<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>分割と分け叉：独立性と次元のモデル論的側面の証明</td>
</tr>
<tr>
<td>集合</td>
<td>作成者</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>タイトル</td>
<td>数理解析研究所講究録</td>
</tr>
</tbody>
</table>
Dividing and Forking
– A Proof of the Equivalence –

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1 Introduction and Preliminaries

Let $T$ be a simple complete theory. Then the two notions forking and dividing are equivalent. (See [1].) The usual proof of this fact needs Erdős-Rado theorem, a basic result in combinatorial set theory. Erdős-Rado theorem is a theorem on uncountable cardinals, extending Ramsey's theorem. So it is somewhat strange to use such a theorem for proving the fact when the language is countable. In this article, we present a new proof that will only use compactness and a Ramsey-type argument.

We explain the notations in this article and recall some basic definitions. $L$ is a language and $T$ is a complete theory formulated in $L$. Although the countability of $L$ is not necessary, we assume that $L$ is countable for simplicity. We fix a big saturated model $\mathcal{M}$ of $T$ and we work in $\mathcal{M}$. Small subsets of $\mathcal{M}$ are denoted by $A, B, \ldots$. Finite tuples in $\mathcal{M}$ are denoted by $a, b, \ldots$. Variables are $x, y, \ldots$. Formulas are denoted by $\varphi, \psi, \ldots$. If all the free variables of $\varphi$ are contained in $x$, $\varphi$ is sometimes written as $\varphi(x)$. For a set $A$, $L(A)$ is the language $L$ augmented by the names (constants) for $a \in A$. For simplicity of the notation, we write $\varphi \in L(A)$, if $\varphi$ is a formula in $L(A)$. In general, a formula $\varphi \in L(A)$ has the form $\psi(x, a)$, where $\psi(x, y)$ is an $L$-formula with $xy$ free and $a$ is the list of parameters (from $A$) appearing in $\varphi$. $a$ will be called the parameters of $\varphi$.

A sequence $\{a_{i} : i \in \omega\}$ is called an indiscernible sequence over $A$, if for any strictly increasing $f : \omega \to \omega$, there is an automorphism $\sigma$ of $\mathcal{M}$ that extends the mapping $id_{A} \cup \{(a_{i}, a_{f(i)})\}_{i \in \omega}$. We say that $\{a_{i} : i \in \omega\}$ starts with $a$, if $a_{0} = a$.

Definition 1. A formula $\varphi(x, a)$ divides over $A$ if there is an indiscernible sequence $\{a_{i} : i \in \omega\}$ starting with $a$ such that $\{\varphi(x, a_{i}) : i \in \omega\}$ is inconsistent.

A set $\Phi$ of formulas is said to be $k$-inconsistent, if every subset $\Psi_{0} \subset \Phi$ of size $k$ is inconsistent. If $\varphi(x, a)$ divides over $A$, by the indiscernibility of $\{a_{i} : i \in \omega\}$, there is some $k \in \omega$ such that $\{\varphi(x, a_{i}) : i \in \omega\}$ is $k$-inconsistent. In this case we say that $\varphi(x, a)$ $k$-divides over $A$.

Definition 2. A formula $\varphi(x, a)$ forks over $A$ if it is covered by a finite number of dividing formulas, more precisely, if there is a finite number of formulas $\psi_{i}(x, b_{i})$ ($i = 1, \ldots, n$) with the following properties:
1. \( \mathcal{M} \models \forall x[\varphi(x, a) \rightarrow \bigvee_{i=1,\ldots,n} \psi_i(x, b_i)] \);

2. Each \( \psi_i(x, b_i) \) divides over \( A \).

\( T \) is called simple if there is a bound for the length of a dividing sequence of complete types. The simplicity of \( T \) is equivalent to the finiteness of the rank defined below:

**Definition 3.** Let \( \Sigma(x) \) be a set of formulas with parameters (with \( x \) free). Let \( \Phi(x, y) \) be a finite set of \( L \)-formulas and let \( k \in \omega \). The rank \( D(\Sigma(x), \Phi(x, y), k) \) is defined by:

1. \( D(\Sigma(x), \Phi(x, y), k) \geq 0 \) if \( \Sigma(x) \) is consistent;
2. \( D(\Sigma(x), \Phi(x, y), k) \geq \alpha + 1 \) if there is \( a \) and \( \varphi \in \Phi \) such that \( D(\Sigma(x) \cup \{ \varphi(x, a) \}, \Phi(x, y), k) \geq \alpha \) and such that \( \varphi(x, a) \) \( k \)-divides over the parameter set of \( \Sigma \);
3. \( D(\Sigma(x), \Phi(x, y), k) \geq \delta \) (a limit ordinal) if \( D(\Sigma(x), \Phi(x, y), k) \geq \alpha \) for any \( \alpha < \delta \).

In the same manner,

## 2 Simple theories

In what follows, \( T \) is a simple complete theory. Let us begin with the following lemma. A proof here is essentially the same as the one presented in Ziegler’s book [3].

**Lemma 4.** Let \( \varphi(x) \in L(A) \). Then \( \varphi(x) \) does not fork over \( A \).

**Proof.** For simplicity we assume \( A = \emptyset \). Suppose otherwise and choose \( \psi_i(x, b) \) (\( i = 1, \ldots, n \)) and \( k \in \omega \) such that

1. each \( \psi_i(x, b) \) \( k \)-divides over \( \emptyset \);
2. \( \forall x[\varphi(x) \rightarrow \bigvee_{i=1,\ldots,n} \psi_i(x, b)] \) holds.

Then we choose \( n_1, \ldots, n_m \leq n \) and \( b_1, \ldots, b_m \) (copies of \( b \)) such that

3. \( \psi_{n_i}(x, b_i) \) \( k \)-divides over \( \{ b_j : j < i \} \), for each \( i = 1, \ldots, n \);
4. \( \varphi(x) \wedge \bigwedge_{i=1,\ldots,m} \psi_{n_i}(x, b_i) \) is consistent, and its \( D(*, \{ \psi_i : i = 1, \ldots, n \}, k) \)-rank is minimum among such.

By moving the \( b_i \)'s, we can assume that each \( \psi_{n_i}(x, b_i) \) \( k \)-divides over \( \{ b \} \cup \{ b_j : j < i \} \).

By conditions 2 and 4, there is \( n_{m+1} \leq n \) such that

\( \varphi(x) \wedge \bigwedge_{i=1,\ldots,m} \psi_{n_i}(x, b_i) \wedge \psi_{n_{m+1}}(x, b) \) is consistent.

Since \( \psi_{n_{m+1}}(x, b) \) divides, by letting \( b_{m+1} = b \), we have

\[
D(\varphi(x) \wedge \bigwedge_{i=1,\ldots,m} \psi_{n_i}(x, b_i), \Psi, k) > D(\varphi(x) \wedge \bigwedge_{i=1,\ldots,m+1} \psi_{n_i}(x, b_i), \Psi, k)
\]

where \( \Psi = \{ \psi_i : i = 1, \ldots, n \} \). This contradicts our choice of \( n_i \) (\( i \leq m \)) and \( b_i \) (\( i \leq m \)) (condition 4). \( \square \)
Remark 5. 1. Let $A \subset B$ and $p(x) \in S(A)$. Then there is an extension $q(x) \in S(B)$ of $p(x)$ such that $q(x)$ does not divide over $A$. This can be shown as follows: Let $\Gamma(x) = p(x) \cup \{ \neg \varphi(x) \in L(B) : \varphi(x) \text{ does not divide over } A \}$. Then $\Gamma(x)$ is consistent, since otherwise we would have $p(x) \models \varphi_1(x) \lor \cdots \lor \varphi_n(x)$, for some $\varphi_i$ dividing over $A$. So $p(x) \in S(A)$ forks over $A$, contradicting the above lemma. Choose $a \models \Gamma$, and let $q(x) = \text{tp}(a/B)$. Then, clearly $q(x)$ does not divide over $A$.

2. Suppose that $\text{tp}(a/Abc)$ does not divide over $A$ and that $\text{tp}(b/Ac)$ does not divide over $A$. Then $\text{tp}(ab/Ac)$ does not divide over $A$: Let $\varphi(x, y, c) \in \text{tp}(ab/Ac)$. Let $I = \{ c_i : i \in \omega \}$ be an arbitrary indiscernible sequence with $c_0 = c$. Since $\text{tp}(b/Ac)$ does not divide over $A$, there is $b'$ (a copy of $b$ over $Ac$) such that $I$ is $Ab'$-indiscernible. For an $A$-automorphism $\sigma : b' \to b$, $\sigma(I)$ is an $Ab$-indiscernible sequence. Notice then that $J = \{ b\sigma(c_i) : i \in \omega \}$ is an $A$-indiscernible sequence with $b\sigma(c_0) = b$. Since $\text{tp}(a/Abc)$ does not divide over $A$, there is $a'$ (a copy of $a$ over $A$) such that $a' \models \bigwedge_{d \in J} \varphi(x, d)$. So $\sigma^{-1}(a') = \bigwedge_{i \in \omega} \varphi(x, b', c_i)$. In particular, $\{ \varphi(x, y, c_i) : i \in \omega \}$ is satisfiable.

Lemma 6. For each non-algebraic type $p(x) \in S(A)$, there is an $A$-indiscernible sequence $J = \{ b_i : i \in \omega \}$ in $p$ such that $\text{tp}(J \setminus \{ b_0 \}/Ab_0)$ does not divide over $A$.

Proof. First we inductively choose $a_i$'s realizing $p$ such that, for each $i \in \omega$,

$$\text{tp}(a_i/A_i) \text{ does not divide over } A,$$

where $A_i = A \cup \{ a_j : j < i \}$. Then, by an iterative use of Remark above, $\text{tp}(\{ a_j : j > 0 \}/Aa_0)$ does not divide over $A$. Similarly we can show that $\text{tp}(\{ a_j : j > i \}/Aa_i)$ does not divide over $A$, for each $i$.

Now let $\Gamma(\{ x_i : i \in \omega \})$ be the following set of $L(A)$-formulas:

$$\bigcup_{i \in \omega} p(x_i) \cup \bigcup_{i \in \omega, F \subseteq \omega} \{ \neg \varphi(x_F, x_i) : \varphi(x_F, a_0) \text{ divides over } A \},$$

where $x_F = x_{i_0}, \ldots, x_{i_k}$ if $F = \{ i_0 < \cdots < i_k \}$. Clearly $\Gamma$ is realized by $I = \{ a_i : i \in \omega \}$.

Moreover, since each $a_i$ realizes $p$, any infinite subsequence of $I$ realizes $\Gamma$. In other words, $\Gamma$ has the subsequence property. So there is an $A$-indiscernible sequence $J = \{ b_i : i \in \omega \}$ realizing $\Gamma$. It is clear that $\text{tp}(J \setminus \{ b_0 \}/Ab_0)$ does not divide over $A$. \hfill $\Box$

Lemma 7. Suppose that $\varphi(x, a)$ divides over $A$. Let $p(x) = \text{tp}(a/A)$ and choose an $A$-indiscernible sequence $J = \{ b_i : i \in \omega \}$ in $p$ having the property described in Lemma 6. Then $\{ \varphi(x, b_i) : i \in \omega \}$ is inconsistent.

Proof. Choose $k$ such that $\varphi(x, a)$ $k$-divides over $A$, and choose an $A$-indiscernible sequence $I = \{ a_i : i \in \omega \}$ such that $\{ \varphi(x, a_i) : i \in \omega \}$ is $k$-inconsistent. By moving $J$ by an $A$-automorphism, we may assume that $b_0 = a_0$. Since $\text{tp}(J \setminus \{ b_0 \}/b_0)$ does not divide, there is $\{ b'_i : i > 0 \}$ (a copy of $J \setminus \{ b_0 \}$ over $Ab_0$) such that

$$a_i(b'_i : i > 0) \equiv_A a_j(b'_j : i > 0) \equiv_A J$$

holds for any $i, j$. Moreover, by Ramsey's theorem, we can assume that $J$ is indiscernible over $A\{ b'_i : i > 0 \}$. 

Claim A. \( \Phi(x) = \{ \varphi(x, b_i) : i \in \omega \} \) is \( k \)-inconsistent.

Suppose otherwise and let \( \alpha = D(\Phi(x), \varphi(x, y), k) \). Since \( J \) is an indiscernible sequence, we have \( J \equiv_A J \smallsetminus \{ b_0 \} \equiv_A \{ b'_i : i > 0 \} \). So we have

\[
\alpha = D(\{ \varphi(x, b_i) : i > 0 \}, \varphi(x, y), k).
\]

However, \( \varphi(x, a_0) \) divides over \( A \{ b'_i : i > 0 \} \), so we must have \( D(\{ \varphi(x, a_0) \} \cup \{ \varphi(x, b_i) : i > 0 \}, \varphi(x, y), k) < \alpha \). This is a contradiction. \( \square \)

Proposition 8. For \( i = 1, \ldots, m \), let \( \varphi_i(x, a) \) be a formula that divides over \( A \). Then

\[
\bigvee_{i=1,\ldots,m} \varphi_i(x, a) \text{ divides over } A.
\]

Proof. Choose \( J \) as in Lemma 7, then for each \( i \) there is \( k_i \) such that \( \{ \varphi_i(x, b) : b \in J \} \) is \( k_i \)-inconsistent. Let \( k = \max \{ k_1, \ldots, k_m \} \). Then \( \{ \varphi_i(x, b) : b \in J \} \) is \( mk \)-inconsistent. Hence \( \bigvee_{i=1,\ldots,m} \varphi_i(x, a) \) divides over \( A \). \( \square \)

Lemma 9. Suppose that \( p(x) \in S(A) \) does not divide (fork) over \( A_0 \subset A \). For any \( B \supset A \), there is \( a \models p \) such that \( \text{tp}(a/B) \) does not divide over \( A_0 \).

Proof. Let \( \Psi(x) \) be the following set of \( L(B) \)-formulas:

\[
p(x) \cup \{ \neg \varphi(x) \in L(B) : \varphi(x) \text{ divides over } A_0 \}.
\]

\( \Psi(x) \) is consistent, since otherwise we would have that \( p(x) \) forks over \( A_0 \). Let \( a \models \Psi(x) \). Then it follows that \( \text{tp}(a/B) \) does not divide over \( A_0 \). \( \square \)

Proposition 10 (Symmetry). \( \text{tp}(a/Ab) \) does not divide over \( A \) \( \Rightarrow \) \( \text{tp}(b/Aa) \) does not divide over \( A \).

Proof. First we inductively choose \( a_i \)'s such that

- \( a_i \models \text{tp}(a/Ab) \);
- \( \text{tp}(a_i/A \cup \{ a_j : j < i \}) \) does not divide over \( A \).

This process can be done by an iterative use of Lemma 9. As in the proof of Lemma 7, by compactness, we can assume that \( I = \{ a_i : i \in \omega \} \) is an \( A \)-indiscernible sequence satisfying the conditions

1. \( \text{tp}(\{ a_i : i > 0 \}/Aa_0) \) does not divide over \( A \) (\( i \in \omega \));
2. By letting \( q_a(x) = \text{tp}(b/Aa) \), \( b \) is a common solution of \( q_{a_i}(x) \) (\( i \in \omega \)).

Thus \( q_a(x) \) does not divide over \( A \), by Lemma 7. \( \square \)
References


