**Title**

ISOMETRIES ON THE SYMMETRIC PRODUCTS OF THE EUCLIDEAN SPACES WITH USUAL METRICS (The present situation of set-theoretic and geometric topology and its prospects)

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**Citation**

数理解析研究所講究録 1884: 7-12

**Issue Date**

2014-04

**URL**

http://hdl.handle.net/2433/195696

**Type**

Departmental Bulletin Paper

**Textversion**

publisher

Kyoto University
ISOMETRIES ON THE SYMMETRIC PRODUCTS OF THE EUCLIDEAN SPACES WITH USUAL METRICS

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1. INTRODUCTION

As an interesting construction in topology, Borsuk and Ulam [3] introduced the $n$-th symmetric product of a metric space $(X, d)$, denoted by $F_n(X)$. Namely $F_n(X)$ is the space of non-empty finite subsets of $X$ with at most $n$ elements endowed with the Hausdorff metric $d_H$, i.e.,

$$F_n(X) = \{ A \subset X \mid 1 \leq |A| \leq n \}$$

and $d_H(A, B) = \inf \{ \epsilon \mid A \subset B_{d}(B, \epsilon) \text{ and } B \subset B_{d}(A, \epsilon) \} = \max \{ d(a, B), d(b, A) \mid a \in A, b \in B \}$ for any $A, B \in F_n(X)$ (see [10, p.6]).

For the symmetric products of $\mathbb{R}$, it is known that $F_2(\mathbb{R}) \approx \mathbb{R} \times [0, \infty)$ and $F_3(\mathbb{R}) \approx \mathbb{R}^3$ (see Section 3). It was proved in [3] that $F_n(\mathbb{I})$ is homeomorphic to $\mathbb{I}^n$ (written $F_n(\mathbb{I}) \approx \mathbb{I}^n$) if and only if $1 \leq n \leq 3$, and that for $n \geq 4$, $F_n(\mathbb{I})$ can not be embedded into $\mathbb{R}^n$, where $\mathbb{I} = [0, 1]$ has the usual metric. Thus, for $n \geq 4$, $F_n(\mathbb{R}) \not\approx \mathbb{R}^n$. Molski [12] showed that $F_2(\mathbb{I}^2) \approx \mathbb{I}^4$, and that for $n \geq 3$ neither $F_n(\mathbb{I}^2)$ nor $F_2(\mathbb{I}^n)$ can be embedded into $\mathbb{R}^{2n}$. Thus, for $n \geq 3$, $F_n(\mathbb{R}^2) \not\approx \mathbb{R}^{2n}$ and $F_2(\mathbb{R}^n) \not\approx \mathbb{R}^{2n}$.

Turning toward the symmetric product $F_n(S^1)$ of the circle $S^1$, Chinen and Koyama [9] prove that for $n \in \mathbb{N}$, both $F_{2n-1}(S^1)$ and $F_{2n}(S^1)$ have the same homotopy type of the $(2n - 1)$-sphere $S^{2n-1}$. In [7] Bott corrected Borsuk’s statement [4] and showed that $F_3(S^1) \approx S^3$. In [9], another proof of it is given.

For a metric space $(X, d)$, we denote by $\text{Isom}_d(X)$ (Isom$(X)$ for short) the group of all isometries from $X$ into itself, i.e., $\phi : X \to X \in \text{Isom}_d(X)$ if $\phi$ is a bijection satisfying that $d(x, x') = d(\phi(x), \phi(x'))$ for any $x, x' \in X$. Let $n \in \mathbb{N}$. Every isometry $\phi : X \to X$ induces an isometry $\chi_n(\phi) : F_n(X), d_H \to (F_n(X), d_H)$ defined by $\chi_n(\phi)(A) = \phi(A)$ for each $A \in F_n(X)$. Thus, there exists a natural monomorphism $\chi_n : \text{Isom}_d(X) \to \text{Isom}_d(F_n(X))$. It is clear that $\chi_n : \text{Isom}_d(X) \to \text{Isom}_d(F_n(X))$ is an isomorphism if and only if $\chi_n$ is an epimorphism, i.e., for every $\Phi \in \text{Isom}_d(F_n(X))$ there exists $\phi \in \text{Isom}_d(X)$ such that $\Phi = \chi_n(\phi)$.

In this paper, it is of interest to know whether $\chi_n : \text{Isom}_d(X) \to \text{Isom}_d(F_n(X))$ is an isomorphism for a metric space $(X, d)$. Recently, Borovikova and Ibragimov [5] prove that $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$ and that $\chi_3 : \text{Isom}_d(\mathbb{R}) \to \text{Isom}_d(F_3(\mathbb{R}))$ is an isomorphism, where $\mathbb{R}$ has the usual metric $d$. The following result is a generalization of the result above and the affirmative answer to [6, p.60, Conjecture 2.1].
Theorem 1.1. Let $l \in \mathbb{N}$ and let $X = \mathbb{R}^l$ or $X = \mathbb{S}^l$ with the usual metric $d$. Then $\chi_n : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X))$ is an isomorphism for each $n \in \mathbb{N}$.

In Section 4, we give the main ideas of proof of Theorem 1.1. In Example 5.2 below, we present a compact metric space $(X, d)$ such that $\chi_n(\text{Isom}_d(X)) \neq \text{Isom}_{d_H}(F_n(X))$ for all $n \geq 2$, i.e., $\chi_n : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X))$ is not an isomorphism. And, in Section 3, we provide another proof of [5, Theorem 6]. Its proof is based on the proof of [11, Lemma 2.4].

2. Preliminaries

Notation 2.1. Let denote the set of all natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let $d$ be the usual metric on $\mathbb{R}^l$, i.e., $d(x, y) = \left\{ \sum_{i=1}^{l}(x_i - y_i)^2 \right\}^{1/2}$ for any $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in \mathbb{R}^l$. Write $\mathbb{S}^l = \{x = (x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1} | \sum_{i=1}^{l+1}x_i^2 = 1\}$ with the length metric $d$. Denote the identity map from $X$ into itself by $id_X$.

Definition 2.2. Let $(X, d)$ be a metric space, let $x \in X$, let $Y, Z$ be subsets of $X$ and let $\epsilon > 0$. Set $d(Y, Z) = \inf\{d(y, x) | y \in Y, z \in Z\}$, and $B_d(Y, \epsilon) = \{x \in X | d(x, Y) \leq \epsilon\}$. If $Y = \{y\}$, for simplicity of notation, we write $B_d(y, \epsilon) = \mathbb{S}_d(Y, \epsilon)$.

For $n \in \mathbb{N}$, the $n$-th symmetric product of $X$ is defined by

$$F_n(X) = \{A \subset X | 1 \leq |A| \leq n\},$$

where $|A|$ is the cardinality of $A$. Write $F_m^*(I) = \{A \subset 2^I | |A| = m\}$ for each $m \in \mathbb{N}$. Let $\text{Isom}(X, Y) = \{\phi \in \text{Isom}(X) | \phi(y) = y$ for each $y \in Y\}$ for $Y \subset X$. Set $r(A) = \min\{\{1\} \cup \{d(a, a') | a, a' \in A, a \neq a'\}\}$ for each $A \in F_n(X)$.

3. A metric space is bi-Lipschitz equivalent to the symmetric product of $\mathbb{R}$

In this section, we give another proof of [5, Theorem 6] which is based on the proof of [11, Lemma 2.4].

Definition 3.1. Let $n \in \mathbb{N}$. Set $F_n^*(I) = \{A \in F_n(I) | 0, 1 \in A\}$. It is known that $F_2^*(I) = \{(0, 1)\}$, $F_3^*(I) = \{(0, t, 1) | 0 \leq t \leq 1\} \approx \mathbb{S}^1$, and, $F_4^*(I) = \{(0, s, t, 1) | 0 \leq s \leq t \leq 1\}$ is homeomorphic to the dance hat (see [16]). In general, $F_{2n}^*(I)$ is contractible but not collapsible, and $F_{2n+1}^*(I)$ has the same homotopy type of $\mathbb{S}^{2n+1}$. In [1], it is called the spaces $F_{2n}^*(I)$, $n \geq 2$, higher dimensional dunce hats (see [1]).
Definition 3.2 ([11]). Let \((X, d)\) be a metric space with \(\text{diam} \ X \leq 2\). Set \(\text{Cone}^o(X) = X \times [0, \infty) / (X \times \{0\})\), is said to be the open cone over \(X\), with the metric \(d_C([[x_1, t_1]], [[x_2, t_2]]) = |t_1 - t_2| + \min\{t_1, t_2\} \cdot d(x_1, x_2)\).

Definition 3.3. Let \(f : (X, d) \to (Y, d')\) be a map. The map \(f\) is said to be Lipschitz (bi-Lipschitz, respectively) if there exists \(L > 0\) such that
\[
d'(f(x_1), f(x_2)) \leq L \cdot d(x_1, x_2)
\]
\[
(L^{-1} \cdot d(x_1, x_2) \leq d'(f(x_1), f(x_2)) \leq L \cdot d(x_1, x_2), \text{respectively})
\]
for any \(x_1, x_2 \in X\). \((X, d)\) is said to be bi-Lipschitz equivalent to \((Y, d')\) if there exists a surjective bi-Lipschitz map from \((X, d)\) to \((Y, d')\).

Theorem 3.4 ([11]). Let \(n \in \mathbb{N}\) with \(n \geq 2\). Then \((F_n(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times \text{Cone}^o(F_n^*([1])), \rho)\), where \(\rho = \sqrt{d^2 + (d_H)_{C}^2}\).

Sketch of Proof. Let \(Z = \{A \in F_n(\mathbb{R}) \mid \min A = 0\}\). For every \(A \in Z\) there exists the unique \(E \in F_n^*([1])\) such that \(A = tE\), where \(t = \max A\).

Step 1: \((F_n(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times Z, \rho_1)\), where \(\rho_1 = \sqrt{d^2 + (d_H)^2}\). In fact, we can show the following.

Step 1.1: A map \(f : F_n(\mathbb{R}) \to \mathbb{R} \times Z : A \mapsto (\min A, A - \min A)\) is \(\sqrt{5}\)-Lipschitz.

Step 1.2: A map \(f^{-1} : \mathbb{R} \times Z \to F_n(\mathbb{R}) : (b, A) \mapsto A + b\) is 2-Lipschitz.

Step 2: \((Z, d_H)\) is bi-Lipschitz equivalent to \((\text{Cone}^o(F_n^*([1])), (d_H)_C)\). In fact, we can show the following.

Step 2.1: A map \(g : Z \to \text{Cone}^o(F_n^*([1])) : tE \mapsto [(E, t)]\) is 1-Lipschitz.

Step 2.2: A map \(g^{-1} : \text{Cone}^o(F_n^*([1])) \to Z : [(E, t)] \mapsto tE\) is 3-Lipschitz.

By the above, \((\text{id}_\mathbb{R} \times g) \circ f : F_n(\mathbb{R}) \to \mathbb{R} \times Z \to \mathbb{R} \times \text{Cone}^o(F_n^*([1]))\) is a bi-Lipschitz equivalence. \(\Box\)

Corollary 3.5. \((F_2(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times [0, \infty), d)\).

Proof. By Definition 3.1, \(F_2([1])\) is one point, thus \((\text{Cone}^o(F_2([1])), (d_H)_C)\) is corresponding to \(([0, \infty), d)\). By Theorem 3.4, \((F_2(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R} \times [0, \infty), d)\). \(\Box\)

The following result is first proved in [5, Theorem 6]. We give another proof by use of Theorem 3.4.

Corollary 3.6 ([5]). \((F_3(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R}^3, d)\).

Sketch of Proof. We note \(F_3^*([1]) = \{\{0, t, 1\} \mid 0 \leq t \leq 1\} \approx S^1\).

Step 1: We can show that \((\text{Cone}^o(F_n^*([1])), (d_H)_C)\) is bi-Lipschitz equivalent to \((\text{Cone}^o(S^1), (d_{|S^1})_C)\).

Step 2: We can show that \((\mathbb{R}^2, d)\) is bi-Lipschitz equivalent to \((\text{Cone}^o(S^1), (d_{|S^1})_C)\).

By Theorem 3.4, \((F_3(\mathbb{R}), d_H)\) is bi-Lipschitz equivalent to \((\mathbb{R}^3, d)\). \(\Box\)
Remark 3.7. We note that $F_2(\mathbb{R}^2) \approx \mathbb{R}^4$. Indeed, we can define a homeomorphism $h : F_2(\mathbb{R}^2) \rightarrow \mathbb{R}^2 \times \text{Cone}^o(\mathbb{S}^1/x \sim -x) (\approx \mathbb{R}^4)$ by

$$h(A) = \begin{cases} (m(A), \left[ \frac{2(A-m(A))}{\text{diam } A}, \text{diam } A \right]) & \text{if } \text{diam } A \neq 0, \\ (m(A), \text{the cone point}) & \text{if } \text{diam } A = 0, \end{cases}$$

where $m(A) = a$ if $A = \{a\}$ and $m(A) = (a + a')/2$ if $A = \{a, a'\}$. In general, we see that $F_2(\mathbb{R}^l) \approx \mathbb{R}^l \times \text{Cone}^o(\mathbb{S}^{l-1}/x \sim -x)$ for each $l \in \mathbb{N}$.

4. Isometries

Lemma 4.1. Let $n \in \mathbb{N}$ and let $(X, d)$ be a metric space such that

(1) $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$ for each $\Phi \in \text{Isom}(F_n(X))$, and that
(2) $\text{Isom}(F_n(X), F_1(X)) = \{\text{id}_{F_n(X)}\}$.

Then, $\chi_{(n)} : \text{Isom}(X) \rightarrow \text{Isom}(F_n(X))$ is an isomorphism.

Proof. Let $\Phi \in \text{Isom}(F_n(X))$ and let $A_x = \{x\} \in F_1(X)$ for each $x \in X$. By assumption, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. Denote $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$ by $\{\phi(x)\}$ for each $x \in X$. Then, $\phi : X \rightarrow X : x \mapsto \phi(x)$ is an isometry. Set $\Phi' = \chi_{(n)}(\phi^{-1}) \circ \Phi \in \text{Isom}(F_n(X))$. We claim that $\Phi'|_{F_1(X)} = \text{id}|_{F_1(X)}$. Indeed, $\Phi|_{F_1(X)} = \chi_{(n)}(\phi)|_{F_1(X)}$ and $\chi_{(n)}(\phi^{-1}) = (\chi_{(n)}(\phi))^{-1}$. By assumption, we have that $\Phi' = \text{id}_{F_n(X)}$, therefore, $\Phi = \chi_{(n)}(\phi)$, which completes the proof. \hfill $\square$

Definition 4.2. Let $(X, d)$ be a metric space, let $n \in \mathbb{N}$, let $\epsilon > 0$ and let $A \in F_n(X)$. Define

$$D_n(A, \epsilon) = \sup\{k \in \mathbb{N} | A_1, \ldots, A_k \in S_d(A, \epsilon), d(A_i, A_j) = 2\epsilon (i \neq j)\} \in \mathbb{N} \cup \{\infty\}.$$

Lemma 4.3. Let $l, n \in \mathbb{N}$, let $X = \mathbb{R}^l$ or $X = \mathbb{S}^l$ and let $\Phi \in \text{Isom}(F_n(X))$. Then, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$.

Sketch of Proof. Let $n \in \mathbb{N}$ with $n \geq 2$.
Step1: Let $A = \{a_1\} \in F_1(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)$. We can show that $D_n(A, \epsilon) = 3$.
Step2: Let $m \in \mathbb{N}$ with $m \geq 2$, let $A = \{a_1, \ldots, a_m\} \in F_m(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)/5$. We can show that $D_n(A, \epsilon) > 3$.

Let $\Phi \in \text{Isom}(F_n(X))$ and let $A \in F_n(X)$. From the definition of $D_n(A, \epsilon)$, we obtain $D_n(A, \epsilon) = D_n(\Phi(A), \epsilon)$ for each $0 < \epsilon < \min\{r(A), r(\Phi(A))\}$. By the above, we see that $A \in F_1(X)$ if and only if $\Phi(A) \in F_1(X)$. Therefore, $\Phi|_{F_1(X)} \in \text{Isom}(F_1(X))$. \hfill $\square$

Lemma 4.4. Let $l, n \in \mathbb{N}$. Then, $\text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) = \{\text{id}_{F_n(\mathbb{R}^l)}\}$. 

Sketch of Proof.

Step 1: Let \( l, n \in \mathbb{N} \) and let \( \Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) \). Then, \( \Phi|_{F_1(\mathbb{R}^l)} = \text{id}_{F_1(\mathbb{R}^l)} \).

Step 2: Let \( n \in \mathbb{N} \) with \( n \geq 2 \) and let \( \Phi \in \text{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) \) and let \( A \in F_{(m)}(\mathbb{R}^l) \). We can show that \( \Phi(A) \subset A \). If similar arguments apply to \( \Phi(A) \) and \( \Phi^{-1} \), we obtain \( A = \Phi^{-1}(\Phi(A)) \subset \Phi(A) \), therefore, \( A = \Phi(A) \).

\[ \square \]

Lemma 4.5. Let \( l, n \in \mathbb{N} \). Then \( \text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) = \{ \text{id}_{F_n(\mathbb{S}^l)} \} \).

Proof. Let \( \Phi \in \text{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) \), \( m \in \mathbb{N} \) with \( 2 \leq m \leq n \) and let \( A \in F_{(m)}(\mathbb{S}^l) \). We show that \( A = \Phi(A) \). Let \( a \in A \) and let \( a' \in \mathbb{S}^l \) be the anti-point of \( a \). Since \( d_H(\{a'\}, \Phi(A)) = d_H(\Phi(\{a'\}), \Phi(A)) = d_H(\{a'\}, A) = \pi \), we have \( a \in \Phi(A) \), therefore, \( A \subset \Phi(A) \). If similar arguments apply to \( \Phi(A) \) and \( \Phi^{-1} \), we obtain \( \Phi(A) \subset \Phi^{-1}(\Phi(A)) = A \), therefore, \( A = \Phi(A) \), which completes the proof.

\[ \square \]

The proof of Theorem 1.1. By Lemmas 4.3, 4.4 and 4.5, the conditions in Lemma 4.1 hold for \((X,d)\), which completes the proof.

\[ \square \]

5. Questions

Question 5.1. Let \( l, n \in \mathbb{N} \) with \( n \geq 2 \). When \((X,d)\) is a following space, is \( \chi_n : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X)) \) an isomorphism?

1. \( X = \mathbb{R}^l \) has a metric \( d_{\infty} \), where \( d_{\infty}(x,y) = \max\{|x_i - y_i| | i = 1, \ldots, l\} \) for any \( x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in X \).

2. \( X \) is a convex subset of \( \mathbb{R}^l \).

3. \( X \) is an \( \mathbb{R} \)-tree (see [2] for \( \mathbb{R} \)-trees).

4. \( X \) is the hyperbolic \( l \)-space (see [8] for the hyperbolic \( l \)-space).

Example 5.2. Let \( n, m \in \mathbb{N} \) with \( 2 \leq n \leq m \) and let \((X,d)\) be an \( m \)-points discrete metric space satisfying that \( d(x,x') = 1 \) whenever \( x \neq x' \). Then, \( F_n(X) \) is a discrete metric space such that \( d_H(A,A') = 1 \) for any \( A, A' \in F_n(X) \) with \( A \neq A' \). Thus, \( |\text{Isom}(X)| = |X|! < |F_n(X)|! = |\text{Isom}(F_n(X))| \), therefore, \( \chi(n) : \text{Isom}_d(X) \to \text{Isom}_{d_H}(F_n(X)) \) is not an isomorphism.

This drives us to the following question as the generalization of Theorem 1.1.

Question 5.3. Let \((X,d)\) be a connected metric space. Then, is \( \chi(n) : \text{Isom}(X) \to \text{Isom}(F_n(X)) \) an isomorphism?

Question 5.4. It is known that \( F_3(\mathbb{S}^1) \approx \mathbb{S}^3 \). Is \( F_3(\mathbb{S}^1) \) bi-Lipschitz equivalent to \( \mathbb{S}^3 \)?
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