi(A) \subset \Phi^{-1}(\Phi(A)) = A$, therefore, $A = \Phi(A)$, which completes the proof.

The proof of Theorem 1.1. By Lemmas 4.3, 4.4 and 4.5, the conditions in Lemma 4.1 hold for $(X,d)$, which completes the proof.

5. Questions

Question 5.1. Let $l,n \in \mathbb{N}$ with $n \geq 2$. When $(X,d)$ is a following space, is $\chi_n : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ an isomorphism?

1. $X = \mathbb{R}^l$ has a metric $d_{\infty}$, where $d_{\infty}(x,y) = \max\{|x_i - y_i| | i = 1, \ldots, l\}$ for any $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in X$.
2. $X$ is a convex subset of $\mathbb{R}^l$.
3. $X$ is an $\mathbb{R}$-tree (see [2] for $\mathbb{R}$-trees).
4. $X$ is the hyperbolic $l$-space (see [8] for the hyperbolic $l$-space).

Example 5.2. Let $n,m \in \mathbb{N}$ with $2 \leq n \leq m$ and let $(X,d)$ be an $m$-points discrete metric space satisfying that $d(x,x') = 1$ whenever $x \neq x'$. Then, $F_n(X)$ is a discrete metric space such that $d_{H}(A,A') = 1$ for any $A,A' \in F_n(X)$ with $A \neq A'$. Thus, $|\text{Isom}(X)| = |X|! < |F_n(X)|! = |\text{Isom}(F_n(X))|$, therefore, $\chi(n) : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ is not an isomorphism.

This drives us to the following question as the generalization of Theorem 1.1.

Question 5.3. Let $(X,d)$ be a connected metric space. Then, is $\chi_n : \text{Isom}(X) \rightarrow \text{Isom}(F_n(X))$ an isomorphism?

Question 5.4. It is known that $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$. Is $F_3(\mathbb{S}^1)$ bi-Lipschitz equivalent to $\mathbb{S}^3$?
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