数理解析研究所講究録
第 1886 巻 2014 年 16-22

A Two-Period Model of Capital Investment under Ambiguity*

Motoh Tsujimura
Faculty of Commerce, Doshisha University

1 Introduction

An important factor in economic decision making is the treatment of uncertainty. Knight (1921) defines two kinds of uncertainty: risk, under which the probability of an outcome is uniquely determined; and uncertainty, under which it is not. The latter is termed Knightian uncertainty or deep uncertainty. In this paper, following Ellseberg (1961), we term Knightian uncertainty ambiguity. For a survey of decision making under uncertainty, see, for example, Camerer and Weber (1992), Etner et. al. (2012) and Guidolin and Rinaldi (2013).

We examine capital investment under ambiguity in a two-period setting. We extend the model of Miao (2004), who investigates optimal consumption under ambiguity in a two-period setting. We analyze a production economy and derive optimal capital investment in a general-equilibrium setting. Suppose that there are a large number of identical consumers and firms in an economy. For analytical simplicity, the number of consumers is equal to that of firms, and consumers own the firms. This enables us to consider a representative consumer and firm. The representative consumer is risk averse and has a constant absolute risk aversion utility function. Because there is ambiguity, the representative consumer considers a set of probability distributions. Then, we formulate the utility function as the multiple-priors expected utility of Gilboa and Schmeidler (1989). We formulate the central planner's problem and derive the optimal level of capital investment. Furthermore, we analyze the comparative static effects of the model's parameters. We find that ambiguity aversion and risk aversion have different effects on capital investment. The more ambiguity averse the central planner, the higher the capital investment. By contrast, at a low level of risk aversion, the more risk averse the central planner, the lower the capital investment. Once risk aversion has reached a certain level, increased risk aversion stimulates capital investment.

The rest of the paper is organized as follows. In Section 2, we describe the setup of the production economy and formulate the central planner's problem. In Section 3, we solve the central planner's problem. In Section 4, we conduct a numerical analysis. Section 5 concludes the paper.

2 The Model

We consider a two-period production economy. There are a large number of identical consumers and firms. The number of consumers is equal to that of firms. The firms are owned by the consumers and produce identical outputs. We consider a representative consumer and firm.

*This research was supported in part by a Grant-in-Aid for Scientific Research (No. 24510213) from the Japan Society for the Promotion of Science.

The representative consumer receives an endowment Y_t in each period t ($t = 1, 2$). This endowment is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. The representative firm produces output by using capital K. The firm's production function $f(k)$ is given by:

$$f(k) = Ak^{\alpha}, \tag{2.1}$$

where $A > 0$ reflects the level of technology and $\alpha > 0$ is the output elasticity of capital. The consumer receives utility from consumption C_t in each period. The utility function $u(c)$ is assumed to be given by:

$$u(c) = -\frac{1}{\theta}\mathrm{e}^{-\theta c}, \tag{2.2}$$

where the coefficient $\theta > 0$ is the degree of absolute risk aversion. The representative consumer maximizes the utility u from consumption subject to the following intertemporal budget constraint:

$$C_1 + K = Y_1, \tag{2.3}$$

$$C_2 = Y_2 + (1 - \delta)K + f(K), \tag{2.4}$$

where $\delta \in (0, 1)$ is the depreciation rate of capital. Suppose that the representative consumer does not uniquely determine the probability distribution of future endowments but instead considers a set of probability distributions. Then, we formulate the representative consumer's utility function as the multiple-priors expected utility of Gilboa and Schmeidler (1989):

$$U(C_1, C_2) = u(C_1) + \beta \min_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[u(C_2)], \tag{2.5}$$

where $\beta \in (0, 1)$ is a discount factor and $\mathcal{P}$ is a set of priors over $(\Omega, \mathcal{F})$. Following Miao (2004) and Kogan and Wang (2002), we define $\mathcal{P}$ as:

$$\mathcal{P}(\mathbb{P}, \phi) = \left\{\mathbb{Q} \in \mathcal{M}(\Omega); \mathbb{E}_{\mathbb{Q}}\left[\ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] \leq \phi^2\right\}, \tag{2.6}$$

where $\mathcal{M}(\Omega)$ is the set of probability measures on Ω, $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is the Radon–Nikodym derivative and $\mathbb{E}_{\mathbb{Q}}[\ln(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})]$ is the relative entropy index.[2] This specification is based on robust control theory.[3] The parameter $\phi > 0$ represents ambiguity aversion. The higher ϕ, the more ambiguity averse the representative consumer.

We assume that $\mathbb{P}$ is the probability measure of the normal distribution with mean μ and variance σ^2. All probability measures in $\mathcal{P}(\mathbb{P}, \phi)$ have normal distributions. $\mathbb{Q}$ is the probability measure of the normal distribution with mean $\mu - h$ and variance σ^2, where $h > 0$ represents the mean distortion chosen by the decision maker. Then, the relative entropy of $\mathbb{P}$ and $\mathbb{Q}$ is given by:

$$\mathbb{E}_{\mathbb{Q}}\left[\ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] = \frac{h^2}{2\sigma^2}. \tag{2.7}$$

The derivation of (2.7) is in Appendix A.

[2] This is also termed the Kullback–Leibler divergence.

[3] See, for example, Hansen and Sargent (2001).

The representative firm maximizes profits, given prices and technology. We formulate the central planner's problem as:

$$\max_{\{C_1,C_2,K\}} U(C_1,C_2), \tag{2.8}$$
$$\text{s.t. (2.3) and (2.4).}$$

Rewriting the central planner's problem yields:

$$\max_{\{K\}}\left\{-\frac{1}{\theta}\mathrm{e}^{-\theta(Y_1-K)}+\beta\min_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}_{\mathbb{Q}}\left[-\frac{1}{\theta}\mathrm{e}^{-\theta[Y_2+(1-\delta)K+AK^{\alpha}]}\right]\right\}. \tag{2.9}$$

In the next section, we solve problem (2.9) and derive the optimal level of capital investment.

3 Optimal Capital Investment

In this section, we derive optimal capital investment.

From (2.9), we obtain:

$$-\theta Y_1+\theta K=\ln\beta[(1-\delta)+\alpha AK^{\alpha-1}]-\theta[(1-\delta)K+AK^{\alpha}]+\ln\left(\max_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-\theta Y_2}]\right). \tag{3.1}$$

From the relative entropy expression, (2.7), we obtain:

$$\begin{aligned}\ln\left(\max_{\mathbb{Q}\in\mathcal{P}}\mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{-\theta Y_2}]\right)&=\ln\left(\max_{h}[\mathrm{e}^{-\theta(\mu-h)+\theta^2\sigma^2/2}]\right)\\&=-\theta(\mu-\sqrt{2}\sigma\phi)+\theta^2\sigma^2/2.\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1) yields:

$$-\theta AK^{\alpha}+\ln\beta[(1-\delta)+\alpha AK^{\alpha-1}]-\theta(2-\delta)K+\theta[Y_1-(\mu-\sqrt{2}\sigma\phi)]+\frac{\theta^2\sigma^2}{2}=0. \tag{3.3}$$

The optimal level of capital investment K^* is derived from (3.3). Optimal consumption C_1^* is $C_1^*=Y_1-K^*$. If the production function exhibits constant return to scale ($\alpha=1$), we obtain the explicit solution:

$$K^*=\frac{y_1}{2-\delta+A}+\frac{\log\beta(1-\delta+A)}{\theta(2-\delta+A)}-\frac{\mu-\sqrt{2}\sigma\phi}{2-\delta+A}+\frac{\theta\sigma^2}{2(2-\delta+A)}. \tag{3.4}$$

Otherwise, it is impossible to obtain an explicit formula for K^*. In the next section, we numerically derive optimal capital investment.

4 Numerical Examples

In this section, we numerically calculate the optimal level of capital investment K^* and investigate its response to parameter changes. The basic parameter values are as follows: $Y_1=10$; $\delta=0.5$; $A=3$; $\alpha=0.75$; $\beta=0.95$; $\mu=5$; $\sigma=2$; $\theta=1$; $\phi=1$. Given these values, optimal capital investment K^* is 2.767, and optimal consumption in period 1 C_1^* is 7.233.

Figures 1–5 illustrate the results of the comparative statics analysis for optimal capital investment K^*. Figure 1 shows that although optimal capital investment K^* is initially decreasing in the coefficient of absolute risk aversion θ, once risk aversion has reached a certain level ($\theta = 0.621$ in the base case), K^* increases in θ. This result implies that a central planner who is barely risk averse initially will cut capital investment once he or she becomes more risk averse. However, once a certain level of risk aversion has been reached, the central planner's capital investment increases with his or her risk aversion.

Figure 2 shows that optimal capital investment K^* is increasing in the degree of ambiguity aversion ϕ. Recall that a decision maker with a higher value of θ is more averse to uncertainty. Figure 2 implies that a central planner who is more averse to ambiguity invests more in capital. This generates wealth in period 2.

Figure 3 shows that optimal capital investment is increasing in the volatility of endowments σ. Such volatility leads a central planner to invest more in capital so that the risk of having less wealth in period 2 is avoided.

Figure 4 shows that optimal capital investment is decreasing in the technology, represented by the parameter A. Technological advance causes less capital to be needed by raising output.

Figure 5 shows that optimal capital investment is decreasing in the output elasticity of capital α. This is because output increases in the output elasticity of capital.

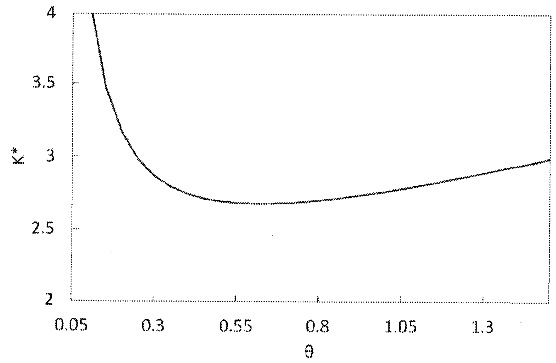

Figure 1: Comparative static effects of θ on optimal capital investment

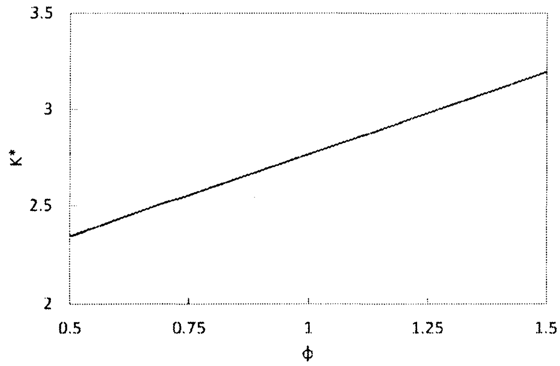

Figure 2: Comparative static effects of ϕ on optimal capital investment

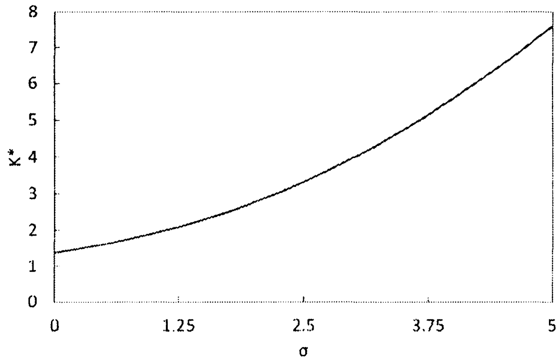

Figure 3: Comparative static effects of σ on optimal capital investment

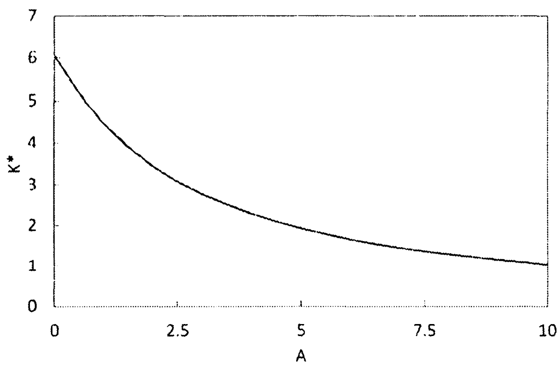

Figure 4: Comparative static effects of A on optimal capital investment

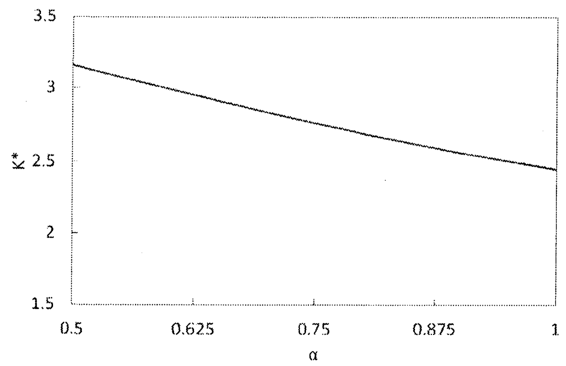

Figure 5: Comparative static effects of α on optimal capital investment

5 Conclusion

In this paper, we analyzed capital investment under ambiguity in a two-period setting. We solved the central planner's problem and numerically derived the optimal level of capital investment. Comparative statics analysis revealed that ambiguity aversion and risk aversion affect capital investment differently.

There are several ways to extend this paper. Although Figure 4 shows that capital investment is affected by technological progress, we did not consider uncertainty about technological progress. Such uncertainty could be formulated by using the Poisson distribution. One could also incorporate capital in addition to that used in production. For example, in cases in which production generates pollution, there is a need to invest in environmental capital that reduces emissions. These important topics are left to future research.

References

Camerer, C. and M. Weber, Recent Developments in Modeling Preferences: Uncertainty and Ambiguity, *Journal of Risk and Uncertainty*, **5**, 325–370, 1992.

Ellseberg, D., Risk, Ambiguity, and the Savage Axioms, *Quarterly Journal of Economics*, **75**, 643–669, 1961.

Etner, J., M. Jeleva and J.-M. Tallon, Decision Theory under Ambiguity, *Journal of Economic Surveys*, **26**, 234–270, 2012.

Gilboa, I. and D. Schmeidler, Maximin Expected Utility with Non-unique Priors, *Journal of Mathematical Economics*, **18**, 141–153, 1989.

Guidolin, M. and F. Rinaldi, Ambiguity in Asset Pricing and Portfolio Choice: A Review of the Literature, *Theory Decision*, **74**, 183–217, 2013.

Hansen, L.P. and T.J. Sargent, Robust Control and Model Uncertainty, *American Economic Review*, **91**, 60–66, 2001.

Knight, F.H., *Risk, Uncertainty, and Profit*, 1921.

Kogan, L. and T. Wang, A Simple Theory of Asset Pricing under Model Uncertainty, *Working Paper, MIT and University of British Columbia*, 2002.

Miao, J., A Note on Consumption and Savings under Knightian Uncertainty, *Annals of Economics and Finance*, **5**, 299–311, 2004.

Appendix A.

Given the assumptions about the probability measures $\mathbb{P}$ and $\mathbb{Q}$, we obtain

$$\begin{aligned}
\ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) &= \ln(\mathbb{Q}) - \ln(\mathbb{P}) \\
&= \ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \left(\frac{(y-(\mu-h))^2}{2\sigma^2}\right) - \ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right) + \left(\frac{(y-\mu)^2}{2\sigma^2}\right) \\
&= \frac{-2h(y-\mu)-h^2}{2\sigma^2}.
\end{aligned} \tag{A.1}$$

Then, relative entropy is

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right] &= \int \ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\mathrm{d}\mathbb{Q} \\
&= \frac{-2h(\mathbb{E}_{\mathbb{Q}}[y]-\mu)-h^2}{2\sigma^2} \\
&= \frac{-2h(\mu-h-\mu)-h^2}{2\sigma^2} \\
&= \frac{h^2}{2\sigma^2}.
\end{aligned} \tag{A.2}$$

Faculty of Commerce, Doshisha University
Kamigyo-ku, Kyoto, 602-8580 Japan
E-mail address: mtsujimu@mail.doshisha.ac.jp

同志社大学商学部　辻村 元男