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QUANTIFIER ELIMINATION IN ADELIC STRUCTURES OVER ALGEBRAICALLY CLOSED VALUED FIELDS

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ABSTRACT. Applying Weispfenning's fundamental work on boolean products, we deduce that the theory of adelic structures over algebraically closed valued fields in the language $\mathcal{L}_{\mathrm{AA}}$ admits quantifier elimination and is complete.

To call a boolean product over algebraically closed valued fields "adelic" one needs to express the product formula in a first-order way. This is not achieved. Here, by an "adelic struture" we just mean a boolean product over algebraically closed valued fields.

We use the Basarab-Kuhlmann style language $\mathcal{L}_{\mathrm{RV}}$ for algebraically closed valued fields of mixed characteristic introduced in [1]. The theory ACVF of algebraically closed valued fields (in any characteristics) admits quantifier elimination in $\mathcal{L}_{\mathrm{RV}}$, a short proof of which may be found in [3].

Theorem. The theory ACVF admits quantifier elimination.

Next we describe an expanded language $\mathcal{L}_{\mathrm{BF}}$ for boolean algebras, which includes the following:

- the language of boolean algebras $\mathcal{L}_{\mathrm{BA}}=\{0,1,\cap,\cup,\sim,\leq\}$;
- a set of unary relations $\{0<_{n}: n\geq 1\}$;
- a unary relation $\mathcal{F}$;
- a constant $\mathbf{a}$.

The theory of infinite atomic boolean algebras with the distinguished set of finite elements in $\mathcal{L}_{\mathrm{BF}}$ (hereafter abbreviated as IABF) states the following:

- the usual axioms for boolean algebras;
- for every $\xi>0$ there is an atom η such that $\xi\geq\eta$;
- $\mathbf{a}$ is an atom;
- axioms for $\mathcal{F}$:
  - $\mathcal{F}(0)$ and $\neg\mathcal{F}(1)$;
  - $\mathcal{F}(\xi\cup\eta)$ if and only if $\mathcal{F}(\xi)$ and $\mathcal{F}(\eta)$;
  - if $\neg\mathcal{F}(\xi)$ then there is an η such that $\neg\mathcal{F}(\xi\cap\eta)$ and $\neg\mathcal{F}(\xi\cap\sim\eta)$;
- $0<_{n}\xi$ if and only if there are $\eta_{1},\ldots,\eta_{n}$ such that $0\leq\eta_{1}<\ldots<\eta_{n}<\xi$ and $\mathcal{F}(\eta_{i})$ for each $i\leq n$.

Theorem (Weispfenning [2], Part II, 1.4(ii, iii)). The theory IABF admits quantifier elimination and is complete.

So IABF axiomatizes the theory of powerset algebras of infinite set, where $\mathcal{F}$ ranges over finite subsets.

In order to formulate a first-order language for adelic structures over algebraically closed valued fields we treat $(\mathcal{L}_{\mathrm{RV}},\mathcal{L}_{\mathrm{BF}})$ as a 2-sorted language (these sorts shall be called the first-order sort, or FO-sort for short, and the boolean algebra sort, or BA-sort for short) and further expand it as follows. For each n-ary relation symbol R (including equality and functions) in $\mathcal{L}_{\mathrm{RV}}$ we add an n-ary function $\mathcal{V}_{R}$ from the FO-sort to the BA-sort. For example, if a,b are two $\mathcal{L}_{\mathrm{RV}}$-terms then $\mathcal{V}_{=}(a,b)$ is considered an $\mathcal{L}_{\mathrm{BF}}$-term. In fact, since the boolean value of each quantifier-free formula in $\mathcal{L}_{\mathrm{RV}}$ is determined by the functions $\mathcal{V}_{R}$, for notational simplicity we may think of one function $\mathcal{V}$ that assigns a boolean value $\mathcal{V}\phi$ to each quantifier-free $\mathcal{L}_{\mathrm{RV}}$-formula ϕ. Let $\mathcal{L}_{\mathrm{AA}}$ denote this expansion of $(\mathcal{L}_{\mathrm{RV}},\mathcal{L}_{\mathrm{BF}})$.

For each $\forall\exists$-formula ϕ in $\mathcal{L}_{\mathrm{RV}}$ of the form $\forall\vec{x}\ \exists\vec{y}\ \psi(\vec{x},\vec{y},\vec{z})$ with ψ quantifier-free, let $\phi^{\mathcal{V}}$ be the $\mathcal{L}_{\mathrm{AA}}$-formula $\forall\vec{x}\ \exists\vec{y}\ \mathcal{V}(\psi(\vec{x},\vec{y},\vec{z}))=1$. Now it is routine to check that ACVF is a $\forall\exists$-theory in $\mathcal{L}_{\mathrm{RV}}$. Let $\mathrm{ACVF}^{\mathcal{V}}=\{\phi^{\mathcal{V}}:\phi$ is an axiom of ACVF$\}$.

The theory of adelic structures over algebraically closed valued field in $\mathcal{L}_{AA}$ (hereafter abbreviated as AACF) states the following:

- ACVF$^{\mathcal{V}}$ and IABF for the corresponding sorts;
- $\mathcal{V}(\operatorname{char} \Bbbk = p)$ is an atom for each prime number p;
- $\mathbf{a} < \mathcal{V}(\operatorname{char} \Bbbk > p)$ for each prime number p;
- the axioms for abstract boolean products (hereafter abbreviated as ABP):
  - $\phi \leftrightarrow \mathcal{V}\phi = 1$ for each atomic formula ϕ in $\mathcal{L}_{RV}$;
  - $\mathcal{V}(x = y) = \mathcal{V}(y = x)$;
  - $\bigcap_{i=1}^{n} \mathcal{V}(x_i = y_i) \cap \mathcal{V}\phi(x_1, \ldots, x_n) \leq \mathcal{V}\phi(y_1, \ldots, y_n)$ for each atomic formula ϕ in $\mathcal{L}_{RV}$;
  - **Finitary gluing**: For all $x, y \in \mathrm{FO}$ and $\alpha, \beta \in \mathrm{BA}$, if $\alpha \cap \beta = 0$ and $\alpha \cup \beta = 1$ then there is a z of the first sort such that $\mathcal{V}(z = x) \geq \alpha$ and $\mathcal{V}(z = y) \geq \beta$.

Theorem. The theory AACF admits quantifier elimination in all sorts and is complete.

Proof. Quantifier elimination is immediate by the theorems above and [2, Part II, 3.7(ii)]. Completeness follows from the representation theorem [2, Part I, 3.27]. □

References

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