A brief survey on a fast Monte Carlo scheme for risk analyses using a probability measure transformation technique

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1 Introduction

Estimation of small probability is one of the most important theme in many application fields such as reliability engineering for structural systems or risk analysis. The well-known and most widely used Monte Carlo simulation method is crucial for such purposes because of its very slow convergence property. Transforming probability measure works quite well for reducing the variance inherent in the Monte Carlo simulation procedure, which enables us to estimate very small probability with good accuracy.

The author has been proposed a method for accelerating the Monte Carlo simulation by applying the probability measure transformation based upon the well-known Maruyama-Girsanov theorem\cite{1} and its variations. In Refs\cite{2}, \cite{3} and \cite{4}, the method has been applied to stochastic systems driven by Wiener processes, in which a systematic procedure has been constructed for selecting the optimal probability measure under which an importance sampling simulation is executed by the use of a concept of design point playing a quite important role in the structural reliability engineering\cite{5}\cite{6}. In Refs\cite{7} and \cite{8}, the method has been extended for treating a stochastic system driven by compound Poisson processes.

Recently, more generalized stochastic models have been applied for modeling various phenomena, especially, Lévy processes have been widely used for modeling dynamics of securities or wealth\cite{9}. For instance, a variance gamma process has been widely used for modeling wealth processes in credit risk analysis\cite{10}. Thus, we need to refine the probability measure transformation technique so that it can be applied to stochastic systems driven by Lévy processes.

In this paper, we give a brief survey of an application of probability measure transformation technique to reduce the variance inherent in Monte Carlo simulations. Further, two examples of its application are shown.

2 Basic Formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t; 0 \leq t \leq T\}$ be a filtration on $(\Omega, \mathcal{F}, P)$. We consider a system such as

- An input noise disturbing the system behavior is described by a real-valued and temporally homogeneous Lévy process denoted by $Z(\omega) = \{Z_t(\omega); 0 \leq t \leq T\}$.

- An output is described by a real-valued stochastic process denoted by $X(\omega) = \{X_t(\omega); 0 \leq t \leq T\}$, which is supposed to be adapted to the filtration.
• The output $X$ is related to the input noise $Z$ as

$$X(\omega) = \mathcal{H}[Z(\omega)] \quad \text{(a.s.)},$$

where $\mathcal{H}$ is a functional representing the system.

Next we introduce an indicator functional $f$ for identifying a risk event which is our main subject of analysis, i.e.,

$$f[X] = \begin{cases} 
1 & \text{(risk event occurs in } [0, T]) \\
0 & \text{(otherwise)} \end{cases}.$$  \hfill (2.2)

The main target of our analysis is to estimate its expectation, denoted by $\psi(T)$, as

$$\psi(T) = \int_{\Omega} f[X(\omega)] P(d\omega) = \int_{\Omega} f[\mathcal{H}[Z(\omega)]] P(d\omega) = E_P \{ f[\mathcal{H}[Z]] \},$$

where $E_P$ denotes an operator to take expectation under the original probability measure $P$. If the risk event represents a failure of a system, $\psi(T)$ represents probability of system failure up to time $T$, which is a main target quantity in the reliability engineering. On the other hand, if $X_t$ represents a wealth of a company at time $t$ and the risk event represents an occurrence of ruin of the company, $\psi(T)$ represents probability of ruin (or frequently called probability of default) with in time $T$, which is a main subject in the field of collective risk theory\cite{11}.

In many fields of application including reliability analysis as well as risk analysis, it is frequently required to estimate $\psi(T)$ when it takes on a very small value. As the well-known Monte Carlo method does not work well for estimating such small probability, we have to execute simulation procedure based upon another probability measure.

Suppose that $Q$ is such a probability measure defined on the same measurable space $(\Omega, \mathcal{F})$, which is equivalent to the original probability measure $P$. Using the measure $Q$, we can rewrite Eq.(2.3) as

$$\psi(T) = \int_{\Omega} f[\mathcal{H}[Z(\omega)]] \frac{dP}{dQ}(\omega) Q(d\omega) = E_Q \{ f[\mathcal{H}[Z]] \frac{dP}{dQ} \},$$

where $dP/dQ$ expresses the Radon-Nikodym derivative and $E_Q$ denotes an operator to take expectation under $Q$. The Monte Carlo estimator under $Q$ based upon Eq.(2.4) is then given as follows;

$$\hat{\psi}(T; N) = \frac{1}{N} \sum_{k=1}^{N} f \left[ \mathcal{H} \left[ Z_{Q}^{(k)} \right] \right] \left( \frac{dP}{dQ} \right)_{Q}^{(k)},$$

where $Z_{Q}^{(k)}$ and $(dP/dQ)_{Q}^{(k)} (k = 1, \ldots, N)$ are independent samples of $Z$ and $dP/dQ$ respectively generated under $Q$.

If we select a suitable measure $Q$ so that we can generate many samples which contribute to the estimation of $\psi(T)$, the variance of $\hat{\psi}(T; N)$ inherent in the Monte Carlo procedure can be effectively reduced, which is one of variance reduction techniques known as importance sampling. Thus, we call the measure $Q$ importance sampling measure in what follows. To execute a Monte Carlo simulation based upon the importance sampling measure, we need to select $Q$ under which we can easily generate independent samples of both $Z$ and $dP/dQ$ with giving an effective reduction of the variance.
3 Probability measure transformation based upon the Lévy-Itô decomposition

In this section, we give a basic framework of the probability measure transformation from $P$ to $Q$ available for Lévy processes by the use of the well-known Lévy-Itô decomposition$^{12}$ based upon Ref.$^{14}$.

3.1 The Lévy-Itô decomposition

The Lévy-Itô decomposition for temporally homogeneous Lévy processes are expressed as follows;

$$Z_t = \sigma_B B_t + q_B t + \int_{|u|>0} \{u \mu^Z_t(du) - t \cdot h(u) m^Z_0(du)\},$$

(3.1)

in which $B_t$ is a Wiener process, $\sigma_B$ and $q_B$ are constants, $\mu^Z_t(A)$ represents the number of discontinuous jumps of $Z_t$ appearing in $[0, t]$ with jump variation belonging to a set $A$. The measure $\mu^Z_t$ is the so-called Poisson random measure such that $\mu^Z_t(A)$ obeys a Poisson distribution with mean

$$E_P \{\mu^Z_t(A)\} = m^Z(A),$$

(3.2)

which determines a measure called Lévy measure. The function $h(u)$ appearing in the integral in the third term is required so that the accumulation of small jumps does not diverge for some subclass of Lévy processes. For example, the following function is frequently used;

$$h(u) = \begin{cases} 
-1 & (u < -1) \\
\mu & (-1 \leq u \leq 1) \\
1 & (1 < u)
\end{cases}$$

(3.3)

Provided that $h(u)$ is given, $\sigma_B$, $q_B$ and the Lévy measure $m_Z$ determine a Lévy process $Z_t$ under the weak uniqueness. Thus, the triplet $(\sigma_B$, $q_B$, $m_Z)$ is called characteristic quantities of Lévy processes.

For example, if $q_B = 0$, $\sigma_B = 1$ and $m_Z \equiv 0$, $Z_t$ gives a Wiener process $B_t$. On the other hand, if $q_B = \sigma_B = 0$ and the Lévy measure is given as, with a positive constant $\lambda$,

$$m^Z(A) = \begin{cases} 
\lambda & (1 \in A) \\
0 & \text{(otherwise)}
\end{cases},$$

(3.4)

$Z_t$ is reduced to a Poisson process with an intensity $\lambda$.

3.2 Probability measure transformation based upon the Lévy-Itô decomposition

Next, we give a probability measure transformation procedure$^{15}$ based upon the Lévy-Itô decomposition given by Eq.(3.1).

The target measure $Q$, which is a probability measure defined on a measurable space $(\Omega, \mathcal{F})$, is assumed to be equivalent with the original probability measure $P$. Then, the Radon-Nikodym
derivative $dQ/dP$, as well as $dP/dQ$, exists, which generates a $P$-martingale $M_t$ given as
\[ M_t = E_P \left\{ \frac{dQ}{dP} \mid \mathcal{F}_t \right\}. \]  
(3.5)
Since $E_P\{M_t\}$ clearly equals to unity, $M_t$ can be expressed, by the use of a suitable $P$-martingale $N_t$, as
\[ M_t = \mathcal{E}(N)_t, \]  \(3.6\)
where $\mathcal{E}(N)_t$ is the well-known Doléans-Dade exponential defined as follows;
\[ \mathcal{E}(N)_t = \exp \left\{ N_t - \frac{1}{2} [N, N]_t^{c} \right\} \prod_{s \leq t} (1 + \Delta N_s) e^{-\Delta N_s}, \]  \(3.7\)
in which $[N, N]_t$ represents a quadratic variation of $N_t$, $[N, N]_t^{c}$ represents its continuous part and $\Delta N_s = N_s - N_{s-}$ represents a discontinuous jump of $N_t$ at time $s$. In this study, we confine ourselves to the case in which the martingale $N_t$ is a Lévy process with mean zero under $P$, whose Lévy-Itô decomposition is supposed to be given as
\[ N_t = \eta \sigma B_t + \int_{|u|>0} \{ g(u) - 1 \} \{ \mu_t^{Z}(du) - t \cdot m_{Z}(du) \}, \]  \(3.8\)
where $\eta$ is a constant, $g(u)$ is a certain deterministic, as well as integrable, function. Substituting Eq.(3.8) into Eq.(3.7), we can obtain
\[ M_t = e^{\hat{N}_t}, \]  \(3.9\)
\[ \hat{N}_t = \eta \sigma B_t - \frac{1}{2} \eta^2 \sigma^2 t + \int_{|u|>0} \{ \log g(u) \mu_t^{Z}(du) - t(g(u) - 1)m_{Z}(du) \}. \]  \(3.10\)
The probability measure $Q$ is finally constructed by substituting Eq.(3.10) into Eq.(3.5). In what follows, we call $(\eta, g(u))$ characteristics of the probability measure transformation from $P$ to $Q$.

If the probability measure can be fully determined by the information up to $t = T$, we can obtain, from Eq.(3.6) and Eq.(3.10), as
\[ \frac{dQ}{dP} = e^{\hat{N}_T}, \]  \(3.11\)
since $dQ/dP$ is $\mathcal{F}_T$-measurable. Thus, we can give an analytical formula for the Radon-Nikodym derivative between two measures.

Since the Wiener process and the accumulation of discontinuous jumps characterized by the Lévy measure, appearing in the Lévy-Itô decomposition, are statistically independent, the above measure transformation is reduced to a combination of the following two independent measure transformation as

(a) The process $B_t^{Q}$ defined as
\[ B_t^{Q} = B_t - \eta \sigma B_t \]  \(3.12\)
is a Wiener process under $Q$.

(b) The Lévy measure under $Q$, denoted by $m_t^{Q}$, is given as follows;
\[ m_t^{Q}(A) = \int_{A} g(u)m_{Z}(du). \]  \(3.13\)
The transformation given by Eq.(3.12) is the well-known Maruyama-Girsanov transformation\[1\].
4 Optimal selection of $Q$

In this section, we briefly review a method\cite{14} for selecting optimal measure $Q$ so that the variance of Monte Carlo simulation under $Q$ can be most effectively reduced.

To make discussion clear, we suppose that the risk event occurs when the system process arrives at a certain risk set $A$, i.e., the indicator functional $f$ is given as

$$f[X] = \begin{cases} 1 & (0 \leq \exists t \leq T \text{ s.t. } X_t \in A) \\ 0 & \text{(otherwise)} \end{cases}.$$  \hspace{1cm} (4.1)

In Ref.\cite{14}, the author has clarified that the convergence is effectively accelerated when the mean behavior of $X$ under $Q$ arrives at the risk set $A$ at time $t_d$, which is called design time in Ref.\cite{14}. That is,

$$\tilde{E}_Q\{X_{t_d}\} = x_c,$$  \hspace{1cm} (4.2)

in which $\tilde{E}_Q\{X_{t_d}\}$ represents an approximated mean value of $X_{t_d}$ under $Q$ and $x_c$ represents a point on the boundary of $A$ nearest to the initial state. If $Q$ is selected so that Eq.(4.2) is satisfied, risk event occurs for about 50\% of generated samples under $Q$, which has been used for realizing the most effective reduction of simulation time in structural system reliability analysis\cite{6}.

Next, a variance of the estimator under $Q$ is given as

$$\text{Var}_Q\{\hat{\psi}(T;N)\} = \frac{1}{N} \left( E_P\left\{ f [\mathcal{X} \mathcal{Z}]^2 \frac{dP}{dQ} \right\} - \psi(T)^2 \right).$$  \hspace{1cm} (4.3)

It should be noted that, although we can perfectly reduce the variance given by Eq.(4.3) by selecting $Q$ as

$$dQ = \frac{1}{\psi(T)} f [\mathcal{X} \mathcal{Z}] dP,$$  \hspace{1cm} (4.4)

which is clearly an impossible selection for estimating $\psi(T)$\cite{16}. That is, we can not reduce the variance itself in selecting the measure $Q$.

Equation(4.3) can be rewritten as the following inequality;

$$\text{Var}_Q\{\hat{\psi}(T;N)\} \leq \frac{1}{N} E_P\left\{ f [\mathcal{X} \mathcal{Z}]^2 \frac{dP}{dQ} \right\}.$$  \hspace{1cm} (4.5)

Further applying the Schwarz inequality, we can obtain

$$\text{Var}_Q\{\hat{\psi}(T;N)\} \leq \frac{1}{N} \left( E_P\left\{ f [\mathcal{X} \mathcal{Z}]^4 \right\} \right)^{1/2} \left( E_P\left\{ \left( \frac{dP}{dQ} \right)^2 \right\} \right)^{1/2},$$  \hspace{1cm} (4.6)

which gives one of upper bounds of the variance. Since $E_P\{f [\mathcal{X} \mathcal{Z}]^4\}$ does not depend on the measure $Q$, we can minimize the upper bound by minimizing $E_P\{(dP/dQ)^2\}$.

Consequently, we can determine the optimal measure $Q$ by solving the following conditional minimizing problem for the characteristics $(\eta, g(u))$;

$$\text{minimize} \quad E_P\left\{ \left( \frac{dP}{dQ} \right)^2 \right\}$$

subject to \quad $\tilde{E}_Q\{X_{t_d}\} - x_c = 0$  \hspace{1cm} (4.7)

(4.8)
5 Application to risk analysis for infrastructures

First, we apply the importance sampling simulation scheme constructed in this paper to a random damage growth model recently developed for tunnel concrete linings\[17\] as an important example of maintenance for infrastructures.

Let $X_t$ be a quantified damage degree at time $t$ for tunnel concrete linings, which is here supposed to obey the following stochastic differential equation;

$$dX_t = \mu X_t + X_{t-}dC_t, \quad X_0 = x_0 \text{ (a.s.)},$$  \hspace{1cm} (5.1)

where $\mu$ is a positive constant representing a damage growth resistance and $C = \{C_t; t \geq 0\}$ is a compound Poisson process drives random damage growth. A compound Poisson process $C$ is a Lévy process whose Lévy measure, here denoted by $m_C$, is uniformly integrable, i.e.,

$$\int_{|u|>0} m_C(du) \equiv \lambda < +\infty,$$  \hspace{1cm} (5.2)

which indicates that a measure $\nu_C$ defined as

$$\nu_C(A) = \frac{1}{\lambda} m_C(A),$$  \hspace{1cm} (5.3)

is a probability measure. Therefore, according to the basic property of Lévy processes, $C$ can be expressed as

$$C_t = \sum_{k=1}^{N_{t}^{\text{HPP}}} Y_k,$$  \hspace{1cm} (5.4)

in which $N_{t}^{\text{HPP}}$ is a temporally homogeneous Poisson process with an intensity $\lambda$ given by Eq.(5.2) and $\{Y_k\}$ is a set of i.i.d. random variables obeying the probability measure defined by Eq.(5.3). Since $C$ drives the damage growth, $Y_k > 0$ (a.s.) for $\forall k$.

The risk event is here supposed to be a failure of tunnel concrete linings, which is here supposed to occur when the damage degree exceeds a certain critical level $x_c (> x_0)$. Hence, the indicator functional is given by Eq.(4.1) with $A = \{X; X > x_c\}$.

As a Lévy measure of any compound Poisson process is uniformly integrable, $C$ is again a compound Poisson process under $Q$ provided that the function $g$ satisfies

$$\int_{u>0} g(u)m_C(du) \equiv \frac{\lambda^Q}{\lambda} < +\infty,$$  \hspace{1cm} (5.5)

where $\lambda^Q$ gives an intensity of $C$ under $Q$ and jumps $\{Y_k\}$ obeys a probability measure defined by

$$\nu_C^Q(A) = \frac{1}{\lambda^Q} m_C^Q(A) = \frac{1}{\lambda^Q} \int_{A} g(u)m_C(du).$$  \hspace{1cm} (5.6)

Applying the result obtained in Section 3, we can obtain the Radon-Nikodym derivative as

$$\frac{dP}{dQ} = \exp\left\{\left(\lambda^Q - \lambda\right)T - \hat{C}_T\right\}$$  \hspace{1cm} (5.7)

$$\hat{C}_t = \int_{|u|>0} \log g(u)\mu^C(du) = \sum_{k=1}^{N_{t}^{\text{HPP}}} \log g(Y_k)$$  \hspace{1cm} (5.8)
It should be noted that the characteristic $\eta$ is not needed in this measure transformation.

Equation (4.2) is here approximated as

$$\tilde{E}_Q(X_{t_d}) = x_0 \exp \left\{ (a + \lambda^Q q_1^Q) t_d \right\} = x_c, \quad (5.9)$$

$$q_1^Q = E_Q(Y_1) = \int_{0}^{\infty} u \nu_C^Q(du) = \int_{0}^{\infty} yg(y) \nu_C(dy), \quad (5.10)$$

where $x_0 = X_0$. Further, we can calculate Eq.(4.7) as

$$E_P \left\{ \left( \frac{dP}{dQ} \right)^2 \right\} = \exp \left\{ -3 \lambda T + \lambda T \int_{0}^{\infty} \{2g(y) + g(y)^{-2}\} \nu_C(dy) \right\} \quad (5.11)$$

Here we assume that $\{Y_k\}$ obeys an exponential distribution with mean $q_1$ under $P$, i.e.,

$$\nu_C(A) = \int_{A} \frac{1}{q_1} \exp \left\{ -\frac{y}{q_1} \right\} dy. \quad (5.12)$$

Then, if we assume that $\{Y_k\}$ also obeys an exponential distribution with mean $q_1^Q$, i.e.,

$$\nu_C(A) = \int_{A} \frac{1}{q_1^Q} \exp \left\{ -\frac{y}{q_1^Q} \right\} dy, \quad (5.13)$$

the function $g$ is obtained as

$$g(y) = \frac{\lambda^Q q_1}{\lambda q_1^Q} \exp \left\{ \left( \frac{1}{q_1} - \frac{1}{q_1^Q} \right) y \right\}. \quad (5.14)$$

Substituting Eqs.(5.9), (5.11) and (5.14) into Eqs.(4.7) and (4.8), we can reduce the optimization procedure as

$$\minimize_{q_1^Q > 2q_1/3} \frac{2w}{q_1^Q} - \frac{(q_1^Q)^5}{w^2 q_1^Q (3q_1^Q - 2q_1)} \quad (5.15)$$

$$w = \frac{1}{\lambda} \left( \frac{1}{T} \log \frac{x_c}{x_0} - a \right) \quad (5.16)$$

The optimal $q_1^Q$ is numerically obtained from Eq.(5.16), which determines the optimal intensity $\lambda^Q$ as

$$\lambda^Q = \frac{\lambda w}{q_1^Q} = \frac{1}{q_1^Q T} \left( \log \frac{x_c}{x_0} - a \right) \quad (5.17)$$

Figure 1 shows estimated $\psi(T)$ under parameters

$$x_0 = 2.0, \ x_c = 15.0, \ a = 5.0 \times 10^{-3}, \ q_1 = 0.06, \ \lambda = 0.5,$$

for $T = 10, 20$ and $30$, where vertical axis is logarithmically plotted. In Fig.1, solid crosses represent estimated $\psi(T)$ obtained by our proposed scheme with 100 samples and error bars show range of estimated values for ten times independent simulations. On the other hand, gray triangle represent estimated $\psi(T)$ obtained by crude Monte Carlo simulation, i.e., Monte Carlo simulation executed under the original measure $P$ with 5000 samples.
Fig.1 Estimated $\psi(T)$ by proposed scheme (solid crosses) and crude Monte Carlo scheme (gray triangle).

From the result, we can see that the proposed scheme works quite well for estimating very small $\psi(T)$ with only 100 samples for each simulation. However, the crude Monte Carlo simulation can not give estimation for $T \leq 20$ even though 5000 samples are generated in the simulation.

Table 1 shows optimally selected $\lambda^Q$ and $q_1^Q$ for each $T$. When $T$ is small, the intensity $\lambda^Q$ and mean $q_1^Q$ are magnified so that we can strongly accelerate the growth of $X$. The magnification ratio gradually decreases as $T$ increases.

<table>
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<th>$T = 10$</th>
<th>$T = 20$</th>
<th>$T = 30$</th>
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<td>$\lambda$</td>
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<td></td>
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</tr>
<tr>
<td>$q_1$</td>
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<td>$q_1^Q$</td>
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<td>0.1239</td>
<td>0.0939</td>
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Table 1 Comparison of parameters associated with the compound Poisson process $C_t$ under the original measure $P$ and the importance sampling measure $Q$.

6 Application to Credit Default Swap pricing

6.1 Firm asset dynamics

Next, we discuss an application of the proposed simulation scheme to pricing of Credit Default Swaps (CDS).

We suppose that dynamics of a firm asset, denoted by $X_t$, is given as a solution of the following stochastic differential equation[18];

$$dX_t = X_t - d\tilde{Z}_t,$$

(6.1)
where the driving noise $\tilde{Z}_t$ is given as
\[
\tilde{Z}_t = Z_t + \frac{1}{2} \langle Z, Z \rangle^c_t + \sum_{u \in (0,t]} \{ e^{\Delta Z_u} - 1 - \Delta Z_u \},
\] (6.2)
where $Z_t$ is supposed to be a temporally homogeneous Lévy process whose decomposition is given by Eq.(3.1). The explicit form of the solution $X_t$ is given as
\[
X_t = X_0 \exp(Z_t),
\] (6.3)
which gives an explicit expression of Eq.(2.1). It should be noted that the firm asset dynamics given by Eq.(6.3) is a natural extension of the well-known Black-Scholes model.

The risk event discussed here is a default of the firm. It is here assumed to occur when the firm asset $X_t$ falls below a prespecified default boundary $x_d(<X_0)$, i.e., the indicator functional is given as
\[
f[X] = \begin{cases} 
1 & (0 \leq t \leq T \text{ s.t. } X_t < x_d) \\
0 & \text{(otherwise)}
\end{cases}.
\] (6.4)
Then, the probability $\psi(T)$ represents probability of default up to time $T$. If we introduce a default time, denoted by $\tau_D$, as
\[
\tau_D = \inf\{ t; X_t < x_d \},
\] (6.5)
$\psi(T)$ can be expressed by the use of $\tau_D$ as
\[
\psi(T) = P(\tau_D \leq T).
\] (6.6)
Thus, $\psi(T)$ as a function of $T$ can be regarded as a probability distribution function of the default time $\tau_D$.

6.2 Pricing of CDS

Let us consider a CDS with maturity $T_0$. A discounted income of the protection buyer, denoted by $c_B$, is given as
\[
c_B = ye^{-rt}1_{(\tau_D \leq T_0)},
\] (6.7)
where $y$ is the protection value, $r$ is a risk-free interest rate and $1_A$ is an indicator function of event $A$. On the other hand, a discounted income of the protection seller, denoted by $c_S$, is given as
\[
c_S = \int_0^{T_0} qye^{-rs}1_{(\tau_D \geq s)} ds + Rey^{-rT_0}1_{(\tau_D \leq T_0)},
\] (6.8)
where $q$ is a premium rate of the CDS and $R$ is a recovery rate.

The CDS premium rate is determined under the condition that the expectation of $c_B$ coincides with the expectation of $c_S$ under the so-called equivalent martingale measure denoted by $P^*$, which can realize a kind of economical equilibrium. We denote $\psi(t)$ under $P^*$ as $\psi^*(t)$, then the expectations of $c_B$ and $c_S$ under $P^*$ are given as follows;
\[
E_{P^*}\{ c_B \} = \int_0^{T_0} ye^{-rt} d\psi^*(t),
\] (6.9)
\[ E_{P^*} \{ c_s \} = \int_{0}^{T_0} q y e^{-rt}(1 - \psi^*(t)) dt + R \int_{0}^{T_0} y e^{-rt} d\psi^*(t). \]  

(6.10)

The equilibrium condition is given as

\[ E_{P^*} \{ c_B \} = E_{P^*} \{ c_s \}. \]  

(6.11)

Therefore we can express the CDS premium rate \( q \) as follows;

\[ q = \frac{(1-R) \int_{0}^{T_0} e^{-rt} d\psi^*(t)}{\int_{0}^{T_0} e^{-rt}(1 - \psi^*(t)) dt}. \]  

(6.12)

### 6.3 Application of the proposed simulation scheme

The proposed simulation scheme can be applied to estimate the fair price of CDS through estimation of the probability of default \( \psi(T) \) as the author discussed in Ref.[19]. The estimation procedure consists of two steps. The first step is to transform the original probability measure \( P \) to the equivalent martingale measure \( P^* \), which is executed based upon a probability measure transformation discussed in Section 3. To avoid the so-called incompleteness, the minimal entropy principle[15] is applied. The second step is to transform the equivalent martingale measure \( P^* \) to the importance sampling measure \( Q \), which is executed just as discussed in Section 4.

### 6.4 Numerical examples

Here, we give numerical examples in which the Lévy process \( Z \) is a variance gamma process.

The variance gamma process is a Lévy process in which \( \sigma_B = q_B = 0 \) and its Lévy measure is given as

\[ m_{Z}(A) = \int_{A} \frac{p}{|u|} \exp\left(-\sqrt{\frac{2}{\sigma}}|u|\right) du, \]  

(6.13)

where \( p \) and \( \sigma \) are positive parameters characterizing the VG process. The VG process is frequently used for modeling random variation of stock price or firm asset. It should also be mentioned that more general VG process has been studied including three parameters[10].

We calculate the CDS premium rate by approximating integrals in Eq.(6.12) by the use of the trapezoidal rule by a time mesh \( T_i = i\Delta t, \Delta t = \frac{T_0}{M} \), i.e., its estimator with sample size \( N_M \), denoted by \( \hat{q}(N_M) \) is given as follows;

\[ \hat{q}(N_M) = \frac{(1-R) \sum_{i=1}^{M} \frac{e^{-rT_i} + e^{-rT_{i-1}}}{2}(\hat{\psi}^*(T_i;N_M) - \hat{\psi}^*(T_{i-1};N_M))}{\sum_{i=1}^{M} \frac{e^{-rT_i}(1 - \hat{\psi}^*(T_i;N_M)) + e^{-rT_{i-1}}(1 - \hat{\psi}^*(T_{i-1};N_M))}{2}(T_i - T_{i-1})} \]

\[ = \frac{(1-R) \sum_{i=1}^{M} \frac{e^{-rT_i} + e^{-rT_{i-1}}}{2} (\hat{\psi}^*(T_i;N_M) - \hat{\psi}^*(T_{i-1};N_M))}{\sum_{i=1}^{M} (e^{-rT_i}(1 - \hat{\psi}^*(T_i;N_M)) + e^{-rT_{i-1}}(1 - \hat{\psi}^*(T_{i-1};N_M)))} \]  

(6.14)
where $T_i = i \cdot \frac{T_0}{M}$ ($i = 1, 2, \cdots, M$).

Table 2 shows the estimated CDS premium rate with parameters

$$T_0 = 0.1, b = 0.099, p = 5.0, \sigma = 0.05, R = 0.$$  \hspace{1cm} (6.15)

<table>
<thead>
<tr>
<th>estimated $q$</th>
<th>crude MC ($N_M = 10^5 \times 10$)</th>
<th>proposed MC ($N_M = 10^4 \times 10$)</th>
<th>crude MC ($N_M = 10^8$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum value</td>
<td>0.00169715</td>
<td>0.00127754</td>
<td>0.00117775</td>
</tr>
<tr>
<td>average value</td>
<td>0.00111931</td>
<td>0.00113313</td>
<td></td>
</tr>
<tr>
<td>minimum value</td>
<td>0.00080056</td>
<td>0.00097704</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 CDS premium rate estimated by the proposed method compared with the crude Monte Carlo method\[^{19}\]. (Parameters are set as Eq.(6.15))

Table 2 shows that the proposed simulation method with $10^4$ samples can give quite good estimations of small CDS premium rate compared to the crude Monte Carlo method with $10^5$ samples. Therefore, we can expect that the proposed method is effective in the case when the value of the credit derivative in interest is derived from the credit risk of many firms, since the probability of the conjunction of many defaults is regularly quite small, even if these defaults are considered to be correlated with each other.

Next, supposing a long term case compared to Table 2, we show estimated CDS premium rate in Table 3, where parameters are set as

$$T_0 = 1.0, b = 0.099, p = 3.0, \sigma = 0.1, R = 0.$$ \hspace{1cm} (6.16)

in the same way as Table 2. Even though the accuracy of the default probability estimated by

<table>
<thead>
<tr>
<th>estimated $q$</th>
<th>crude MC ($N_M = 10^4 \times 10$)</th>
<th>proposed MC ($N_M = 10^4 \times 10$)</th>
<th>crude MC ($N_M = 10^8$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum value</td>
<td>0.00651128</td>
<td>0.00653323</td>
<td>0.00568284</td>
</tr>
<tr>
<td>average value</td>
<td>0.00552585</td>
<td>0.00571317</td>
<td></td>
</tr>
<tr>
<td>minimum value</td>
<td>0.00494841</td>
<td>0.00495594</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 CDS premium rate estimated by the proposed importance sampling method compared with the crude Monte Carlo method. (Parameters are set as Eq.(6.16))

the proposed method is quite good for small $T_i$, the accuracy of the estimated premium rate is almost same as the crude Monte Carlo method. It is due to that the integral in Eq.(6.14) mainly depends on the large default probability in the supposed time interval.
7 Conclusion

In this paper, we have briefly discussed a variance reduction technique realizing importance sampling in the Monte Carlo simulation based upon a probability measure transformation available for stochastic systems driven by Lévy processes. Two practical application have been shown for demonstrating the proposed simulation scheme which works quite well for estimating very small probability of risk event.

References


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