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QUANTIFIER ELIMINATION IN ADELIC STRUCTURES OVER ALGEBRAICALLY CLOSED VALUED FIELDS

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Abstract. Applying Weispfenning’s fundamental work on boolean products, we deduce that the theory of adelic structures over algebraically closed valued fields in the language $\mathcal{L}_{AA}$ admits quantifier elimination and is complete.

To call a boolean product over algebraically closed valued fields “adelic” one needs to express the product formula in a first-order way. This is not achieved. Here, by an “adelic strcture” we just mean a boolean product over algebraically closed valued fields.

We use the Basarab-Kuhlmann style language $\mathcal{L}_{RV}$ for algebraically closed valued fields of mixed characteristic introduced in [1]. The theory ACVF of algebraically closed valued fields (in any characteristics) admits quantifier elimination in $\mathcal{L}_{RV}$, a short proof of which may be found in [3].

Theorem. The theory ACVF admits quantifier elimination.

Next we expand an described language $\mathcal{L}_{BF}$ for boolean algebras, which includes the following:

- the language of boolean algebras $\mathcal{L}_{BA} = \{0, 1, \cap, \cup, \sim, \leq\}$;
- a set of unary relations $\{0 <_{n}: n \geq 1\}$;
- a unary relation $\mathcal{F}$;
- a constant $a$.

The theory of infinite atomic boolean algebras with the distinguished set of finite elements in $\mathcal{L}_{BF}$ (hereafter abbreviated as IABF) states the following:

- the usual axioms for boolean algebras;
- for every $\xi > 0$ there is an atom $\eta$ such that $\xi \geq \eta$;
- $a$ is an atom;
- axioms for $\mathcal{F}$:
  - $\mathcal{F}(0)$ and $\neg \mathcal{F}(1)$;
  - $\mathcal{F}(\xi \cup \eta)$ if and only if $\mathcal{F}(\xi)$ and $\mathcal{F}(\eta)$;
  - if $\neg \mathcal{F}(\xi)$ then there is an $\eta$ such that $\neg \mathcal{F}(\xi \cap \eta)$ and $\neg \mathcal{F}(\xi \sim \eta)$;
- $0 <_{n} \xi$ if and only if there are $\eta_{1}, \ldots, \eta_{n}$ such that $0 \leq \eta_{1} < \ldots < \eta_{n} < \xi$ and $\mathcal{F}(\eta_{i})$ for each $i \leq n$.

Theorem (Weispfenning [2], Part II, 1.4(ii, iii)). The theory IABF admits quantifier elimination and is complete.

So IABF axiomatizes the theory of powerset algebras of infinite set, where $\mathcal{F}$ ranges over finite subsets.

In order to formulate a first-order language for adelic structures over algebraically closed valued fields we treat $(\mathcal{L}_{RV}, \mathcal{L}_{BF})$ as a 2-sorted language (these sorts shall be called the first-order sort, or FO-sort for short, and the boolean algebra sort, or BA-sort for short) and further expand it as follows. For each $n$-ary relation symbol $R$ (including equality and functions) in $\mathcal{L}_{RV}$ we add an $n$-ary function $V_{R}$ from the FO-sort to the BA-sort. For example, if $a, b$ are two $\mathcal{L}_{RV}$-terms then $V_{=(a, b)}$ is considered an $\mathcal{L}_{BF}$-term. In fact, since the boolean value of each quantifier-free formula in $\mathcal{L}_{RV}$ is determined by the functions $V_{R}$, for notational simplicity we may think of one function $\mathcal{V}$ that assigns a boolean value $\mathcal{V}_{\phi}$ to each quantifier-free $\mathcal{L}_{RV}$-formula $\phi$. Let $\mathcal{L}_{AA}$ denote this expansion of $(\mathcal{L}_{RV}, \mathcal{L}_{BF})$.

For each $\exists \exists$-formula $\psi$ in $\mathcal{L}_{RV}$ of the form $\forall \vec{x} \exists \vec{y} \psi(\vec{x}, \vec{y}, \vec{z})$ with $\psi$ quantifier-free, let $\phi^{\psi}$ be the $\mathcal{L}_{AA}$-formula $\forall \vec{x} \exists \vec{y} \mathcal{V}(\psi(\vec{x}, \vec{y}, \vec{z})) = 1$. Now it is routine to check that ACVF is a $\forall \exists$-theory in $\mathcal{L}_{RV}$. Let $\mathcal{ACVF}^{\forall} = \{ \phi^{\psi} : \phi$ is an axiom of ACVF$\}$. 
The theory of adelic structures over algebraically closed valued field in $\mathcal{L}_{AA}$ (hereafter abbreviated as AACF) states the following:

- $ACVF^\mathcal{V}$ and IABF for the corresponding sorts;
- $\forall (\text{char } k = p)$ is an atom for each prime number $p$;
- $a < \forall (\text{char } k > p)$ for each prime number $p$;
- the axioms for abstract boolean products (hereafter abbreviated as ABP): 
  - $\phi \leftrightarrow \forall \phi = 1$ for each atomic formula $\phi$ in $\mathcal{L}_{RV}$;
  - $\forall (x = y) = \forall (y = x)$;
  - $\bigcap_{i=1}^{n} \forall (x_i = y_i) \cap \forall \phi(x_1, \ldots, x_n) \leq \forall \phi(y_1, \ldots, y_n)$ for each atomic formula $\phi$ in $\mathcal{L}_{RV}$;
  - **Finitary gluing**: For all $x, y \in FO$ and $\alpha, \beta \in BA$, if $\alpha \cap \beta = 0$ and $\alpha \cup \beta = 1$ then there is a $z$ of the first sort such that $\forall (z = x) \geq \alpha$ and $\forall (z = y) \geq \beta$.

**Theorem.** The theory AACF admits quantifier elimination in all sorts and is complete.

**Proof.** Quantifier elimination is immediate by the theorems above and [2, Part II, 3.7(ii)]. Completeness follows from the representation theorem [2, Part I, 3.27].

**REFERENCES**

