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QUANTIFIER ELIMINATION IN ADELIC STRUCTURES
OVER ALGEBRAICALLY CLOSED VALUED FIELDS
(Model theoretic aspects of the notion of independence and dimension)

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QUANTIFIER ELIMINATION IN ADELIC STRUCTURES OVER ALGEBRAICALLY CLOSED VALUED FIELDS

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Abstract. Applying Weispfenning’s fundamental work on boolean products, we deduce that the theory of adelic structures over algebraically closed valued fields in the language $\mathcal{L}_{AA}$ admits quantifier elimination and is complete.

To call a boolean product over algebraically closed valued fields “adelic” one needs to express the product formula in a first-order way. This is not achieved. Here, by an “adelic structure” we just mean a boolean product over algebraically closed valued fields.

We use the Basarab-Kuhlmann style language $\mathcal{L}_{RV}$ for algebraically closed valued fields of mixed characteristic introduced in [1]. The theory ACVF of algebraically closed valued fields (in any characteristics) admits quantifier elimination in $\mathcal{L}_{RV}$, a short proof of which may be found in [3].

Theorem. The theory ACVF admits quantifier elimination.

Next we expand an described language $\mathcal{L}_{BF}$ for boolean algebras, which includes the following:
- the language of boolean algebras $\mathcal{L}_{BA} = \{0, 1, \land, \lor, \neg\}$;
- a set of unary relations $\{<_{n}: n \geq 1\}$;
- a unary relation $\mathcal{F}$;
- a constant $a$.

The theory of infinite atomic boolean algebras with the distinguished set of finite elements in $\mathcal{L}_{BF}$ (hereafter abbreviated as IABF) states the following:
- the usual axioms for boolean algebras;
- for every $\xi > 0$ there is an atom $\eta$ such that $\xi \geq \eta$;
- $a$ is an atom;
- axioms for $\mathcal{F}$:
  - $\mathcal{F}(0)$ and $\neg \mathcal{F}(1)$;
  - $\mathcal{F}(\xi \lor \eta)$ if and only if $\mathcal{F}(\xi)$ and $\mathcal{F}(\eta)$;
  - if $\neg \mathcal{F}(\xi)$ then there is an $\eta$ such that $\neg \mathcal{F}(\xi \land \eta)$ and $\neg \mathcal{F}(\xi \land \neg \eta)$;
- $0 <_{n} \xi$ if and only if there are $\eta_{1}, \ldots, \eta_{n}$ such that $0 \leq \eta_{1} < \ldots < \eta_{n} < \xi$ and $\mathcal{F}(\eta_{i})$ for each $i \leq n$.

Theorem (Weispfenning [2], Part II, 1.4(ii, iii)). The theory IABF admits quantifier elimination and is complete.

So IABF axiomatizes the theory of powerset algebras of infinite set, where $\mathcal{F}$ ranges over finite subsets.

In order to formulate a first-order language for adelic structures over algebraically closed valued fields we treat $(\mathcal{L}_{RV}, \mathcal{L}_{BF})$ as a 2-sorted language (these sorts shall be called the first-order sort, or FO-sort for short, and the boolean algebra sort, or BA-sort for short) and further expand it as follows. For each $n$-ary relation symbol $R$ (including equality and functions) in $\mathcal{L}_{RV}$ we add an $n$-ary function $\mathcal{V}_{R}$ from the FO-sort to the BA-sort. For example, if $a, b$ are two $\mathcal{L}_{RV}$-terms then $\mathcal{V}_{=}(a, b)$ is considered an $\mathcal{L}_{BF}$-term. In fact, since the boolean value of each quantifier-free formula in $\mathcal{L}_{RV}$ is determined by the functions $\mathcal{V}_{R}$, for notational simplicity we may think of one function $\mathcal{V}$ that assigns a boolean value $\mathcal{V}_{\phi}$ to each quantifier-free $\mathcal{L}_{RV}$-formula $\phi$. Let $\mathcal{L}_{AA}$ denote this expansion of $(\mathcal{L}_{RV}, \mathcal{L}_{BF})$.

For each $\exists \forall$-formula $\phi$ in $\mathcal{L}_{RV}$ of the form $\forall \vec{z} \exists \vec{y} \psi(\vec{x}, \vec{y}, \vec{z})$ with $\psi$ quantifier-free, let $\phi^{\exists}$ be the $\mathcal{L}_{AA}$-formula $\forall \vec{y} \forall \vec{z} \mathcal{V}_{\psi(\vec{x}, \vec{y}, \vec{z})} = 1$. Now it is routine to check that ACVF is a $\forall \exists$-theory in $\mathcal{L}_{RV}$. Let $ACVF^{\exists} = \{ \phi^{\exists} : \phi$ is an axiom of ACVF$\}$. 
The theory of adelic structures over algebraically closed valued field in $\mathcal{L}_{AA}$ (hereafter abbreviated as AACF) states the following:

- ACVF$^\mathcal{V}$ and IABF for the corresponding sorts;
- $\mathcal{V}(\text{char } k = p)$ is an atom for each prime number $p$;
- an atom for each prime number $p$;
- the axioms for abstract boolean products (hereafter abbreviated as ABP):
  - $\phi \leftrightarrow \mathcal{V}\phi = 1$ for each atomic formula $\phi$ in $\mathcal{L}_{RV}$;
  - $\mathcal{V}(x = y) = \mathcal{V}(y = x)$;
  - $\bigcap_{i=1}^{n} \mathcal{V}(x_i = y_i) \cap \mathcal{V}\phi(x_1, \ldots, x_n) \leq \mathcal{V}\phi(y_1, \ldots, y_n)$ for each atomic formula $\phi$ in $\mathcal{L}_{RV}$;
  - Finitary gluing: For all $x, y \in FO$ and $\alpha, \beta \in BA$, if $\alpha \cap \beta = 0$ and $\alpha \cup \beta = 1$ then there is a $z$ of the first sort such that $\mathcal{V}(z = x) \geq \alpha$ and $\mathcal{V}(z = y) \geq \beta$.

**Theorem.** The theory AACF admits quantifier elimination in all sorts and is complete.

**Proof.** Quantifier elimination is immediate by the theorems above and [2, Part II, 3.7(ii)]. Completeness follows from the representation theorem [2, Part I, 3.27].

## References

