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A Note on Optimal Multiple Stopping in Spectrally Negative Lévy Models*

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1 Introduction

Consider a firm facing a decision of when to abandon or contract a project so as to maximize the total expected future cash flows. This problem is often referred to as the *abandonment option* or the *contraction option*. A typical formulation reduces to a standard optimal stopping problem, where the uncertainty of the future cash flow is driven by a stochastic process and the objective is to find a stopping time that maximizes the total expected cash flows realized until then. A more realistic extension is its multiple-stage version where the firm can withdraw from a project in stages.

In a standard formulation, given a discount rate $r>0$ and $X_t = x + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ for a standard Brownian motion W, $\mu \in \mathbb{R}$ and $\sigma > 0$, one wants to obtain a stopping time τ of X that maximizes the expectation $\mathbb{E}\left[\int_0^\tau e^{-rt}(e^{X_t}-\delta)\mathrm{d}t + e^{-r\tau}K1_{\{\tau<\infty\}}\right]$. The profit collected continuously is modeled as the geometric Brownian motion e^{X_t} less the constant operating expense $\delta \geq 0$. The value $K \in \mathbb{R}$ corresponds to the lump-sum benefits attained (or the costs incurred) at the time of abandonment. Here a technical assumption $r > \mu$ is commonly imposed so that the expectation is finite and the problem is non-trivial. The problem is rather simple mathematically; it reduces to the well-known perpetual American option. An explicit solution can be attained even when X is generalized to a Lévy process (see, e.g., Mordecki [18]).

In this paper, we generalize the classical model by extending from Brownian motion to a general Lévy process with negative jumps (spectrally negative Lévy process), and consider the optimal stopping problem of the form:

$$\sup_\tau \mathbb{E}\left[\int_0^\tau e^{-rt} f(X_t)\mathrm{d}t + e^{-r\tau} g(X_\tau) 1_{\{\tau<\infty\}}\right].$$

We obtain the optimal stopping time as well as the value function for the case f is increasing and g admits the form $g(x) = K - bx - \sum_{i=1}^N c_i e^{a_i x}$ for some positive constants a, b and c.

We further extend to the multiple-stage case where one wants to obtain a set of stopping times $\{\tau^{(m)}; 1 \leq m \leq M\}$ such that $0 = \tau^{(0)} \leq \tau^{(1)} \leq \cdots \leq \tau^{(M)}$ a.s. and achieve

$$\sup_{\tau^{(1)}\leq\cdots\leq\tau^{(M)}} \sum_{m=1}^M \mathbb{E}\left[\int_{\tau^{(m-1)}}^{\tau^{(m)}} e^{-rt} F_m(X_t)\mathrm{d}t + e^{-r\tau^{(m)}} g_m(X_{\tau^{(m)}}) 1_{\{\tau^{(m)}<\infty\}}\right]$$

when g_m and $f_m := F_m - F_{m+1}$ (with $F_{M+1} = 0$), for each $1 \leq m \leq M$, satisfy the same assumptions as in the one-stage case. The multiple-stopping problem arises frequently in real options (see e.g. [9]) and is well-studied particularly for the case X is driven by Brownian motion.

*This note is a short summary of [20].

In mathematical finance, similar problems are dealt in the valuation of swing options [7, 8] with refraction times between any consecutive stoppings.

In this paper, we take advantage of the recent advances in the theory of the spectrally negative Lévy process (see, e.g., [5, 15]). In particular, we use the results by Egami and Yamazaki [12], where they obtained and showed the equivalence of the continuous/smooth fit condition and the first-order condition in a general optimal stopping problem. Unlike the two-sided jump case, the identification of the candidate optimal stopping time can be conducted efficiently without intricate computation. The resulting value function can be written in terms of the scale function, which also can be computed efficiently by using, e.g., [13, 19]. The extension to the multiple-stage can be carried out without losing generality. The resulting optimal stopping times are of threshold type with possibly simultaneous stoppings, and the value function again admits the form in terms of the scale function. We refer the reader to, among others, [1, 2, 11, 16, 17] for optimal stopping problems and [3, 4, 10, 14] for optimal stopping games of spectrally negative Lévy processes.

The rest of the paper is organized as follows. In Section 2, we review the spectrally negative Lévy process and the scale function and then solve the one-stage problem. In Section 3, we extend it to the multiple-stage problem. For the proofs omitted in this note, see [20].

2 One-stage Problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a *spectrally negative Lévy process* $X = \{X_t : t \geq 0\}$ characterized uniquely by the *Laplace exponent*

$$\psi(\beta) := \log \mathbb{E}^0 \left[e^{\beta X_1} \right] = c\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(0,\infty)} (e^{-\beta z} - 1 + \beta z 1_{\{0<z<1\}})\,\Pi(\mathrm{d}z), \quad \beta \geq 0, \qquad (2.1)$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ and Π is a Lévy measure concentrated on $(0,\infty)$ such that $\int_{(0,\infty)}(1 \wedge z^2)\Pi(\mathrm{d}z) < \infty$. Here and throughout the paper $\mathbb{P}^x$ is the conditional probability where $X_0 = x \in \mathbb{R}$ and $\mathbb{E}^x$ is its expectation. We exclude the case X is a negative subordinator (decreasing a.s.) and we shall further assume that the Lévy measure does not have an atom.

2.1 Scale functions

For any spectrally negative Lévy process, there exists a function called the *(r-)scale function*

$$W^{(r)} : \mathbb{R} \to \mathbb{R},$$

which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0,\infty)$, and is characterized by the Laplace transform:

$$\int_0^\infty e^{-sx} W^{(r)}(x)\mathrm{d}x = \frac{1}{\psi(s) - r}, \qquad s > \Phi_r,$$

where

$$\Phi_r := \sup\{\lambda \geq 0 : \psi(\lambda) = r\}, \quad r \geq 0.$$

Here, the Laplace exponent ψ in (2.1) is known to be zero at the origin, convex on $[0,\infty)$; Φ_r is well-defined and is strictly positive whenever $r>0$. We also define

$$Z^{(r)}(x) := 1 + r\int_0^x W^{(r)}(y)\mathrm{d}y, \quad x \in \mathbb{R},$$

which is also called the scale function.

Define the first down- and up-crossing times of X by, respectively,

$$\tau_A := \inf\{t>0 : X_t \le A\} \quad \text{and} \quad \tau_A^+ := \inf\{t \ge 0 : X_t \ge A\} \quad A \in \mathbb{R},$$

with $\inf\emptyset = \infty$ by convention. Then, for any $0<x<b$,

$$\mathbb{E}^x\left[e^{-r\tau_b^+}1_{\{\tau_b^+<\tau_0\}}\right] = \frac{W^{(r)}(x)}{W^{(r)}(b)},$$
$$\mathbb{E}^x\left[e^{-r\tau_0}1_{\{\tau_b^+>\tau_0\}}\right] = Z^{(r)}(x) - Z^{(r)}(b)\frac{W^{(r)}(x)}{W^{(r)}(b)}.$$

As in Lemmas 8.3 and 8.5 of Kyprianou [15], for each $x>0$, the functions $r \mapsto W^{(r)}(x)$ and $r \mapsto Z^{(r)}(x)$ can be analytically extended to $r \in \mathbb{C}$. Fix $a \ge 0$ and define $\psi_a(\cdot)$, as the Laplace exponent of X under $\mathbb{P}_a$ with the change of measure $\left.\frac{\mathrm{d}\mathbb{P}_a^0}{\mathrm{d}\mathbb{P}^0}\right|_{\mathcal{F}_t} = \exp(aX_t - \psi(a)t)$, $t \ge 0$. If W_a and Z_a are the scale functions associated with X under $\mathbb{P}_a$ (or equivalently with $\psi_a(\cdot)$). Then, by Lemma 8.4 of [15],

$$W_a^{(r-\psi(a))}(x) = e^{-ax}W^{(r)}(x), \quad x \ge 0.$$

2.2 Results on the single stopping problem

This section considers the *one-stage optimal stopping* problem of the form

$$u(x) := \sup_{\tau\in\mathcal{S}} u(x,\tau), \tag{2.2}$$

with

$$u(x,\tau) \equiv u(x,\tau;f,g) := \mathbb{E}^x\left[\int_0^\tau e^{-rt}f(X_t)\mathrm{d}t + e^{-r\tau}g(X_\tau)1_{\{\tau<\infty\}}\right], \quad x \in \mathbb{R}, \tag{2.3}$$

where the supremum is taken over the set of stopping times with respect to the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge0}$ generated by X. Regarding the running payoff f and the terminal reward g, we assume the following.

Assumption 2.1 *1. $f(\cdot)$ is continuous, increasing, $f(-\infty) := \lim_{x\downarrow-\infty} f(x) > -\infty$ and $\int_0^\infty e^{-\Phi_r y}f_+(y)\mathrm{d}y < \infty$;*

2. $g(\cdot)$ admits the form

$$g(x) = K - bx - \sum_{i=1}^N c_i e^{a_i x}, \quad x \in \mathbb{R},$$

for some $K \in \mathbb{R}$, $b \ge 0$, and strictly positive constants a_i and c_i, $1 \le i \le N$, $N \ge 0$ such that $a_i \ne a_j$ for any $i \ne j$,

3. *for the case $b > 0$, $\mathbb{E}^0 X_1 = \psi'(0+) \in (-\infty, \infty)$.*

The results discussed in this section are applications of Egami and Yamazaki [12] and will be extended to the multiple-stage problem in the next section.

For every $A \in \mathbb{R}$, we define

$$\Lambda(A) := -\frac{r}{\Phi_r}K + b\Big(\frac{r}{\Phi_r^2} + \frac{rA - \psi'(0+)}{\Phi_r}\Big) + \sum_{i=1}^{N} c_i e^{a_i A}\varpi_r(a_i) + \Psi_f(A), \tag{2.4}$$

where

$$\varpi_r(a) := \left\{ \begin{array}{ll} \frac{r-\psi(a)}{\Phi_r - a}, & a \neq \Phi_r \\ \psi'(\Phi_r) = \lim_{a\to\Phi_r} \frac{r-\psi(a)}{\Phi_r - a}, & a = \Phi_r \end{array} \right\}, \quad a > 0,$$

$$\Psi_f(A) := \int_0^\infty e^{-\Phi_r y} f(y + A)\mathrm{d}y, \quad A \in \mathbb{R}.$$

By the monotonicity of ψ, $\varpi_r(a) > 0$ for any $a > 0$. Moreover, by Assumption 2.1, $\Psi_f(A)$ is monotonically increasing. Now, in view of (2.4) above, the function $\Lambda(A)$ is continuous and increasing. Therefore, if $\lim_{A\downarrow-\infty}\Lambda(A) < 0 < \lim_{A\uparrow\infty}\Lambda(A)$, there exists a unique root $A^* \in \mathbb{R}$ such that $\Lambda(A^*) = 0$. Otherwise, let $A^* = -\infty$ if $\lim_{A\downarrow-\infty}\Lambda(A) \geq 0$ and let $A^* = \infty$ if $\lim_{A\uparrow\infty}\Lambda(A) \leq 0$.

Remark 2.1 *Except for the case g is a constant, because $\Lambda(A)$ increases to ∞, we have $A^* < \infty$.*

The following is an application of Egami and Yamazaki [12].

Proposition 2.1 *1. If $-\infty < A^* < \infty$, the stopping time $\tau_{A^*} := \inf\{t \geq 0 : X_t \leq A^*\}$ is optimal and the value function is $u(x) = u_{A^*}(x)$ where*

$$u_{A^*}(x) = KZ^{(r)}(x - A^*) - b\Big[\overline{Z}^{(r)}(x - A^*) + \big(A^* - \frac{\psi'(0+)}{r}\big)Z^{(r)}(x - A^*) + \frac{\psi'(0+)}{r}\Big]$$
$$- \sum_{i=1}^{N} c_i e^{a_i x} Z_{a_i}^{(r-\psi(a_i))}(x - A^*) - \Theta_f(x; A^*),$$

with $\overline{Z}^{(r)}(x) := \int_0^x Z^{(r)}(y)\mathrm{d}y$, $x \in \mathbb{R}$, and

$$\Theta_f(x; A) := \left\{ \begin{array}{ll} \int_A^x W^{(r)}(x - y) f(y)\mathrm{d}y, & x > A, \\ 0, & x \leq A. \end{array} \right.$$

2. *If $A^* = \infty$, immediate stopping is always optimal and $u(x) = g(x)$ for any $x \in \mathbb{R}$.*

3. *If $A^* = -\infty$, it is never optimal to stop (i.e. $\tau^* = \infty$ is optimal), and the value function is given by*

$$u(x) = \int_{-\infty}^{\infty} \Big(\Phi_r' e^{-\Phi_r(x-y)} - W^{(r)}(y - x)\Big) f(y)\mathrm{d}y,$$

where Φ_r' is the derivative of Φ_r with respect to r.

3 Multiple-stage Problem

In this section, we extend to the scenario where the firm can decrease its involvement in the project in multiple stages; we solve

$$U^{(M)}(x) := \sup_{(\tau^{(1)},\ldots,\tau^{(M)})\in\mathcal{S}_M} \sum_{m=1}^{M} \mathbb{E}^x\left[\int_{\tau^{(m-1)}}^{\tau^{(m)}} e^{-rt}F_m(X_t)\mathrm{d}t + e^{-r\tau^{(m)}} g_m(X_{\tau^{(m)}})1_{\{\tau^{(m)}<\infty\}}\right], \tag{3.1}$$

for all $x \in \mathbb{R}$ where we define $\tau^{(0)} := 0$ for notational brevity and the supremum is over the set of increasing sequences of M stopping times,

$$\mathcal{S}_M := \{\tau^{(m)} \in \mathcal{S}, 1 \le m \le M : \tau^{(1)} \le \cdots \le \tau^{(M)}\}.$$

We first consider the case g_m admits the form

$$g_m(x) := K_m - b_m x - \sum_{i=1}^{N_m} c_{mi} e^{a_{mi}x} \quad 1 \le m \le M,$$

for some constants $K_m \in \mathbb{R}$, $b_m \ge 0$, $c_{mi}, a_{mi} > 0$, $1 \le i \le N_m$, and show the optimality as an extension of Proposition 2.1. Regarding the running reward function F, define the differences:

$$f_m := F_m - F_{m+1}, \quad 1 \le m \le M,$$

with $F_{M+1} \equiv 0$. As is also assumed in [6], we consider the case f_m is increasing. Using the notation as in (2.3), we can then write for all $x \in \mathbb{R}$

$$U^{(M)}(x) = \sup_{(\tau^{(1)},\ldots,\tau^{(M)})\in\mathcal{S}_M} \sum_{m=1}^{M} u(x, \tau^{(m)}; f_m, g_m). \tag{3.2}$$

Assumption 3.1 *For each $1 \le m \le M$, we assume that f_m and g_m satisfy Assumption 2.1.*

As is clear from the problem structure, simultaneous stoppings (i.e. $\tau_k = \cdots = \tau_{k+l}$ a.s. for some k and l) may be optimal in case it is not advantageous to stay in some intermediate stages (i.e. stages $k+1, \ldots, k+l$). For this reason, define, for any subinterval $\mathcal{I} = \{\min\mathcal{I}, \min\mathcal{I} + 1, \ldots, \max\mathcal{I}\} \subset \{1, \ldots, M\}$,

$$g_{\mathcal{I}} := \sum_{i\in\mathcal{I}} g_i \quad \text{and} \quad f_{\mathcal{I}} := F_{\min\mathcal{I}} - F_{\max\mathcal{I}+1}, \tag{3.3}$$

and consider an *auxiliary one-stage problem* (2.2) with $g = g_{\mathcal{I}}$ and $f = f_{\mathcal{I}}$. Notice that Assumption 3.1 guarantees that $f_{\mathcal{I}}$ and $g_{\mathcal{I}}$ also satisfy Assumption 2.1 for any $\mathcal{I}$. Hence Proposition 2.1 applies.

Let

$$\Lambda_m(A) := \Lambda(A; f_m, g_m), \quad A \in \mathbb{R}, 1 \le m \le M,$$

as the function (2.4) for (f_m, g_m). We see that

$$\Lambda_{\mathcal{I}}(A) := \Lambda(A; f_{\mathcal{I}}, g_{\mathcal{I}}) = \Lambda\Big(A; \sum_{m\in\mathcal{I}} f_m, \sum_{m\in\mathcal{I}} g_m\Big) = \sum_{m\in\mathcal{I}} \Lambda_m(A) \tag{3.4}$$

is increasing and corresponds to the function (2.4) for $(f_{\mathcal{I}}, g_{\mathcal{I}})$. Now let $A^*_{\mathcal{I}}$ be the root of $\Lambda_{\mathcal{I}}(A) = 0$ if it exists. If $\lim_{A\uparrow\infty} \Lambda_{\mathcal{I}}(A) \leq 0$, we set $A^*_{\mathcal{I}} = \infty$; if $\lim_{A\downarrow-\infty} \Lambda_{\mathcal{I}}(A) \geq 0$, we set $A^*_{\mathcal{I}} = -\infty$. For simplicity, let $A^*_m := A^*_{\{m\}}$ for any $1 \leq m \leq M$. Also define

$$u^{\mathcal{I}}_A(x) := u(x, \tau_A; f_{\mathcal{I}}, g_{\mathcal{I}}), \quad x, A \in \mathbb{R}.$$

With these notations, the following is immediate by Proposition 2.1.

Corollary 3.1 *Fix any $\mathcal{I}$ and $x \in \mathbb{R}$, and consider the problem:*

$$u_{\mathcal{I}}(x) := \sup_{\tau\in\mathcal{S}} u(x, \tau; f_{\mathcal{I}}, g_{\mathcal{I}}).$$

1. *If $-\infty < A^*_{\mathcal{I}} < \infty$, then*

$$\begin{aligned} u_{\mathcal{I}}(x) &= u^{\mathcal{I}}_{A^*_{\mathcal{I}}}(x) \\ &= \sum_{m\in\mathcal{I}} \Big(K_m Z^{(r)}(x - A^*_{\mathcal{I}}) - b_m\Big[\overline{Z}^{(r)}(x - A^*_{\mathcal{I}}) + \big(A^*_{\mathcal{I}} - \frac{\psi'(0+)}{r}\big)Z^{(r)}(x - A^*_{\mathcal{I}}) + \frac{\psi'(0+)}{r}\Big] \\ &\quad - \sum_{i=1}^{N_m} c_{mi} e^{a_{mi}x} Z^{(r-\psi(a_{mi}))}_{a_{mi}}(x - A^*_{\mathcal{I}})\Big) - \Theta_{f_{\mathcal{I}}}(x; A^*_{\mathcal{I}}), \end{aligned}$$

and the stopping time $\tau_{A^*_{\mathcal{I}}} := \inf\{t > 0 : X_t \leq A^*_{\mathcal{I}}\}$ is optimal.

2. *If $A^*_{\mathcal{I}} = \infty$, $u_{\mathcal{I}}(x) = g_{\mathcal{I}}(x)$ for any $x \in \mathbb{R}$ with optimal stopping time $\tau^* = 0$.*

3. *If $A^*_{\mathcal{I}} = -\infty$, it is never optimal to stop, and the value function is given by*

$$u_{\mathcal{I}}(x) = \int_{-\infty}^{\infty} \big(\Phi'_r e^{-\Phi_r(x-y)} - W^{(r)}(y - x)\big) f_{\mathcal{I}}(y) \mathrm{d}y.$$

3.1 Two-stage problem

In order to gain intuition, we first consider the case with $M = 2$ and obtain $U^{(2)}(x)$ under Assumption 3.1. Following the procedures discussed above, $A^*_m \in [-\infty, \infty]$, or the root of $\Lambda_m(A) = 0$, is well-defined for $m = 1, 2$. As a special case of (3.3),

$$f_2 \equiv F_2, \quad f_1 \equiv F_1 - F_2 \equiv F_1 - f_2, \quad \text{and} \quad f_{\{1,2\}} \equiv F_1 \equiv f_1 + f_2.$$

We shall consider the cases (i) $A^*_1 > A^*_2$ and (ii) $A^*_1 \leq A^*_2$, separately. For (i), we shall show that $(\tau_{A^*_1}, \tau_{A^*_2})$ is optimal. For (ii), we shall show that simultaneous stoppings are optimal. We first consider the former.

Proposition 3.1 *If $\infty \geq A_1^* > A_2^* \geq -\infty$, then $(\tau_{A_1^*}, \tau_{A_2^*})$ is optimal; the value function is given by $U^{(2)}(x) = \sum_{m=1,2} u_{A_m^*}^{\{m\}}(x)$. In particular, if $\infty > A_1^* > A_2^* > -\infty$,*

$$U^{(2)}(x) = \sum_{m=1,2} \Big(K_m Z^{(r)}(x - A_m^*) - b_m \Big[\overline{Z}^{(r)}(x - A_m^*) + \big(A_m^* - \frac{\psi'(0+)}{r}\big) Z^{(r)}(x - A_m^*) + \frac{\psi'(0+)}{r} \Big] \\ - \sum_{i=1}^{N_m} c_{mi} e^{a_{mi} x} Z_{a_{mi}}^{(r-\psi(a_{mi}))}(x - A_m^*) \Big) \\ - \int_{A_1^*}^{x} W^{(r)}(x-y) F_1(y) \mathrm{d}y - \int_{A_2^*}^{A_1^*} W^{(r)}(x-y) F_2(y) \mathrm{d}y.$$

Now consider the case $-\infty \leq A_1^* \leq A_2^* \leq \infty$.

Lemma 3.1 *Suppose $-\infty \leq A_1^* \leq A_2^* \leq \infty$. Under Assumption 3.1, the first optimal stopping cannot occur on (A_2^*, ∞); namely if $\tau^{*(1)}$ is the optimal first stopping time in the sense of* (3.1), *then $X_{\tau^{*(1)}} \in (-\infty, A_2^*]$ a.s. on $\{\tau^{*(1)} < \infty\}$.*

The following lemma suggests under $-\infty \leq A_1^* \leq A_2^* \leq \infty$ that the optimal strategy is the simultaneous stopping corresponding to the threshold level $A_{\{1,2\}}^*$, which is the value that makes $\Lambda_{\{1,2\}} \equiv \Lambda_1 + \Lambda_2$ as in (3.4) vanish.

Proposition 3.2 *Suppose $-\infty \leq A_1^* \leq A_2^* \leq \infty$.*

1. *We have $A_1^* \leq A_{\{1,2\}}^* \leq A_2^*$.*

2. *It is optimal to stop simultaneously and the value function is given by $U^{(2)}(x) = u_{A_{\{1,2\}}^*}^{\{1,2\}}(x)$ under Assumption 3.1.*

3.2 Multiple-stage problem

We now generalize to the multiple-stage problem and solve (3.1) or equivalently (3.2) with $M \geq 3$. For $1 \leq m \leq M$, let

$$U_m^{(M)}(x) := \sup_{(\tau^{(m)}, \ldots, \tau^{(M)}) \in \mathcal{S}_{M-m+1}} \sum_{k=m}^{M} u(x, \tau^{(k)}; f_k, g_k). \tag{3.5}$$

In particular, $U^{(M)} \equiv U_1^{(M)}$ and by Corollary 3.1

$$U_M^{(M)}(x) = \sup_{\tau \in \mathcal{S}} u(x, \tau; f_M, g_M) = u_{A_M^*}^{\{M\}}(x).$$

The expressions for $U_{M-1}^{(M)}$ can also be obtained as in the two-stage case.

Given $1 \leq m \leq M$, let us partition $\{m, m+1, \ldots, M\}$ to an $L(m)$ number of (non-empty) disjoint sets $\mathcal{I}_m := \{\mathcal{I}(k; m), 1 \leq k \leq L(m)\}$ such that

$$\{m, m+1, \ldots, M\} = \mathcal{I}(1; m) \cup \cdots \cup \mathcal{I}(L(m); m)$$

where, if $L(m) = 1$, $\mathcal{I}(1;m) = \{m, \ldots, M\}$ and, if $L(m) \geq 2$,

$$\begin{aligned} \mathcal{I}(1;m) &:= \{m, \ldots, n_{1,m} - 1\}, \\ \mathcal{I}(l;m) &:= \{n_{l-1,m}, \ldots, n_{l,m} - 1\}, \quad 2 \leq l \leq L(m) - 1, \\ \mathcal{I}(L(m);m) &:= \{n_{L(m)-1,m}, \ldots, M\}, \end{aligned}$$

for some integers $m < n_{1,m} < \cdots < n_{L(m)-1,m} < M$. We consider the strategy such that, if k and l are in the same set, then the k-th and l-th stops occur simultaneously a.s.

We shall show that (3.5), for any $1 \leq m \leq M$, can be solved by a strategy with some partition $\mathcal{I}_m^* := \{\mathcal{I}^*(k;m), 1 \leq k \leq L^*(m)\}$ satisfying

$$A^*_{\mathcal{I}^*(1;m)} > \cdots > A^*_{\mathcal{I}^*(L^*(m);m)},$$

where $A^*_{\mathcal{I}}$ is defined as in (3.4) for any set $\mathcal{I}$. The corresponding expected value becomes

$$U^{(M)}_{m,\mathcal{I}^*_m}(x) := \sum_{k=1}^{L^*(m)} u(x, \tau_{A^*_{\mathcal{I}^*(k;m)}}; f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k;m)}) = \sum_{k=1}^{L^*(m)} u^{\mathcal{I}^*(k;m)}_{A^*_{\mathcal{I}^*(k;m)}}(x), \tag{3.6}$$

whose strategy is given by for any $m \leq n \leq M$,

$$\tau^{*(n)} = \tau_{A^*_{\mathcal{I}^*(k;m)}} \quad \text{for the unique } 1 \leq k \leq L^*(m) \text{ such that } n \in \mathcal{I}^*(k;m).$$

We shall show that (3.6) is optimal, i.e. $U_m^{(M)} = U^{(M)}_{m,\mathcal{I}^*_m}$ under Assumption 3.1 for any $1 \leq m \leq M$. Moreover, $\mathcal{I}^*_m$ can be obtained inductively moving backwards starting from $\mathcal{I}^*_M$ with $L^*(M) = 1$ and $\mathcal{I}^*(1;M) = \{M\}$. For the inductive step, the following algorithm outputs $\mathcal{I}^*_{m-1}$ from $\mathcal{I}^*_m$ for any $2 \leq m \leq M$. By repeating this, we can obtain the partition $\mathcal{I}^*_1$; the resulting $U^{(M)}_{1,\mathcal{I}^*_1}$ as in (3.6) becomes the value function $U^{(M)} = U^{(M)}_1$.

Algorithm $\mathcal{I}^*_{m-1}$ = Update($\mathcal{I}^*_m, m$)

Step 1 Set $i = 1$.

Step 2 Set

$$\hat{\mathcal{I}} := \begin{cases} \{m-1\}, & i = 1, \\ \{m-1\} \cup \mathcal{I}^*(1;m) \cup \cdots \cup \mathcal{I}^*(i-1;m), & i \geq 2. \end{cases}$$

Step 3 Compute $A^*_{\hat{\mathcal{I}}}$ and

1. if $i = L^*(m) + 1$, then **stop and return** $\mathcal{I}^*_{m-1} = \{\mathcal{I}^*(1;m-1)\}$ with $L^*(m-1) = 1$ and $\mathcal{I}^*(1;m-1) = \{m-1, \ldots, M\}$;
2. if $A^*_{\hat{\mathcal{I}}} > A^*_{\mathcal{I}^*(i;m)}$, then **stop and return** $\mathcal{I}^*_{m-1} = \{\mathcal{I}^*(k;m-1), 1 \leq k \leq L^*(m-1)\}$ with $L^*(m-1) = L^*(m) - i + 2$ and

$$\mathcal{I}^*(1;m-1) = \hat{\mathcal{I}} \quad \text{and} \quad \mathcal{I}^*(l;m-1) = \mathcal{I}^*(l+i-2;m), \quad 2 \leq l \leq L^*(m-1);$$

3. if $A^*_{\hat{\mathcal{I}}} \leq A^*_{\mathcal{I}^*(i;m)}$, set $i = i+1$ and go back to **Step 2**.

The role of the algorithm is in words to extend from $n(= M - m + 1)$-stage problem to $n + 1(= M - m + 2)$-stage problem. The idea is similar to what we discussed in the previous section on how to extend from a one-stage problem to a two-stage problem. When a new initial stage is added, the corresponding threshold value $A^*_{\hat{\mathcal{I}}}$ is first calculated. Depending on whether its value is higher than that of the subsequent stages or not, simultaneous stoppings may become optimal. For n larger than two, we must solve it recursively by keeping updating the set $\hat{\mathcal{I}}$, or the set of the first (simultaneous) stoppings, as given in this algorithm. If $A^*_{\hat{\mathcal{I}}}$ is low, the strategy of the new initial stage may naturally depend on the strategies of all the subsequent stages. Unlike the extension to the two-stage problem which only needs to take into account the strategy of the stage immediately next, it needs to reflect the strategies of all subsequent stages.

Lemma 3.2 *In view of the algorithm above, suppose Assumption 3.1 and fix* $2 \leq m \leq M$. *Given that* $\mathcal{I}^*_m$ *satisfies, for every* $1 \leq l \leq L^*(m)$,

$$U^{(M)}_{\min \mathcal{I}^*(l;m)}(x) = \sum_{k=l}^{L^*(m)} u^{\mathcal{I}^*(k;m)}_{A^*_{\mathcal{I}^*(k;m)}}(x), \tag{3.7}$$

and is used as an input in the algorithm. Then, we have the following.

1. *At the end of* ***Step 2***, *if* $1 \leq i \leq L^*(m)$,

$$U^{(M)}_{m-1}(x) \leq \sup_{\tau \in \mathcal{S}} \Big[u(x, \tau; f_{\hat{\mathcal{I}}}, g_{\hat{\mathcal{I}}}) + \sum_{k=i}^{L^*(m)} u(x, \vartheta_\tau(A^*_{\mathcal{I}^*(k;m)}); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k;m)}) \Big]$$

where $\vartheta_\nu(A) := \nu + \tau_A \circ \theta_\nu$ *for any* $\nu \in \mathcal{S}$ *and* $A \in \mathbb{R}$ *with the time-shift operator* θ_t, *and if* $i = L^*(m) + 1$

$$U^{(M)}_{m-1}(x) = \sup_{\tau \in \mathcal{S}} u(x, \tau; f_{\hat{\mathcal{I}}}, g_{\hat{\mathcal{I}}}) = u^{\{m-1,\dots,M\}}_{A^*_{\{m-1,\dots,M\}}}(x).$$

2. *Let* $\mathcal{I}^*_{m-1}$ *be produced by the algorithm. For any* $1 \leq l \leq L^*(m-1)$,

$$U^{(M)}_{\min \mathcal{I}^*(l;m-1)}(x) = \sum_{k=l}^{L^*(m-1)} u^{\mathcal{I}^*(k;m-1)}_{A^*_{\mathcal{I}^*(k;m-1)}}(x).$$

Using Lemma 3.2 as an inductive step, the main theorem is immediate. Indeed, (3.7) holds trivially for M by Corollary 3.1. By applying the algorithm $M - 1$ times, we can obtain (3.1) for $M - 1, M - 2, \dots, 1$.

Theorem 3.1 *Let* $\{\mathcal{I}^*_m; 1 \leq m \leq M\}$ *be produced by the algorithm. Under Assumption 3.1, for every* $1 \leq m \leq M$ *and* $1 \leq i \leq L^*(m)$,

$$U^{(M)}_{\min \mathcal{I}^*(i;m)}(x) = \sum_{k=i}^{L^*(m)} u^{\mathcal{I}^*(k;m)}_{A^*_{\mathcal{I}^*(k;m)}}(x).$$

In particular,

$$U^{(M)}(x) \equiv U^{(M)}_{\min \mathcal{I}^*(1;1)}(x) = \sum_{k=1}^{L^*(1)} u^{\mathcal{I}^*(k;1)}_{A^*_{\mathcal{I}^*(k;1)}}(x).$$

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