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Basic properties of definable isomorphisms of quantum 2-tori $T_q^2(\mathbb{C})$

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Abstract

In [5] we constructed quantum 2-tori and studied their first-order theories. Here we discuss basic properties of the definable isomorphisms between quantum 2-tori $T_q^2(\mathbb{C})$ and show that $SL_2(\mathbb{Z})$ is the set of definable isomorphisms of them.

1 Definable isomorphisms

Let $\theta \in \mathbb{R}$ be a transcendental element and put $q = \exp(2\pi i\theta)$ where $i = \sqrt{-1}$. Let also Γ_θ be an infinite multiplicative group generated by q. We denote $\mathcal{A}_\theta$ the non-commutative algebra $\mathcal{O}_q((\mathbb{C}^\times)^2)$ with generators written as U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV.$$

For each pair $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*/\Gamma_\theta$, we construct two $\mathcal{A}_\theta$-modules $M_{|u,v\rangle}$ and $M_{\langle v,u|}$.

The module $M_{|u,v\rangle}$ generated by elements labeled $\{\mathbf{u}(\gamma u, v) : \gamma \in \Gamma_\theta\}$ satisfies

$$\begin{array}{lcl} U & : & \mathbf{u}(\gamma u, v) \mapsto \gamma u \mathbf{u}(\gamma u, v), \\ V & : & \mathbf{u}(\gamma u, v) \mapsto v\mathbf{u}(q^{-1}\gamma u, v). \end{array} \tag{1}$$

We also define the module $M_{\langle v,u|}$ generated by elements labeled $\{\mathbf{v}(\gamma v, u) : \gamma \in \Gamma_\theta\}$ satisfying

$$\begin{array}{lcl} U & : & \mathbf{v}(\gamma v, u) \mapsto u\mathbf{v}(q\gamma v, u), \\ V & : & \mathbf{v}(\gamma v, u) \mapsto \gamma v \mathbf{v}(\gamma v, u). \end{array} \tag{2}$$

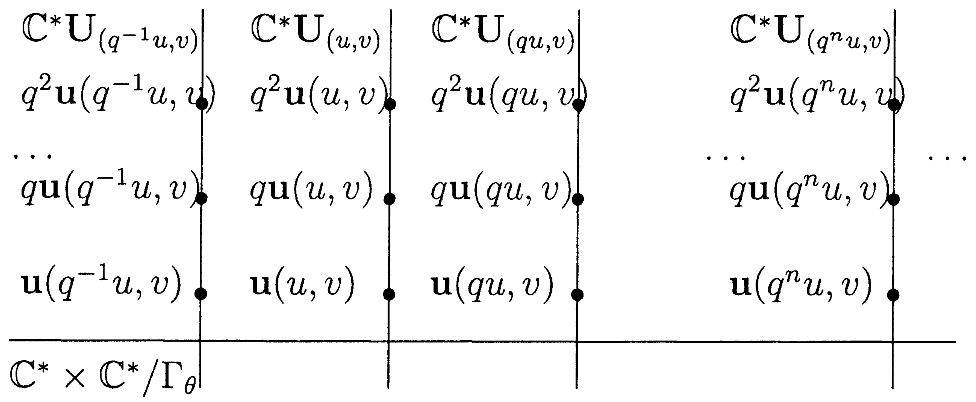

Figure 1: Γ_θ-bundle over $\mathbb{C} \times \mathbb{C}/\Gamma_\theta$ inside an ambient $\mathbb{C}$-module

Notice that we have

$$\begin{array}{lcl} U^{-1} & : & \mathbf{u}(\gamma u, v) \mapsto \gamma^{-1}u^{-1}\mathbf{u}(\gamma u, v), \\ V^{-1} & : & \mathbf{u}(\gamma u, v) \mapsto v^{-1}\mathbf{u}(q\gamma u, v), \end{array} \tag{3}$$

and

$$\begin{array}{lcl} U^{-1} & : & \mathbf{v}(\gamma v, u) \mapsto u^{-1}\mathbf{v}(q^{-1}\gamma v, u), \\ V^{-1} & : & \mathbf{v}(\gamma v, u) \mapsto \gamma^{-1}v^{-1}\mathbf{v}(\gamma v, u). \end{array} \tag{4}$$

Now let $\phi : \mathbb{C}^*/\Gamma_\theta \to \mathbb{C}^*$. Put $\Phi = \mathrm{ran}(\phi)$. Set

$$\begin{array}{lcl} \Gamma_\theta \cdot \mathbf{u}(u,v) & := & \{\gamma\mathbf{u}(u,v) : \gamma \in \Gamma_\theta\}, \\ \mathbf{U}_{\langle u,v\rangle} & := & \bigcup_{\gamma\in\Gamma_\theta} \Gamma_\theta \cdot \mathbf{u}(\gamma u, v) = \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \gamma_1, \gamma_2 \in \Gamma_\theta\}, \\ \mathbf{U}_\phi & := & \bigcup_{\langle u,v\rangle\in\Phi^2} \mathbf{U}_{\langle u,v\rangle} \\ & = & \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \langle u,v\rangle \in \Phi^2, \gamma_1.\gamma_2 \in \Gamma_\theta\}. \end{array} \tag{5}$$

We call $\Gamma_\theta \cdot \mathbf{u}(u,v)$ a Γ_θ-**set** over $\mathbf{u}(u,v)$, and $\mathbf{U}_\phi$ a Γ_θ-bundle over $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_\theta$. Notice that $\mathbf{U}_\phi$ is a subset of the $\mathcal{A}_\theta$-module

$$\bigcup_{\langle u,v\rangle} M_{|u,v\rangle}.$$

Similarly a Γ_θ-set $\Gamma_\theta \cdot \mathbf{v}(v,u)$ over $\mathbf{v}(v,u)$, and a Γ_θ-bundle $\mathbf{V}_\phi$ over $\mathbb{C}^*/\Gamma_\theta \times \mathbb{C}^*$ is defined. As before the Γ_θ-bundle $\mathbf{V}_\phi$ is a subset of the $\mathcal{A}_\theta$-module

$$\bigcup_{\langle v,u\rangle} M_{\langle v,u|}.$$

Definition 1 *Given θ and $q = \exp(2\pi i\theta)$. Let $\phi : \mathbb{C}^*/\Gamma_\theta \to \mathbb{C}^*$. We call the structure $(\mathbf{U}_\phi, \mathbf{V}_\phi, \mathbb{C})$ with actions U and V satisfying (1) and (2) a* **quantum 2-torus** *$T_q^2(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$. We denote $T_q^2(\mathbb{C})$ as T_θ in this note.*

Notice that the structure of the quantum 2-torus T_θ is essentially decided by θ (in fact $q = e^{2\pi i\theta}$). More precisely $\Gamma_q = q^{\mathbb{Z}} = \Gamma_\theta = e^{2\pi i\theta\mathbb{Z}} = \langle e^{2\pi i\theta}\rangle$ is the key ingredient of Γ-bundles.

Remark 2 *It is clear that we have $\Gamma_{\theta\pm 1} = \Gamma_\theta$, hence for any $n \in \mathbb{Z}$ we have $\Gamma_{\theta+n} = \Gamma_\theta$. It is also clear that $\Gamma_{-\theta} = \Gamma_\theta$. For $n \in \mathbb{Z}$, it is easy to see that $\Gamma_{n\theta}$ and Γ_θ are both infinite multiplicative groups generated by single elements hence isomorphic as such groups.*

Definition 3 *Given $\theta_1, \theta_2 \in \mathbb{R}$, both transcendental. We say that the quantum 2-tori T_{θ_1} and T_{θ_2} are <u>definably</u> isomorphic in the structure $(\mathbb{C}, +, \cdot, x^\theta)$ if there is an definable $f : \mathbb{R} \to \mathbb{R}$ such that $f(\theta_1) = \theta_2$ and*

1. *f induces a definable isomorphism between $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_1}$ and $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_2}$,*
2. *f induces a definable isomorphism between Γ_{θ_1}-bundle over $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_1}$ and Γ_{θ_2}-bundle over $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_2}$*

Proposition 4 *From the definition of definable isomorphism of quantum 2-torus, we see that there are three cases for quantum 2-tori to be definably isomorphic:*

1. *$\Gamma_{\theta_1} = \Gamma_{\theta_2}$, or*
2. *Γ_{θ_1} and Γ_{θ_2} are isomorphic, or*
3. *$\mathbb{C}/\Gamma_{\theta_1}$ is definably isomorphic to $\mathbb{C}/\Gamma_{\theta_2}$, more precisely there is a definable one-to-one correspondence between the cosets of Γ_{θ_1} and the cesets of Γ_{θ_2}.*

Remark 5 *Our defintion of two quantum 2-tori being isomorphic is essentialy the same as the notion of Morita equivarence. The idea of using the Morita equivalence as the definiton of isomorphim of quantum tori comes from Manin's argument.*

We investigate here the structure of the set $\mathrm{DefIso}(T_{\theta_1}, T_{\theta_2})$ of all *definable* isomorphisms f between quantum tori T_{θ_1} and T_{θ_2}.

Proposition 6 *It is clear that from Remark 2, we have (1) $T_{-\theta}$ and T_θ are definably isomorphic and (2) $T_{\theta+n}$ and T_θ are definably isomorphic for $n \in \mathbb{Z}$. It is not hard to see that $T_{n\theta}$ and T_θ are definably isomorphic since $\Gamma_{n\theta}$ and Γ_θ are isomorphic.*

1.1 x^θ is a definable isomorphism between $T_{\frac{1}{\theta}}$ and T_θ

Here we show that the function x^θ gives rise to a definable isomorphim in the sense of Definition 3 between quantum tori $T_{\frac{1}{\theta}}$ and T_θ.

Recall that Γ_θ is an infinite multipicative group generated by $q = \exp(2\pi i\theta)$.

Recall that over the complex numbers we have

$$x^\theta = e^{\theta \mathrm{Log} x},$$

where $\mathrm{Log} x = \ln x + 2\pi i\mathbb{Z}$.

Remark 7 *x^θ is not a function but a multivalued operation.*

First observation to make is the following:

Claim Suppose $\Gamma_{\theta^{-1}} = \{q_1^k : q_1 = \exp(2\pi i\theta^{-1}), k \in \mathbb{Z}\}$, and $\Gamma_\theta = \{q_2^m : q_2 = \exp(2\pi i\theta), m \in \mathbb{Z}\}$. Then x^θ sends a coset of $\Gamma_{\frac{1}{\theta}}$ to a coset of Γ_θ as follows;

$$\begin{aligned} x^\theta : a \cdot \exp(2\pi i\theta^{-1}k) &\mapsto \exp\Big(\theta(\ln(a\exp(2\pi i\theta^{-1}k)) + 2\pi im)\Big) \\ &= \exp\Big(\theta(\ln a + 2\pi i\theta^{-1}k + 2\pi im)\Big) \\ &= \exp\Big(\theta(\ln a + 2\pi im) + 2\pi ik)\Big) \\ &= \exp(\theta(\ln a + 2\pi im)) \cdot \exp(2\pi ik) \\ &= \exp(\theta \ln a) \cdot \exp(2\pi i\theta m) \\ &= a^\theta \cdot \exp(2\pi i\theta m) \\ &\in a^\theta\Gamma_\theta \end{aligned}$$

On the other hand, for given $a^\theta \cdot \exp(2\pi i\theta m) \in a^\theta \cdot \Gamma_\theta$ we have $a \cdot \exp(2\pi i\theta^{-1}m) \in a \cdot \Gamma_{\frac{1}{\theta}}$ and

$$\begin{aligned} x^\theta : a \cdot \exp(2\pi i\theta^{-1}m) &\mapsto \exp\Big(\theta(\ln(a\exp(2\pi i\theta^{-1}m)) + 2\pi il)\Big) \\ &= a^\theta \cdot \exp(2\pi i\theta l) \\ &\in a^\theta \cdot \Gamma_\theta \end{aligned}$$

These computations show that x^θ gives a one-to-one correspondence between a coset of $\Gamma_{\frac{1}{\theta}}$ and a coset of Γ_θ. From this we see that x^θ defines a definable bijection between $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\frac{1}{\theta}}$ and $\mathbb{C}^* \times \mathbb{C}^*/\Gamma_{\theta_1}$.

Recall that the module $M_{|u,v\rangle,\frac{1}{\theta}}$ is generated by elements labeled $\{\mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) : \gamma \in \Gamma_\theta\}$. Set first

$$x^\theta : \mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) \mapsto \mathbf{u}(\gamma u, v)$$

Then the operators U, V act as follows:

$$\begin{array}{rcl} U & : & \mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) \mapsto \gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}\mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}), \\ V & : & \mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) \mapsto v^{\frac{1}{\theta}}\mathbf{u}(q^{-1}\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}), \end{array} \tag{6}$$

where $q = e^{2\pi i\theta}$.

Hence we have the following diagrams:

$$\begin{array}{ccc} \mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & \mathbf{u}(\gamma u, v) \\ \downarrow U & \circlearrowleft & U \downarrow \\ \gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}\mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & \gamma u\mathbf{u}(\gamma u, v) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{u}(\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & \mathbf{u}(\gamma u, v) \\ \downarrow V & \circlearrowleft & V \downarrow \\ v^{\frac{1}{\theta}}\mathbf{u}(q^{-1}\gamma^{\frac{1}{\theta}}u^{\frac{1}{\theta}}, v^{\frac{1}{\theta}}) & \xrightarrow{x^\theta} & v\mathbf{u}(q^{-1}\gamma u, v) \end{array}$$

These diagrams tell us that we have a definable correpondence of vectors in $\Gamma_{\frac{1}{\theta}}$-set and Γ_θ-set and vectors in $\Gamma_{\frac{1}{\theta}}$-bundle and Γ_θ-and, eventually vectors in line-bundles of both sides. Observe that x^θ preserves actions of U, V.

Proposition 8 *In the structure* $(\mathbb{C}, +, \cdot, x^\theta)$, *by sending* $\frac{1}{\theta}$ *to* θ, *we construct a definable isomorphism* f *between* $\Gamma_{\frac{1}{\theta}}$ *and* Γ_θ *which give rise to the definable*

$$f : \mathbb{C}^*/\Gamma_{\frac{1}{\theta}} \overset{x^\theta}{\simeq} \mathbb{C}/\Gamma_\theta.$$

Therefore the quantum 2-tori $T_{\frac{1}{\theta}}$ *and* T_θ *are definably isomorphic.*

1.2 $\mathrm{SL}_2(\mathbb{Z})$ is the group of definable isomorphisms

We have shown that Γ_θ and $\Gamma_{\frac{1}{\theta}}$ define isomorphic quantum 2-tori. Since $\Gamma_{\theta+1} = \Gamma_\theta$, it is reasonable to have the following statement.

Proposition 9 *$\mathrm{SL}_2(\mathbb{Z})$ is the set of all definable isomorphisms. More precisely, for any $f \in \mathrm{SL}_2(\mathbb{Z})$, Γ_θ and $\Gamma_{f(\theta)}$ define isomorphic quantum 2-tori.*

Proof: 1) first show that for any $f \in \mathrm{SL}_2(\mathbb{Z})$, Γ_θ and $\Gamma_{f(\theta)}$ give rise to isomorphic quantum 2-tori. Consider two functions

$$\begin{array}{rcl} f_S: \; \theta & \mapsto & -\dfrac{1}{\theta}, \\ f_T: \; \theta & \mapsto & \theta + 1. \end{array}$$

View both f_S and f_T as Möbius transformations;

$$\theta \mapsto \frac{a\theta + b}{c\theta + d},$$

which is identified with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then f_S corresponds to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S$, and f_T corresponds to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T$. Then $S, T \in \mathrm{SL}_2(\mathbb{Z})$ and S and T generate $\mathrm{SL}_2(\mathbb{Z})$, i.e., $\langle S, T\rangle = \mathrm{SL}_2(\mathbb{Z})$. From the argument in the previous subsection we have that $SL_2(\mathbb{Z})$ is a subgroup of the group of definable ismorphisms.

We still need to show that any definable isomorphism in fact belongs to $\mathrm{SL}_2(\mathbb{Z})$. For this we have

Lemma 10 *Suppose T_θ and T_{θ_1} are definably isomorphic quantum tori in $(\mathbb{C}, +, \cdot, x^\theta)$. Then there is a $f \in \langle S, T\rangle$ such that $\theta_1 = f(\theta)$.*

Proof: Recall Propositon 4. From the fact that θ_1 being *definable* in the structure $(\mathbb{C}, +, \cdot, x^\theta)$, there are three cases to consider;

1. $\Gamma_\theta = \Gamma_{\theta_1}$, or
2. Γ_θ and Γ_{θ_1} are isomorphic, or

3. $\mathbb{C}/\Gamma_\theta$ is definably isomorphic to $\mathbb{C}/\Gamma_{\theta_1}$, more precisely there is a definable one-to-one correspondence between the cosets of Γ_θ and the cesets of Γ_{θ_1}.

Hence we *may assume* there are $a, b, c, d \in \mathbb{Z}$ such that

$$\theta_1 = \frac{a\theta + b}{c\theta + d},$$

thus there is an $f \in \mathrm{SL}_2(\mathbb{Z})$ such that $\theta_1 = f(\theta)$.

References

[1] John Baldwin, **Fundamentals of Stability Theory**, Springer, 1988

[2] Henrique Bursztyn, *A Survey on Morita equivalence of Quantum Tori, preprint*

[3] David Marker, **Model Theory: An Introduction**, Springer 2002

[4] Katrin Tent, Martin Ziegler, **A course in Model Theory**, Lecture Notes in Logic, Cambridge, 2012

[5] Masanori Itai, Boris Zilber, *Notes on a model theory of a quantum 2-torus*, submitted 2012

[6] Boris Zilber, *A note on the model theory of the complex field with roots of unity*, preprint, 1990

[7] Boris Zilber, *Structural approximation*, preprint, 2010

[8] Boris Zilber, **Zariski Geometries Geometry from the Logician's Point of View**, Cambridge University Press, 2010

[9] Boris Zilber, *A class of quantum Zariski geometries* in **Model Theory with Applications to Algebra and Analysis**, London Math. Soc. Lecture Note Series, Vol. 349, 293-326, 2008

[10] Boris Zilber, *Pseudo-analytic structures, quantum tori and non-commutative geometry*, preprint, 2005

[11] Boris Zilber,*Pseudo-exponentiation on algebraically closed fields of characteristic zero.* Ann. Pure Appl. Logic, 132(1):pp. 67 - 95, 2005

[12] Boris Zilber, *The noncommutative torus and Dirac calculus*, preprint, 2010

[13] Boris Zilber, *Raising to powers in algebraically closed fields*, Journal of Mathematical Logic, Vol. 3, No. 2, 217-238, 2003

[14] Boris Zilber, *The theory of exponential sums*, preprint

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