Resplendent models of o-minimal expansions of RCOF

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Abstract

In this paper, the author gives a characterization of resplendent models of the axioms, formulated by van den Dries, of restricted analytic real fields.

1 Introduction.

In classical model theory, we usually investigate properties of first order theories $T$, using their models. The properties that we are interested in are, for example, those concerning the existence of special types of models of $T$, such as prime models, saturated models and compact models, and so on. Of course, such models as listed above do not always exist. A saturated model of a complete $T$ exists under the assumption of G.C.H., but it does not exist in general without such assumptions. However, if we replace the definition of saturation by a weaker version, we can sometimes show its existence without set theoretic assumptions. Especially, every theory has a recursively saturated model. J.P. Ressayre shows the following important fact on recursive saturation, which states that resplendence and recursively saturation coincide for countable structures.

1 Fact. (J.P. Ressayre(1972)[5]) For each countable structure $M$ of finite language, $M$ is resplendent if and only if $M$ is recursively saturated.

It is not hard to show the existence of a recursively saturated model. From the fact above, we know that a resplendent model also exists for any countable theory. Resplendence seems a useful property to be studied. In Ressayer's proof of the only if part of the fact above, he finds some consistent sentence $\varphi(P)$ with a new unary predicate $P$ such that if a structure has a solution of $P$, then the structure is recursively saturated. There are some works aiming to get a more concrete $\varphi(P)$, when the axioms are specified. For example, P. D’Aquino, J.F. Knight and S. Starchenko find a characterization of recursively saturated model in the theory of real closed field([1]). Moreover, the author and A. Tsuboi found a characterization of recursively saturation in an o-minimal effectively model complete theory of real closed fields with a finite number of functions. This can be applied to A. J. Wilkie’s exponential fields([8]). However, we cannot apply this result to van den Dries’s restricted analytic field because the restricted analytic field is not a constructive object. The author considered a constructive fragment of theories for restricted analytic fields and find a characterization of recursive saturation for models of such theories.
2 Preliminaries and basic facts.

Let $L$ be a finite language, $M$ an $L$-structure, $T$ an $L$-theory (not necessarily complete). Let $L_{or}$ be the language $\{+, \cdot, 0, 1, <\}$ of ordered rings, $RCOF$ the theory of real closed fields, $PA$ the theory of first order arithmetic. Let $Th(M) := \{ \phi : \phi$ is an $L$-sentence, $M \models \phi \}$ be a theory of $M$. $Diag_{el}(M) := \{ \phi : \phi$ is an $L(M)$-sentence, $M \models \phi \}$ an elementary diagram of $M$.

2 Definition. (1). We say that $M$ is resplendent if for any new relational symbol $R \notin L$ and any $L(M) \cup \{ R \}$-sentence $\phi(R)$ if $Diag_{el}(M) \cup \{ \phi(R) \}$ is consistent, then there is an interpretation $R^M$ on $M$ such that $(M, R^M) \models \phi(R)$.

(2). We say that $M$ is recursively saturated if every recursive type (with finite parameters) is realized in $M$.

3 Fact. (J.P. Ressayre(1972)[5]) For each countable structure $M$ of finite language, $M$ is resplendent if and only if $M$ is recursively saturated.

In Ressayer's proof of the only if part of the fact above, he finds some consistent sentence $\varphi(P)$ with a new unary predicate $P$ such that if a structure has a solution of $P$, then the structure is recursively saturated. By the meaning of $\varphi(P)$ in Ressayer's proof, we can construct a model of arithmetic from a solution of $\varphi(P)$.

4 Question. If a theory $T$ naturally involves some arithmetic structure, then $\varphi(P)$ can be taken as a natural form under $T$.

Next fact is an answer in the case of $T = RCOF$ for this question.

5 Definition. Let $K$ be an ordered field. We call an ordered subring $Z \subset K$ an integer part if it satisfies $\forall x \in K, \exists ! n \in Z$ s.t. $n \leq x < n + 1$.

6 Fact. (P. D’Aquino, J.F. Knight and S. Starchenko (2010)[1]) For a countable ordered field $K$, the followings are equivalent:

- $K$ is a recursively saturated model of $RCOF$;
- $K$ has a non-archimedean integer part whose the non-negative part satisfies $PA$.

3 Background.

In this section, we introduce the previous investigation (A. Tsuboi and T.(2013)[8]). Firstly, we show a characterization of recursively saturated model of $\sigma$-minimal expansion of the theory $RCOF$ as like Fact 6. Secondly, we will construct recursively saturated models by using nonstandard analysis.
3.1 $\omega$-minimal analogue

In the proof of Fact 6, we use $\omega$-minimality and quantifier elimination of the theory $RCOF$.

7 Question. Are there any analogue for $\omega$-minimal expantion of $RCOF$?

To answer the question above, we introduce definitions of $\omega$-minimality and weak form of quantifier elimination.

8 Definition. ($\omega$-minimal) We say that a theory $T$ is $\omega$-minimal if for any model $M$ of $T$ and any definable set $A \subset M$ (with parameters from $M$), $A$ can be described some finite union of open intervals and points.

9 Example. The following theories are $\omega$-minimal.
- The theory of real closed field: $RCOF$.
- $T_{exp} = Th(\mathbb{R}, +, \cdot, 0, 1, <, \exp)$.
- $T_{an} = Th(\mathbb{R}, +, \cdot, 0, 1, <, (f_{i})_{i})$.
  Where $(f_{i})_{i}$ is an enumeration of all analytic functions defined on closed box.

Next definition is a weak form of quantifier elimination.

10 Definition. We say that a theory $T$ is model complete if every $L$-formula $\phi(\overline{x})$ is equivalent to some existential $L$-formula $\psi(\overline{x})$ modulo $T$:

\[ \forall \phi(\overline{x}) \exists \psi(\overline{x}), T \models \forall \overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x})). \]

11 Example. $RCOF$, $T_{exp}$ and $T_{an}$ are model complete.

This definition is not sufficient to prove Fact. 6. We need an effective version of model completeness. Since $RCOF$ is recursively axiomatized, we can effectively obtain an equivalent existential formula $\psi(\overline{x})$ for above setting. In general, a decidable and model complete theory has same property.

12 Definition. We say that a theory $T$ is effectively model complete if there is a effective procedure finding an existential $L$-formula $\psi(\overline{x})$ which equivalent to any given $L$-formula $\phi(\overline{x})$ modulo $T$.

A. Macintyre and A. J. Wilkie defined the effectively model completeness for finding a decidability result of $T_{exp}$.

13 Fact. (A. Macintyre and A. J. Wilkie (1996)[4]) $T_{exp}$ is effectively model complete.

Lastly we will define a notion of definably approximation which means a relevance of an integer part and additional functions, e.g. an exponential function.

14 Definition. Let $R$ be a real closed ordered field with an integer part $Z$ and let $Q \subset R$ be the quotient field of $Z$. Suppose that $N$ (the nonnegative part of $Z$) satisfies $PA$. Finally, let $E : R^{n} \to R$ be a continuous function. We say that $E$ is $Z$-definably approximated if there exists a continuous function $F : N \times Q^{n} \to Q$ such that
• \(F\) is definable in the ordered field \(Q\);

• \(\{F(m, \bar{x}) : m \in \mathbb{N}\}\) converges uniformly to \(E(\bar{x})\) on closed bounded subsets of \(Q\).

More precisely, for all closed bounded boxes \(B \subset Q^n\) and \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that, for all \(n \in \mathbb{N}\) with \(n \geq n_0\) and all \(\bar{x} \in B\), \(R \models |E(\bar{x}) - F(n, \bar{x})| < \varepsilon\).

Then we can state an answer of the question above.

15 Theorem. (A. Tsuboi (2013)[8]) Let \(L\) be a language \(L_{or} \cup \{f_1, \ldots, f_k\}\), \(T\) an \(\omega\)-minimal and effectively model complete \(L\)-theory extended from \(RCOF\). Let \(R\) be a model of \(T\). \(R\) is a recursively saturated if there is an integer part \(Z \subset R\) such that

• the non-negative part of \(Z\) satisfies \(PA\), \(Z \neq \mathbb{Z}\) and

• each \(f_i\) is \(Z\)-definably approximated.

16 Corollary. Let \(R\) be a countable model of \(T_{exp}\). \(R\) is recursively saturated if and only if there is an integer part \(Z \subset R\) such that

• the non-negative part of \(Z\) satisfies \(PA\), \(Z \neq \mathbb{Z}\) and

• \(\exp(x)\) is \(Z\)-definably approximated.

Since \(T_{an}\) is a non-constructive object, we can not consider effective model completeness of \(T_{an}\). For application, we need to consider a constructive sub-theory of \(T_{an}\).

3.2 natural construction of recursively saturated real closed fields

In previous arguments, we give a characterization of recursively saturated model of a fixed theory. We do not consider applications of a given characterization. In this subsection, we will construct a recursively saturated models by using nonstandard analysis. We can easily construct a recursively saturated model by adding ideal elements, but our construction, showed below, is adding elements simultaneously.

17 Question. Is there a "natural" construction of recursively saturated model of \(RCOF\)?

Next theorem is an answer of the question above.

18 Definition. Let \(K\) be an ordered field and \(K^*\) an elementary extension of \(K\). We call following sets finite part and infinitesimal part respectively:

• \(F_K := \{x \in K^* : \exists q \in K \text{ s.t. } |x| < |q|\}\)

• \(I_K := \{x \in K^* : \forall q \in K^* \text{ s.t. } |x| < |q|\}\).

19 Theorem. (A. Tsuboi and T. (2013)[8]) Let \(K\) be an ordered field with an integer part \(Z\) satisfying \(PA\). If \(F_K \neq K^*\), the quotient field \(R := F_K/I_K\) satisfies \(RCOF\). Moreover, if \(Z \neq \mathbb{Z}\), then \(R\) is recursively saturated.
Similarly, we can construct a recursively saturated model of $T_{exp}$. Let $Q^*$ be an $\omega_1$-saturated elementary extension of $Q$. Let $(Q^*,Q) \equiv (Q^*,Q)$ where $Q \not\equiv Q$. Then $\mathbb{R} \equiv F_Q/I_Q \equiv F_Q/I_Q$. Let $\phi_2(x)$ be a defining formula of $Z$ in $Q$. (by J.Robinson) Let $Z := \phi_2(Q)$ and $Z^* := \phi_2(Q^*)$. Fix $n^* \in Z^* - Z$ and define $e(x) := \sum_{k=0}^{n^*} \frac{1}{k!} x^k$. Define $\exp^* : F_Q/I_Q \to F_Q/I_Q$ by $\exp^*(x + I_Q) := e(x) + I_Q$. Then $(\mathbb{R},\exp) \equiv (F_Q/I_Q,\exp^*) \equiv (F_Q/I_Q,\exp^*)$ holds. In $(F_Q/I_Q,\exp^*)$, $\exp^*$ is approximated in its integer part $\cong Z$.

20 Example. $(F_Q/I_Q, \exp^*)$ is a recursively saturated model of $T_{exp}$.

4 Results.

We will review a definition of the restricted analytic field.

21 Definition. Let $L_{an} = L_{or} \cup \{ f_i \}$, where $f_i$ is a function symbol, $R_{an} = (\mathbb{R},+,\cdot,0,1,<,(f_i))$ where $(f_i)$ is an enumeration of all analytic functions defined on closed box, and $T_{an} = Th(R_{an})$.

22 Theorem. $T_{an}$ is model complete and $o$-minimal.

For application of our theorem 15, we need a good fragment of $T_{an}$. Let $F$ be a class of restricted analytic functions. Then $L_{an}|F$ is $L_{or} \cup F$ and $T_{an}|F$ is restriction of $T_{an}$ to $L_{an}|F$. It is easy to show that every complete subtheory of $o$-minimal theory is $o$-minimal, i.e. $T_{an}|F$ is $o$-minimal(for any $F$). For a subtheory of $T_{an}$, A. Gabrièlov finds a condition of $F$ whether $T_{an}|F$ is model complete.

23 Theorem. (A. Gabrièlov(1996)[3]) Let $F$ be a class of restricted analytic functions closed under derivation. Then $T_{an}|F$ is model complete.

This proof is not prefer an effective version because it is a geometric. Since a proof of J.Denef and L.van den Dries (1988)[2] is algorithmic, we based on it. This proof of the model completeness of $T_{an}$ depends on following two basic facts for analytic functions.

- Wierstrass's preparation theorem,
- van den Dries's preparation theorem

In the first subsection, we will give an outline of effective proofs. We will give a coding of restricted analytic functions and statements of an effective form of facts above. Moreover, we give a condition of a set $F$ such that $T_{an}|F$ is eventually effective model complete. In the second subsection, we will give a characterization of recursively saturated model of $T_{an}|F$ for some $F$ and a construction of recursively saturated model of it.

4.1 effective proof of basic facts

We fix notations.

- $O_n$ : a ring of $n$-ary analytic functions on neighborhood of $0$;
• $R[Y]$: a polynomial ring of a new variable $Y$ with coefficients from a ring $R$;
• We use multi-index notations: if $\bar{i} = (i_1, ..., i_n)$, then $\bar{x}^\bar{i} = x_1^{i_1}x_2^{i_2}...x_n^{i_n}$;
• For a function $f(\bar{x}) = \sum_\bar{i} a_{\bar{i}}\bar{x}^\bar{i} \in O_n$ and a tuple of positive reals $\bar{e}$,

\[ ||f||_\bar{e} := \begin{cases} 
\sum_\bar{i} |a_{\bar{i}}\bar{e}^\bar{i}| & \text{if it converges} \\
\infty & \text{otherwise} 
\end{cases} \]

• $|\bar{x}| \leq |\bar{e}|$ means $\land_i |x_i| \leq |e_i|$.

We will define a coding of restricted analytic functions to prove effective results.

24 Definition. (coding of real) Let $(a^n)^n$ be a recursive sequence of rational numbers. We say that a real $\alpha \in \mathbb{R}$ is coded by $(a^n)^n$ if $\forall n, |\alpha - a^n| < 2^{-n}$.

25 Definition. (coding of restricted analytic function) Let $(a^n_i)^i$ be a recursive multi-indexed sequence of rational numbers and $\bar{e}, b, M, M$ are positive rational numbers. We say that a restricted analytic function $f(\bar{x}) = \sum_\bar{i} a_{\bar{i}}\bar{x}^\bar{i} \in O_n$ is coded by a code $C = ((a^n_i)^i; \bar{e}, b, M)$ if $||f||_\bar{e} < M$, $\alpha_i$ is coded by $(a^n_i)^i$, $b > 1$ and $dom(f) = \{\bar{x}: |\bar{x}| \leq |\bar{e}|\}$.

For a code $C = ((a^n_i)^i; \bar{e}, b, M)$, let $a^n_i, (\bar{c})\bar{e}(C), b(C)$ and $M(C)$ denote components $a^n_i, \bar{e}, b$ and $M$ of $C$ respectively.

26 Example. Let $\pi_n$ be $n$ decimal digits of $\pi$ and $M$ a sufficiently large positive number. Then the restricted sine function $\sin(\pi x)|[-1, 1]$ can be coded by $((\frac{1}{2}(\frac{1}{2}i+1)!\cdot\pi_i^n)^i, 1, 2, M)$.

Remark: Let $f \in O_n$ and $g_1, ..., g_n \in O_n$ be coded by $C, D_1, ..., D_n$ respectively. If $M(D_i) \leq \bar{e}(C), (i < n)$, then $f(g_1, ..., g_n)$ can be coded by some $G = C_{\text{com}}(C, D_1, ..., D_n)$.

To state the Wierstrass’s preparation, we define the regularity of an analytic function.

27 Definition. (regularity) We say that a restricted analytic function $f(x_1, ..., x_n) \in O_n$ is regular of order $p$ with respect to $x_n$ if $f(0, 0, ..., x_n) = c \cdot x_n^p + o(x_n^p)$ where $c \neq 0$.

28 Fact. (Wierstrass’s preparation) Let $\Phi \in O_n$ be regular of order $p$ with respect to $x_n$. There exists unique unit $Q \in O_n$ and unique $R \in O_{n-1}[x_n]$ regular of order $p$ with respect to $x_n$ such that $R = \Phi Q$.

29 Lemma. (Effective Wierstrass’s preparation) There exist recursive functions $C_{WQ}(C, n), C_{WR}(C, n)$ which map from pairs of a code and a natural number to codes such that the follows holds: for any given $\Phi \in O_n$ which is regular of order $p$ with respect to $x_n$ and coded by $C$, $Q \in O_n$ and $R \in O_{n-1}[x_n]$ are obtained by the Wierstrass’s preparation; then for any sufficiently large $n \in \mathbb{N}, Q, R$ are coded by $C_{WQ}(C, n), C_{WR}(C, n)$ respectively.

Unfortunately, there is no effective procedure finding sufficiently large $n$. This problem deduce to check $\forall X, R(X) = \Phi(X)Q(X)$. Next, we will state the van den Dries’s preparation and an effective form of this.
30 Fact. (van den Dries’s preparation) Let \( X = (X_1, \ldots, X_n) \), \( Y = (Y_1, \ldots, Y_m) \), \( m > 0 \) and \( \Phi(X, Y) \in O_{n+m} \). There exist \( d \in \mathbb{N} \), \( a_i(X) \in O_n \) and units \( u_i(X, Y) \in O_{n+m} \) \( (|\overline{i}| < d) \) such that:
\[
\Phi(X, Y) = \sum_{|\overline{i}| < d} a_i(X) Y^\overline{i} u_i(X, Y).
\]

31 Lemma. (Effective van den Dries’s preparation) Let \( X = (X_1, \ldots, X_n) \), \( Y = (Y_1, \ldots, Y_m) \), \( m > 0 \). There exist recursive functions \( C_vA(C, d, n, \overline{i}) \), \( C_vU(C, d, n, \overline{i}) \) such that the followings holds: for any given \( \Phi(X, Y) \in O_{n+m} \) be coded by \( C \), for any sufficiently large \( d \in \mathbb{N} \), there exists \( n \in \mathbb{N} \) such that \( \Phi(X, Y) = \sum_{|\overline{i}| < d} a_i(X) Y^\overline{i} u_i(X, Y) \), where \( a_i(X) \), \( u_i(X, Y) \) are coded by \( C_vA(C, d, n, \overline{i}) \), \( C_vU(C, d, n, \overline{i}) \) respectively and each \( u_i \) is a unit.

There is a problem how to find \( d, n \) effectively. This problem deduce to check \( \forall X \in \mathbb{R}, \Phi(X, Y) = \sum_{|\overline{i}| < d} a_i(X) Y^\overline{i} u_i(X, Y) \). Then we will give a condition of a set \( F \) such that \( T_{an}|F \) is eventually effective model complete and a definition of eventually effective model complete.

32 Definition. We say that a set \( S \) of codes closed if it is closed under \( C_{com}, C_vA, C_vU, C_{WQ}, C_{WR} \) and contains codes of bounded polynomial functions. Let \( F_S = \{ f \in \cup_n O_n : f \) is coded by some element of \( S \} \).

33 Definition. We say that an \( L \)-theory \( T \) is eventually effectively nearly model complete if there is an effective procedure, for any given formula \( \Phi(X) \), finding recursive enumeration of boolean combinations of existential \( L \)-formulas \( \{ \psi_n(x) \}_{n \in \omega} \) such that \( T \models \Phi(x) \rightarrow \psi_m(x) \) for any \( m \) and \( T \models \Phi(x) \leftarrow \bigwedge_{m < n} \psi_m(x) \) for any sufficiently large \( n \).

We obtain a weak form of the effective model completeness for some fragment of \( T_{an} \).

34 Theorem. (T. 2013) Let \( S \) be a r.e. closed set of codes, \( L = L_{an}|F_S \). Then \( T_{an}|F_S = Th(R_{an}|F_S) \) is eventually effectively nearly model complete.

4.2 main results

Similarly to a proof of Theorem 15, we will show the main theorem.

35 Theorem. (revisited A. Tsuboi (2013) : modified by T.) Let \( L \) be a language \( L_{or} \cup \{ f_i \}_{i \in \mathbb{N}} \), \( T \) an \( \sigma \)-minimal and eventually effectively nearly model complete \( L \)-theory extended from \( RCOF \). Let \( R \) be a model of \( T \). Then \( R \) is a recursively saturated if there is an integer part \( Z \subset R \) such that:

- the non-negative part of \( Z \) satisfies \( PA, Z \neq \mathbb{Z} \) and

- each \( f_i \) is \( Z \)-definably approximated by a \( \Sigma_{k_0} \)-formula where \( k_0 \) does not depend on \( i \).

We fix \( L, T, R \) and \( Z \) as in Theorem 35, and prove a series of lemmas before proving the theorem. Let \( N \) be the non-negative part of \( Z \), \( Q \) the quotient field of \( Z \) in \( R \). Choose \( k_0 \) such that every \( f_i(\bar{x}) (i \in \omega) \) is \( Z \)-definably approximated by a \( \Sigma_{k_0} \)-formula. To prove Theorem 35, we need following lemmas proved in [8].
36 Lemma. ([8]) Every \( L \)-term (i.e., every term constructed from +, \cdot and the \( f_i \)'s) is \( Z \)-definably approximated by \( \Sigma_{k_0} \)-formulas.

37 Lemma. ([8] modified by T.) Let \( \varphi(\bar{x}) \) be a boolean combination of existential \( L \)-formulas. Then we can effectively find an \( L \)-formula \( \varphi_0(\bar{x}) \) and an \( L_{or} \)-formula \( \varphi'(\bar{x}) \) such that

- \( R \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \varphi_0(\bar{x})) \);
- \( R \models \varphi_0(\bar{b}) \iff Q \models \varphi'(\bar{b}) \), for all \( \bar{b} \in Q \).

The formula \( \varphi' \) obtained in Lemma 37 is a \( \Sigma_{k_0+5} \)-formula.

38 Lemma. ([8] modified by T.) Let \( \varphi(\bar{x}) \) and \( \psi(\bar{x}) \) be boolean combinations of existential \( L \)-formulas such that \( R \models \forall \bar{x}(\varphi \rightarrow \psi) \). Let \( \varphi' \) and \( \psi' \) be the formulas obtained in Lemma 37. Then \( Q \models \forall \bar{x}(\varphi' \rightarrow \psi') \).

39 Lemma. ([8]) For any \( \bar{a} \in R \), \( \text{dcl}(\bar{a}) \) is a bounded subset of \( R \).

Proof. (Proof of Theorem 35) Let \( \Sigma(\bar{x}, \bar{a}) = \{ \varphi_i(\bar{x}, \bar{a}) : i \in \omega \} \) be a (non-algebraic) recursive type with \( \bar{a} \in R \). We can assume that \( \varphi_{i+1}(\bar{x}, \bar{a}) \rightarrow \varphi_i(\bar{x}, \bar{a}) \) holds in \( R \). Since other cases can be treated similarly, we assume that \( \bar{a} \in R \setminus \text{dcl}(\emptyset) \) and that elements in \( \bar{a} \) are mutually non-algebraic. For each \( \varphi_i(\bar{x}, \bar{a}) \in \Sigma \), let \( \theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1) = \theta_i(u_0, u_1, v_{0j}, v_{1j}) \) be the formula

\[
\forall x \forall y \left( u_0 < x < u_1 \land \bigwedge_{j<k} v_{0j} < y_j < v_{1j} \rightarrow \varphi_i(x, y) \right),
\]

where \( k \) is the length of \( \bar{a} \). Notice that \( \exists u_0 u_1 (u_0 < u_1 \land \bigwedge_{j<k} v_{0j} < a_j < v_{1j} \land \theta_i(u_0, u_1, \bar{v}_0, \bar{v}_1)) \) is satisfiable in \( R \). (We can use the cell decomposition theorem to see this.) We can assume that \( \theta_i \) is an boolean combination of existential formulas by the eventually effective nearly model completeness assumption.

By the \( \omega \)-minimality, there exist minimum \( \bar{b}_0 \) and maximum \( \bar{b}_1 \) (in the lexicographic ordering) such that \( \exists u_0 u_1 (u_0 < u_1 \land \bigwedge_{j<k} b_{0j} < a_j < b_{1j} \land \theta_i(u_0, u_1, \bar{b}_0, \bar{b}_1)) \) holds in \( R \). Therefore, \( \bar{b}_0, \bar{b}_1 \in \text{dcl}(\bar{a}) \cup \{ \pm \infty \} \). Using Lemma 39, choose a sufficiently large integer \( n^* \) such that \( \text{dcl}(\bar{a}) < n^* \). We can choose \( \bar{c}_0, \bar{c}_1 \in Q \) with \( \sum_{j<k} |c_{0j} - c_{1j}| < 1/n^* \) such that \( b_{0j} < c_{0j} < a_j < c_{1j} < b_{1j} \) (\( j < k \)). Then \( \exists u_0 u_1 (u_0 < u_1 \land \theta_i(u_0, u_1, \bar{c}_0, \bar{c}_1)) \) holds in \( R \) regardless of the choice of \( i \in \omega \).

For each \( \theta_i \), choose a formula \( \theta'_i \) having the property described in Lemma 37. Namely, choose \( \theta'_i \) such that

1. \( R \models \forall u_0 u_1 \bar{v}(\theta_i \leftrightarrow \theta_{i,0}) \);
2. \( R \models \theta_{i,0}(q_0, q_1, \bar{r}, \bar{s}) \iff Q \models \theta'_i(q_0, q_1, \bar{r}, \bar{s}) \), for any \( q_0, q_1, \bar{r}, \bar{s} \in Q \).

In the present situation, \( \exists u_0 u_1 (u_0 < u_1 \land \theta_{i,0}(u_0, u_1, \bar{c}_0, \bar{c}_1)) \) holds in \( R \). Since \( u_0, u_1 \) can be chosen from \( Q \), \( \exists u_0 u_1 (u_0 < u_1 \land \theta'_i(u_0, u_1, \bar{c}_0, \bar{c}_1)) \) holds in \( Q \). Then, by Lemma 38, \( \{ \theta'_i(u_0, u_1, \bar{c}_0, \bar{c}_1) : i \in \omega \} \) is a recursive \( \Sigma_{k_0+5} \)-type in \( Q \). So, by the \( \Sigma_{k_0+5} \)-recursive saturation of \( Q \), there exists \( (d_1, d_2) \in Q^2 \) such that \( Q \models \bigwedge_{i \in \omega} \theta'_i(d_0, d_1, \bar{c}_0, \bar{c}_1) \). Hence, \( \Sigma(x, \bar{a}) \) is realized in \( R \) by any \( e \) between \( d_0 \) and \( d_1 \). \( \square \)
40 Example. Let $F_{\sin}$ be a closed r.e. set contains a code of $\sin(\pi x)|[-1,1]$. Let $T_{\sin} = T_{an}|F_{\sin}$.

41 Corollary. Let $R$ be a model of $T_{\sin}$. $R$ is recursively saturated if there is an integer part $Z \subset R$ such that:

- the non negative part of $Z$ satisfies $PA$, $Z \neq \mathbb{Z}$ and
- each $f \in F_{\sin}$ is $Z$-definably approximated by a $\Sigma_{k_0}$-formula where $k_0$ does not depend on $f$.

Finally, we will construct a recursive saturated model of $T_{\sin}$ by using nonstandard analysis. Let $Q, Q^*, Z, Z^*, n^*$ be in a construction of Example 20. For any $f \in F_{\sin}$ coded by $((a_i^n)_{i=0}^n; e, b, M)$, define $f^*: F_Q/I_Q \to F_Q/I_Q$ by $f^*(x + I_Q) := \sum_{|i|<n} (\lim_n a_i^n x^i) + I_Q$. Then $(\mathbb{R}, F_{\sin}) \cong (F_Q/I_Q, \{f^* : f \in F_{\sin}\}) \equiv (F_Q/I_Q, \{f^* : f \in F_{\sin}\})$. In $(F_Q/I_Q, \{f^* : f \in F_{\sin}\})$, $f^*$ is approximated in its integer part $\cong Z$.

42 Example. $(F_Q/I_Q, \{f^* : f \in F_{\sin}\})$ is a recursively saturated model of $T_{\sin}$.

References


