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<tr>
<td>Issue Date</td>
<td>2014-04</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195696">http://hdl.handle.net/2433/195696</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
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Kyoto University
ISOMETRIES ON THE SYMMETRIC PRODUCTS OF THE EUCLIDEAN SPACES WITH USUAL METRICS

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1. INTRODUCTION

As an interesting construction in topology, Borsuk and Ulam [3] introduced the n-th symmetric product of a metric space $(X, d)$, denoted by $F_n(X)$. Namely $F_n(X)$ is the space of non-empty finite subsets of $X$ with at most $n$ elements endowed with the Hausdorff metric $d_H$, i.e., $F_n(X) = \{ A \subset X \mid 1 \leq |A| \leq n \}$ and $d_H(A, B) = \inf \{ \epsilon \mid A \subset B_d(B, \epsilon) \text{ and } B \subset B_d(A, \epsilon) \} = \max \{ d(a, B), d(b, A) \mid a \in A, b \in B \}$ for any $A, B \in F_n(X)$ (see [10, p.6]).

For the symmetric products of $\mathbb{R}$, it is known that $F_2(\mathbb{R}) \approx \mathbb{R} \times [0, \infty)$ and $F_3(\mathbb{R}) \approx \mathbb{R}^3$ (see Section 3). It was proved in [3] that $F_n(I)$ is homeomorphic to $I^n$ (written $F_n(I) \approx I^n$) if and only if $1 \leq n \leq 3$, and that for $n \geq 4$, $F_n(I)$ cannot be embedded into $\mathbb{R}^n$, where $I = [0, 1]$ has the usual metric. Thus, for $n \geq 4$, $F_n(\mathbb{R}) \not\approx \mathbb{R}^n$. Molski [12] showed that $F_2(I^2) \approx I^4$, and that for $n \geq 3$ neither $F_n(I^2)$ nor $F_2(I^n)$ can be embedded into $\mathbb{R}^{2n}$. Thus, for $n \geq 3$, $F_n(\mathbb{R}^2) \not\approx \mathbb{R}^{2n}$ and $F_2(\mathbb{R}^n) \not\approx \mathbb{R}^{2n}$.

Turning toward the symmetric product $F_n(S^1)$ of the circle $S^1$, Chinen and Koyama [9] prove that for $n \in \mathbb{N}$, both $F_{2n-1}(S^1)$ and $F_{2n}(S^1)$ have the same homotopy type of the $(2n - 1)$-sphere $S^{2n-1}$. In [7] Bott corrected Borsuk’s statement [4] and showed that $F_3(S^1) \approx S^3$. In [9], another proof of it is given.

For a metric space $(X, d)$, we denote by $\text{Isom}_d(X)$ (Isom$(X)$ for short) the group of all isometries from $X$ into itself, i.e., $\phi : X \rightarrow X \in \text{Isom}_d(X)$ if $\phi$ is a bijection satisfying that $d(x, x') = d(\phi(x), \phi(x'))$ for any $x, x' \in X$. Let $n \in \mathbb{N}$. Every isometry $\phi : X \rightarrow X$ induces an isometry $\chi_{(n)}(\phi) : (F_n(X), d_H) \rightarrow (F_n(X), d_H)$ defined by $\chi_{(n)}(\phi)(A) = \phi(A)$ for each $A \in F_n(X)$. Thus, there exists a natural monomorphism $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_d(F_n(X))$. It is clear that $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_d(F_n(X))$ is an isomorphism if and only if $\chi_{(n)}$ is an epimorphism, i.e., for every $\Phi \in \text{Isom}_d(F_n(X))$ there exists $\phi \in \text{Isom}_d(X)$ such that $\Phi = \chi_{(n)}(\phi)$.

In this paper, it is of interest to know whether $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_d(F_n(X))$ is an isomorphism for a metric space $(X, d)$. Recently, Borovikova and Ibragimov [5] prove that $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$ and that $\chi(3) : \text{Isom}_d(\mathbb{R}) \rightarrow \text{Isom}_d(F_3(\mathbb{R}))$ is an isomorphism, where $\mathbb{R}$ has the usual metric $d$. The following result is a generalization of the result above and the affirmative answer to [6, p.60, Conjecture 2.1].
Theorem 1.1. Let $l \in \mathbb{N}$ and let $X = \mathbb{R}^l$ or $X = S^l$ with the usual metric $d$. Then $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_{H}}(F_n(X))$ is an isomorphism for each $n \in \mathbb{N}$.

In Section 4, we give the main ideas of proof of Theorem 1.1. In Example 5.2 below, we present a compact metric space $(X, d)$ such that $\chi_{(n)}(\text{Isom}_d(X)) \neq \text{Isom}_{d_{H}}(F_{n}(X))$ for all $n \geq 2$, i.e., $\chi_{(n)} : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_{H}}(F_n(X))$ is not an isomorphism. And, in Section 3, we provide another proof of [5, Theorem 6]. Its proof is based on the proof of [11, Lemma 2.4].

2. Preliminaries

Notation 2.1. Let denote the set of all natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let $d$ be the usual metric on $\mathbb{R}^l$, i.e., $d(x, y) = \sqrt{\sum_{i=1}^{l}(x_i - y_i)^2}$ for any $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in \mathbb{R}^l$. Write $S^l = \{x = (x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1} | \sum_{i=1}^{l+1}x_i^2 = 1\}$ with the length metric $d$. Denote the identity map from $X$ into itself by $id_X$.

Definition 2.2. Let $(X, d)$ be a metric space, let $x \in X$, let $Y, Z$ be subsets of $X$ and let $\epsilon > 0$. Set $d(Y, Z) = \inf\{d(y, x) | y \in Y, z \in Z\}$, and $B_d(Y, \epsilon) = \{x \in X | d(x, Y) \leq \epsilon\}$. If $Y = \{y\}$, for simplicity of notation, we write $B_d(y, \epsilon) = B_d(Y, \epsilon)$ and $S_d(y, \epsilon) = S_d(Y, \epsilon)$.

For $n \in \mathbb{N}$, the $n$-th symmetric product of $X$ is defined by

$$F_n(X) = \{A \subset X | 1 \leq |A| \leq n\},$$

where $|A|$ is the cardinality of $A$. Write $F_{(m)}(X) = \{A \in 2^X | |A| = m\}$ for each $m \in \mathbb{N}$. Let $\text{Isom}(X, Y) = \{\phi \in \text{Isom}(X) | \phi(y) = y \text{ for each } y \in Y\}$ for $Y \subset X$. Set $r(A) = \min\{\{1\} \cup \{d(a, a') | a, a' \in A, a \neq a'\}\}$ for each $A \in F_n(X)$.

3. A metric space is bi-Lipschitz equivalent to the symmetric product of $\mathbb{R}$

In this section, we give another proof of [5, Theorem 6] which is based on the proof of [11, Lemma 2.4].

Definition 3.1. Let $n \in \mathbb{N}$. Set $F_n^*(\mathbb{I}) = \{A \in F_n(\mathbb{I}) | 0, 1 \in A\}$. It is known that $F_2^*(\mathbb{I}) = \{(0, 1)\}$, $F_3^*(\mathbb{I}) = \{(0, t, 1) | 0 \leq t \leq 1\} \approx S^1$, and, $F_4^*(\mathbb{I}) = \{(0, s, t, 1) | 0 \leq s \leq t \leq 1\}$ is homeomorphic to the dance hat (see [16]). In general, $F_{2n}(\mathbb{I})$ is contractible but not collapsible, and $F_{2n+1}^*(\mathbb{I})$ has the same homotopy type of $S^{2n+1}$. In [1], it is called the spaces $F_{2n}^*(\mathbb{I})$, $n \geq 2$, higher dimensional dunce hats (see [1]).
Definition 3.2 ([11]). Let \((X, d)\) be a metric space with \(\text{diam} X \leq 2\). Set \(\text{Cone}'(X) = X \times [0, \infty)/(X \times \{0\})\), is said to be the \textit{open cone over} \(X\), with the metric \(d_C([x_1, t_1]), ([x_2, t_2]) = |t_1 - t_2| + \min(t_1, t_2) \cdot d(x_1, x_2)\).

Definition 3.3. Let \(f : (X, d) \rightarrow (Y, d')\) be a map. The map \(f\) is said to be \textit{Lipschitz} (bi-Lipschitz, respectively) if there exists \(L > 0\) such that
\[
d'(f(x_1), f(x_2)) \leq L \, d(x_1, x_2)
\]
\[(L^{-1} \, d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq L \, d(x_1, x_2), \text{ respectively})
\]
for any \(x_1, x_2 \in X\). \((X, d)\) is said to be \textit{bi-Lipschitz equivalent} to \((Y, d')\) if there exists a surjective bi-Lipschitz map from \((X, d)\) to \((Y, d')\).

Theorem 3.4 ([11]). Let \(n \in \mathbb{N}\) with \(n \geq 2\). Then \((F_n(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times \text{Cone}'(F_n^*(\mathbb{I})), \rho)\), where \(\rho = \sqrt{d^2 + (d_H)^2}\).

\textbf{Sketch of Proof.} Let \(Z = \{A \in F_n(\mathbb{R}) \mid \min A = 0\}\). For every \(A \in Z\) there exists the unique \(E \in F_n^*(\mathbb{I})\) such that \(A = tE\), where \(t = \max A\).

Step1: \((F_n(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times Z, \rho_1)\), where \(\rho_1 = \sqrt{d^2 + (d_H)^2}\). In fact, we can show the following.

- Step1.1: A map \(f : F_n(\mathbb{R}) \rightarrow \mathbb{R} \times Z : A \mapsto (\min A, A - \min A)\) is \(\sqrt{5}\)-Lipschitz.
- Step1.2: A map \(f^{-1} : \mathbb{R} \times Z \rightarrow F_n(\mathbb{R}) : (b, \lambda) \mapsto A + b\) is 2-Lipschitz.

Step2: \((Z, d_H)\) is bi-Lipschitz equivalent to \((\text{Cone}'(F_n^*(\mathbb{I})), (d_H)_C)\). In fact, we can show the following.

- Step2.1: A map \(g : Z \rightarrow \text{Cone}'(F_n^*(\mathbb{I})) : tE \mapsto [(E, t)]\) is 1-Lipschitz.
- Step2.2: A map \(g^{-1} : \text{Cone}'(F_n^*(\mathbb{I})) \rightarrow Z : [(E, t)] \mapsto tE\) is 3-Lipschitz.

By the above, \((\text{id}_\mathbb{R} \times g) \circ f : F_n(\mathbb{R}) \rightarrow \mathbb{R} \times Z \rightarrow \mathbb{R} \times \text{Cone}'(F_n^*(\mathbb{I}))\) is a bi-Lipschitz equivalence.

\(\square\)

Corollary 3.5. \((F_2(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times [0, \infty), d)\).

\textbf{Proof.} By Definition 3.1, \(F_2^*(\mathbb{I})\) is one point, thus \((\text{Cone}'(F_2^*(\mathbb{I})), (d_H)_C)\) is corresponding to \(([0, \infty), d)\). By Theorem 3.4, \((F_2(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times [0, \infty), d)\). \(\square\)

The following result is first proved in [5, Theorem 6]. We give another proof by use of Theorem 3.4.

Corollary 3.6 ([5]). \((F_3(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R}^3, d)\).

\textbf{Sketch of Proof.} We note \(F_3^* = \{\{0, t, 1\} \mid 0 \leq t \leq 1\} \approx S^3\).

Step1: We can show that \((\text{Cone}'(F_3^*(\mathbb{I})), (d_H)_C)\) is bi-Lipschitz equivalent to \((\text{Cone}'(S^3), (d_{S^3})_C)\).

Step2: We can show that \((\mathbb{R}^2, d)\) is bi-Lipschitz equivalent to \((\text{Cone}'(S^1), (d_{S^1})_C)\).

By Theorem 3.4, \((F_3(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R}^3, d)\). \(\square\)
Remark 3.7. We note that $F_2(\mathbb{R}^2) \approx \mathbb{R}^4$. Indeed, we can define a homeomorphism $h : F_2(\mathbb{R}^2) \to \mathbb{R}^2 \times \text{Cone}^o(S^1 / x \sim -x) \approx \mathbb{R}^4$ by

$$h(A) = \begin{cases} 
(\text{m}(A), \left[ \frac{2(A-\text{m}(A))}{\text{diam} A}, \text{diam} A \right]) & \text{if diam } A \neq 0
\end{cases}$$

where $m(A) = a$ if $A = \{a\}$ and $m(A) = (a + a')/2$ if $A = \{a, a'\}$. In general, we see that $F_2(\mathbb{R}^l) \approx \mathbb{R}^l \times \text{Cone}^o(S^{l-1} / x \sim -x)$ for each $l \in \mathbb{N}$.

4. ISOMETRIES

Lemma 4.1. Let $n \in \mathbb{N}$ and let $(X, d)$ be a metric space such that

1. $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$ for each $\Phi \in \text{Isom}(F_n(X))$, and that
2. $\text{Isom}(F_n(X), F_1(X)) = \{id_{F_n(X)}\}$.

Then, $\chi(n) : \text{Isom}(X) \to \text{Isom}(F_n(X))$ is an isomorphism.

Proof. Let $\Phi \in \text{Isom}(F_n(X))$ and let $A_x = \{x\} \in F_1(X)$ for each $x \in X$. By assumption, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. Denote $\Phi(A_x) \in F_1(X)$ by $\{\phi(x)\}$ for each $x \in X$. Then, $\phi : X \to X : x \mapsto \phi(x)$ is an isometry. Set $\Phi' = \chi(n)(\phi^{-1}) \circ \Phi \in \text{Isom}(F_n(X))$. We claim that $\Phi'|_{F_1(X)} = id|_{F_1(X)}$. Indeed, $\Phi|_{F_1(X)} = \chi(n)(\phi)|_{F_1(X)}$ and $\chi(n)(\phi^{-1}) = (\chi(n)(\phi))^{-1}$. By assumption, we have that $\Phi' = id_{F_n(X)}$, therefore, $\Phi = \chi(n)(\phi)$, which completes the proof. □

Definition 4.2. Let $(X, d)$ be a metric space, let $n \in \mathbb{N}$, let $\epsilon > 0$ and let $A \in F_n(X)$. Define

$$D_n(A, \epsilon) = \sup \{k \in \mathbb{N} | A_1, \ldots, A_k \in S_{d_H}(A, \epsilon), d_H(A_i, A_j) = 2\epsilon (i \neq j)\} \in \mathbb{N} \cup \{\infty\}.$$

Lemma 4.3. Let $l, n \in \mathbb{N}$, let $X = \mathbb{R}^l$ or $X = S^l$ and let $\Phi \in \text{Isom}(F_n(X))$. Then, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$.

Sketch of Proof. Let $n \in \mathbb{N}$ with $n \geq 2$.

Step 1: Let $A = \{a_1\} \in F_1(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)$. We can show that $D_n(A, \epsilon) = 3$.

Step 2: Let $m \in \mathbb{N}$ with $m \geq 2$, let $A = \{a_1, \ldots, a_m\} \in F_m(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)/5$. We can show that $D_n(A, \epsilon) > 3$.

Let $\Phi \in \text{Isom}(F_n(X))$ and let $A \in F_n(X)$. From the definition of $D_n(A, \epsilon)$, we obtain $D_n(A, \epsilon) = D_n(\Phi(A), \epsilon)$ for each $0 < \epsilon < \min\{r(A), r(\Phi(A))\}$. By the above, we see that $A \in F_1(X)$ if and only if $\Phi(A) \in F_1(X)$. Therefore, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. □

Lemma 4.4. Let $l, n \in \mathbb{N}$. Then, $\text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) = \{id_{F_n(\mathbb{R}^l)}\}$.
Sketch of Proof.
Step1: Let $l, n \in \mathbb{N}$ and let $\Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$. Then, $\Phi|_{F_2(\mathbb{R}^l)} = \text{id}_{F_2(\mathbb{R}^l)}$.
Step2: Let $n \in \mathbb{N}$ with $n \geq 2$ and let $\Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$ and let $A \in F_{(m)}(\mathbb{R}^l)$. We can show that $\Phi(A) \subset A$. If similar arguments apply to $\Phi(A)$ and $\Phi^{-1}$, we obtain $A = \Phi^{-1}(\Phi(A)) \subset \Phi(A)$, therefore, $A = \Phi(A)$.

Lemma 4.5. Let $l, n \in \mathbb{N}$. Then $\text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) = \{ \text{id}_{F_n(\mathbb{S}^l)} \}$.

Proof. Let $\Phi \in \text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)), m \in \mathbb{N}$ with $2 \leq m \leq n$ and let $A \in F_{(m)}(\mathbb{S}^l)$. We show that $A = \Phi(A)$. Let $a \in A$ and let $a' \in \mathbb{S}^l$ be the anti-point of $a$. Since $d_H(\{a'\}, \Phi(A)) = d_H(\Phi(\{a'\}), \Phi(A)) = d_H(\{a'\}, A) = \pi$, we have $a \in \Phi(A)$, therefore, $A \subset \Phi(A)$. If similar arguments apply to $\Phi(A)$ and $\Phi^{-1}$, we obtain $\Phi(A) \subset \Phi^{-1}(\Phi(A)) = A$, therefore, $A = \Phi(A)$, which completes the proof.

The proof of Theorem 1.1. By Lemmas 4.3, 4.4 and 4.5, the conditions in Lemma 4.1 hold for $(X, d)$, which completes the proof.

5. Questions

Question 5.1. Let $l, n \in \mathbb{N}$ with $n \geq 2$. When $(X, d)$ is a following space, is $\chi_n : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ an isomorphism?

1. $X = \mathbb{R}^l$ has a metric $d_\infty$, where $d_\infty(x, y) = \max\{|x_i - y_i| | i = 1, \ldots, l\}$ for any $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in X$.
2. $X$ is a convex subset of $\mathbb{R}^l$.
3. $X$ is an $\mathbb{R}$-tree (see [2] for $\mathbb{R}$-trees).
4. $X$ is the hyperbolic $l$-space (see [8] for the hyperbolic $l$-space).

Example 5.2. Let $n, m \in \mathbb{N}$ with $2 \leq n \leq m$ and let $(X, d)$ be an $m$-points discrete metric space satisfying that $d(x, x') = 1$ whenever $x \neq x'$. Then, $F_n(X)$ is a discrete metric space such that $d_H(A, A') = 1$ for any $A, A' \in F_n(X)$ with $A \neq A'$. Thus, $|\text{Isom}(X)| = |X| < |F_n(X)| = |\text{Isom}(F_n(X))|$, therefore, $\chi(n) : \text{Isom}_d(X) \rightarrow \text{Isom}_{d_H}(F_n(X))$ is not an isomorphism.

This drives us to the following question as the generalization of Theorem 1.1.

Question 5.3. Let $(X, d)$ be a connected metric space. Then, is $\chi(n) : \text{Isom}(X) \rightarrow \text{Isom}(F_n(X))$ an isomorphism?

Question 5.4. It is known that $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$. Is $F_3(\mathbb{S}^1)$ bi-Lipschitz equivalent to $\mathbb{S}^3$?
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