The strong continuity in weakly o-minimal structures (Model theoretic aspects of the notion of independence and dimension)

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The strong continuity in weakly o-minimal structures

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Abstract

In this paper, we study the strong continuity of definable functions in weakly o-minimal structures with the strong cell decomposition property.

Throughout this paper, “definable” means “definable possibly with parameters” and we assume that a structure $\mathcal{M} = (M, <, \ldots)$ is a dense linear ordering $<$ without endpoints.

A subset $A$ of $M$ is said to be convex if $a, b \in A$ and $c \in M$ with $a < c < b$ then $c \in A$. Moreover if $A = \emptyset$ or $\inf A, \sup A \in M \cup \{-\infty, +\infty\}$, then $A$ is called an interval in $M$. We say that $\mathcal{M}$ is o-minimal (weakly o-minimal) if every definable subset of $M$ is a finite union of intervals (convex sets), respectively. A theory $T$ is said to be weakly o-minimal if every model of $T$ is weakly o-minimal. The reader is assumed to be familiar with fundamental results of o-minimality and weak o-minimality; see, for example, [1], [2], [3], or [4].

For any subsets $C, D$ of $M$, we write $C < D$ if $c < d$ whenever $c \in C$ and $d \in D$. A pair $\langle C, D \rangle$ of non-empty subsets of $M$ is called a cut in $M$ if $C < D, C \cup D = M$ and $D$ has no lowest element. A cut $\langle C, D \rangle$ is said

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to be \textit{definable} in $\mathcal{M}$ if the sets $C, D$ are definable in $\mathcal{M}$. The set of all cuts definable in $\mathcal{M}$ will be denoted by $\overline{M}$. Note that we have $M = \overline{M}$ if $\mathcal{M}$ is o-minimal. We define a linear ordering on $\overline{M}$ by $\langle C_1, D_1 \rangle < \langle C_2, D_2 \rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(M, <)$ as a substructure of $(\overline{M}, <)$ by identifying an element $a \in M$ with the definable cut $((-\infty, a], (a, +\infty))$.

We equip $M$ (or $\overline{M}$) with the \textit{interval topology} (the open intervals form a base), and each product $M^n$ (or $\overline{M}^n$) with the corresponding product topology, respectively. For each positive integer $n$ the topological closure in $M^n$ of a set $A \subseteq M^n$ is denoted by $\text{cl}(A)$. We also write $\text{CL}(A)$ for the closure of a set $A \subseteq \overline{M}^n$ in $\overline{M}^n$.

Recall the notion of definable functions from [4]. Let $n$ be a positive integer and $A \subseteq M^n$ definable. A function $f : A \to \overline{M}$ is said to be \textit{definable} if the set $\{\langle x, y \rangle \in M^{n+1} : x \in A, y < f(x)\}$ is definable. A function $f : A \to \overline{M} \cup \{-\infty, +\infty\}$ is said to be \textit{definable} if $f$ is a definable function from $A$ to $\overline{M}$, $f(x) = -\infty$ for all $x \in A$, or $f(x) = +\infty$ for all $x \in A$.

Recall the notion of strong cells from [5].

\textbf{Definition 1.} Suppose that $\mathcal{M} = (M, <, \ldots)$ is a weakly o-minimal structure. For each positive integer $n$, we inductively define \textit{strong cells} in $M^n$ and their completions in $\overline{M}^n$.

(1) A one-element subset of $M$ is called a \textit{strong 0-cell} in $M$. If $C \subseteq M$ is a strong 0-cell, then its completion $\overline{C} := C$.

(2) A non-empty definable convex open subset of $M$ is called a \textit{strong 1-cell} in $M$. If $C \subseteq M$ is a strong 1-cell, then its completion $\overline{C} := \{x \in \overline{M} : (\exists a, b \in C)(a < x < b)\}$.

Assume that $k$ is a non-negative integer, and strong $k$-cells in $M^n$ and their completions in $\overline{M}^n$ are already defined.

(3) Let $C \subseteq M^n$ be a strong $k$-cell in $M^n$ and $f : C \to M$ is a definable continuous function which has a continuous extension $\overline{f} : \overline{C} \to \overline{M}$. Then the graph $\Gamma(f)$ is called a \textit{strong $k$-cell} in $M^{n+1}$ and its completion $\Gamma(\overline{f}) := \Gamma(f)$.
(4) Let $C \subseteq M^n$ be a strong $k$-cell in $M^n$ and $g, h : C \to \overline{M} \cup \{-\infty, +\infty\}$ are definable continuous functions which have continuous extensions $\overline{g}, \overline{h} : \overline{C} \to \overline{M} \cup \{-\infty, +\infty\}$ such that

(a) each of the functions $g, h$ assumes all its values in one of the sets $M, \overline{M} \setminus M, \{\infty\}, \{-\infty\},$

(b) $\overline{g}(x) < \overline{h}(x)$ for all $x \in \overline{C}$.

Then the set $$(g, h)_C := \{(a, b) \in M^{n+1} : a \in C, \ g(a) < b < h(a)\}$$ is called a **strong $(k + 1)$-cell** in $M^{n+1}$. The completion of $(g, h)_C$ is defined as $$\overline{(g, h)_C} := \{(a, b) \in M^{n+1} : a \in \overline{C}, \ \overline{g}(a) < b < \overline{h}(a)\}.$$ 

(5) Let $C$ be a subset of $M^n$. The set $C$ is called a **strong cell** in $M^n$ if there exists some non-negative integer $k$ such that $C$ is a strong $k$-cell in $M^n$.

Let $C$ be a strong cell of $M^n$. A definable function $f : C \to \overline{M}$ is said to be **strongly continuous** if $f$ has a continuous extension $\overline{f} : \overline{C} \to \overline{M}$. A function which is identically equal to $-\infty$ or $+\infty$, and whose domain is a strong cell is also said to be strongly continuous.

**Definition 2.** Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure. For each positive integer $n$, we inductively define a **strong cell decomposition** (or a decomposition into strong cells in $M^n$) of a non-empty definable set $A \subseteq M^n$.

(1) If $A \subseteq M$ is a non-empty definable set and $\mathcal{D} = \{C_1, \ldots, C_k\}$ is a partition of $A$ into strong cells in $M$, then $\mathcal{D}$ is called a decomposition of $A$ into strong cells in $M$.

(2) Suppose that $A \subseteq M^{n+1}$ is a non-empty definable set and $\mathcal{D} = \{C_1, \ldots, C_k\}$ is a partition of $A$ into strong cells in $M^{n+1}$. Then $\mathcal{D}$ is called a decomposition of $A$ into strong cells in $M^{n+1}$ if $\{\pi(C_1), \ldots, \pi(C_k)\}$ is a decomposition of $\pi(A)$ into strong cells in $M^n$, where $\pi : M^{n+1} \to M^n$ is the projection on the first $n$ coordinates.
Definition 3. Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure and $n$ a positive integer. Suppose that $A, B \subseteq M^n$ are definable sets, $A \neq \emptyset$ and $\mathcal{D}$ is a decomposition of $A$ into strong cells in $M^n$. We say that $\mathcal{D}$ partitions $B$ if for each strong cell $C \in \mathcal{D}$, we have either $C \subseteq B$ or $C \cap B = \emptyset$.

Definition 4. A weakly o-minimal structure $\mathcal{M} = (M, <, \ldots)$ is said to have the strong cell decomposition property if for any positive integers $k, n$ and any definable sets $A_1, \ldots, A_k \subseteq M^n$, there exists a decomposition of $M^n$ into strong cells partitioning each of the sets $A_1, \ldots, A_k$.

Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal expansion of an ordered abelian group $(M, <, +)$. Then, the weakly o-minimal structure $\mathcal{M}$ is said to be non-valuational if for any definable cut $(C, D)$ we have $\inf\{d - c : c \in C, d \in D\} = 0$.

Then, the following facts hold.

Fact 5 ([4, Fact 2.5]). Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with the cell decomposition property. Suppose that $X \subseteq M^n$ is definable and $f : X \to \overline{M}$ is definable. Then, there is a decomposition $\mathcal{D}$ of $X$ into strong cells in $M^n$ such that for every $D \in \mathcal{D}$,

1. $f|_D$ assumes all its values in one of the sets $M, \overline{M} \setminus M$,
2. $f|_D$ is strongly continuous.

Fact 6 ([4, Corollary 2.16]). Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal expansion of an ordered abelian group $(M, <, +)$. Then the following conditions are equivalent.

1. $\mathcal{M}$ is non-valuational.
2. $\mathcal{M}$ has the strong cell decomposition property.

Let $\mathcal{M}$ be a weakly o-minimal structure with the cell decomposition property. For any strong cell $C \subseteq M^m$, we denote by $\overline{R}_C$ the $m$-ary relation determined by $\overline{C}$, i.e. if $a \in \overline{M}^m$, then $\overline{R}_C(a)$ holds iff $a \in \overline{C}$. We define the structure $\overline{\mathcal{M}} := (\overline{M}, <, (\overline{R}_C : C \text{ is a strong cell}))$. The following fact is known.
Fact 7 ([4]). Let $\mathcal{M}$ be a weakly o-minimal structure with the cell decomposition property. Then, $\overline{\mathcal{M}}$ is o-minimal, and every set $X \subseteq \overline{M}^n$ definable in $\mathcal{M}$ is a finite Boolean combination of completions of strong cells in $M^n$.

Proposition 8. Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal expansion with the cell decomposition property of an ordered abelian group. Let $X \subseteq M^n$ be definable and $f : X \to \overline{M}$ definable. Suppose that there is a decomposition $\mathcal{D}$ of $X$ into strong cells in $M^n$ such that for every $D \in \mathcal{D}$,

1. $f|_D$ assumes all its values in one of the sets $M$, $\overline{M} \setminus M$,

2. $f|_D$ is strongly continuous,

3. $\overline{f|_D}(\overline{D})$ is bounded.

Then, there exists some continuous extension $\overline{f} : CL(X) \to \overline{M}$ of $f$.

Corollary 9. Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal expansion with the cell decomposition property of an ordered abelian group. Let $C$ be a strong cell and $f : C \to M$ or $f : C \to \overline{M} \setminus M$. Suppose that $f$ is definable and strongly continuous, and $\overline{f}(\overline{C})$ is bounded. Then, for any strong cell $D \subseteq C$, $f|_D$ is strongly continuous.

Remark 10. Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with the cell decomposition property. Then, the following hold.

1. There exist strong cells $C, D_1, D_2$ such that $C = D_1 \cup D_2$ but $\overline{C} \neq \overline{D_1} \cup \overline{D_2}$.

2. There exist strong cells $C, D$ such that $C \subseteq D$ but $\overline{C} \not\subseteq \overline{D}$.

References


