A proof of the existence of indiscernible trees without Erdos-Rado theorem (Model theoretic aspects of the notion of independence and dimension)

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A proof of the existence of indiscernible trees without Erdős-Rado theorem

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Our interest in this paper is to see the similarity between Erdős-Rado theorem and compactness argument using Ramsey theorem in model theory. Erdős-Rado theorem is a theorem in infinitary combinatorics that generalizes Ramsey theorem to handle uncountable situations. In model theory, compactness arguments are available, so arguments tend to be settled in countable situation.

We give a proof without Erdős-Rado theorem to the next theorem.

Theorem 3.1.16. Let $B$ be a set of parameters, and $\Gamma(x_{\omega}^{<\omega})$ be a set of $\mathcal{L}_{B}$-formulas. If $\Gamma(x_{\omega}^{<\omega})$ has $\mathcal{L}_{S}$-subtree property, then $\Gamma$ is realized by an $\mathcal{L}_{S}$-indiscernible tree over $B$.

This theorem is proved with Erdős-Rado theorem in [2] and [3], while we use compactness arguments and Ramsey theorem.

Byunghan Kim, Hyeung-Joon Kim, and Lynn Scow recently revised their preprint[4], and it contains essentially the same argument of this paper. We have constructed the content independently.

We work in a complete theory $T$ in a language $\mathcal{L}$ throughout this paper. Let $M$ be a big model of $T$. We write $\langle n_{1}\ldots n_{k}\rangle$ to refer the element of $\omega^{<\omega}$ of length $k$ whose $i$-th value is $n_{i}$. For $\eta_{1}, \eta_{2} \in \omega^{<\omega}$, we write $\eta_{1}\eta_{2}$ to refer the concatenation of $\eta_{1}$ and $\eta_{2}$. For a set $S$ and an indexed set $(a_{s})_{s\in S}$, we write $a_{S}$ to denote $(a_{s})_{s\in S}$.

1 Theorems in infinitary combinatorics

1.1 Ramsey’s theorem and Erdős-Rado theorem

Infinite Ramsey’s theorem and Erdős-Rado theorem are theorems in infinitary combinatorics. Erdős-Rado theorem is a generalization of Ramsey’s theorem to uncountable situations.

Definition 1.1.1. For cardinals $\alpha, \beta, \gamma$ and for $n < \omega$, we write

$$\alpha \rightarrow (\beta)^{\gamma}_{n}$$

whenever $|X| = \alpha$ and $f : [X]^{n} \rightarrow \gamma$, there exists $Y \subset X$ with $|Y| = \beta$ such that $f([Y]^{n})$ is a singleton.

Theorem 1.1.2 (Infinite Ramsey’s Theorem). For all $k, n \in \omega$,

$$\aleph_{0} \rightarrow (\aleph_{0})^{k}_{n}.$$
Theorem 1.1.3 (Erdős-Rado Theorem). For all $n \in \omega$ and infinite cardinal $\kappa$,
$$\exp_n(\kappa)^+ \rightarrow (\kappa^+)^{n+1}_\kappa,$$
where $\exp_n(\kappa)$ is inductively defined by $\exp_0(\kappa) = \kappa$, $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

2 Indiscernible structures

We introduce indiscernible sequences and $L_S/L_1$-indiscernible trees. We also define subsequence property and $L_S/L_1$-subtree property, which later we prove that they induces the existence of indiscernible structures.

2.1 Indiscernible sequences

Definition 2.1.4 (Indiscernible sequences). Let $L_o = \{<\}$ and $L_o$-structure $I$ be a totally ordered set, and let $B \subset M$. For $a_I \subset M$, we say $a_I$ is an indiscernible sequence over $B$ if for all $I_0, I_1 \subset I$ such that $I_0 \upmodels I_1$, it holds that $\text{tp}(a_{I_0}/B) = \text{tp}(a_{I_1}/B)$.

Be careful the index set $I$ is not a subset of the big model $M$ and the $I$-indexed set $a_I$ is a subset of $M$.

Subsequence property was introduced by Tsuboi in his lecture note in 1999.

Definition 2.1.5 (Subsequence property). Let $L_o = \{<\}$ and $L_o$-structure $I$ be a totally ordered set. For a set of formulae $\Gamma(x_I)$, we say $\Gamma$ has subsequence property if
$$\bigcup \{ \Gamma(x_{\sigma(I)}) \mid \sigma : I \rightarrow I \text{ is an } L_o\text{-embedding} \}$$
is consistent.

Example 2.1.6. Let $\Gamma(x_\omega)$ be the set of formulas expressing "$x_\omega$ is an indiscernible sequence." Then, $\Gamma$ has subsequence property. $\Gamma$ can be concretely written as
$$\left\{ \varphi(x_I) \leftrightarrow \varphi(x_J) \mid \varphi \in L, \ I, J \subset \omega, \ I \upmodels J \right\}.$$

Example 2.1.7. Let $\Gamma(x_\omega, y_\omega)$ be the set of formulas expressing "$(x_i, y_i)_{i \in \omega}$ witnesses the order property of $\varphi(x, y)$." Then, $\Gamma$ has the subsequence property.

$\Gamma$ can be concretely written as
$$\{ \varphi(x_i, y_j) \mid i < j < \omega \} \cup \{ \neg \varphi(x_j, y_i) \mid j \leq i < \omega \}.$$

The following lemma guarantees the existence of indiscernible sequences.

Lemma 2.1.8 (Tsuboi 1999). Let $B$ be a set of parameters, and $\Gamma(x_\omega)$ be a set of $L_B$-formulas. If $\Gamma(x_\omega)$ has subsequence property, then $\Gamma$ is realized by an indiscernible sequence over $B$.

Proof. We show $\Gamma(x_\omega) \cup \text{"}x_\omega\text{" is an indiscernible sequence over } B$" is consistent, where
$$\text{"}x_\omega\text{" is an indiscernible sequence over } B =$$
$$\left\{ \varphi(x_I) \leftrightarrow \varphi(x_J) \mid \varphi \in L_B, \ I, J \subset \omega, \ I \upmodels J \right\}.$$
We use compactness argument. We fix $\mathcal{L}_B$-formulas $\varphi_1, \ldots, \varphi_m$ each of which has $n$ free variables from $x_I$. It is sufficient to show

$$\tilde{\Gamma} = \Gamma \cup \left\{ \varphi_k(x_{I_0}) \leftrightarrow \varphi_k(x_{I_0}) \right\} \quad \left| k = 1, \ldots, m, \ I_0, I_1 \in \omega, \ I_0 _{\text{elem}} \supseteq I_1 \right.$$ 

is consistent. We fix a realization $A \models \Gamma$, and we define $F : A^n \rightarrow 2^n$ by

$$F(\bar{a}) = \sum_{k=1}^{n} i_k 2^k,$$

where

$$\begin{cases} i_k = 0 & \text{if } \neg \varphi_k(\bar{a}) \text{ holds} \\ i_k = 1 & \text{if } \varphi_k(\bar{a}) \text{ holds.} \end{cases}$$

for $\bar{a} \in A^n$

By Ramsey’s theorem, there is an infinite $A' \subset A$ such that $F|_{A'^n}$ is constant. This $A'$ is a witness of $\tilde{\Gamma}$, for $\varphi_k$ have the same truth value on $A'^n$, and $A' \models \Gamma$ by subsequence property. \hfill $\square$

### 2.2 Indiscernible trees

**Definition 2.2.9.** Let $\mathcal{L}_1 = \{ \cap, <\text{len}, <\text{lex}, <\text{ini} \}$, and let $\mathcal{L}_S = \mathcal{L}_1 \cup \{ P_n \mid n \in \omega \}$.

Here, we use the notation $\mathcal{L}_S$ instead of the original notation $\mathcal{L}_0$ in [2].

**Definition 2.2.10.** Let the interpretation of $\mathcal{L}_1$ and $\mathcal{L}_S$ in $\omega^{<\omega}$ as follows:

- $\eta \cap \nu$ is the longest common initial segment of $\eta$ and $\nu$.
- $\eta <\text{len} \nu \iff \eta$ has the less length than $\nu$.
- $\eta <\text{lex} \nu \iff \eta$ is less than $\nu$ in the lexicographic order.
- $\eta <\text{ini} \nu \iff \eta$ is a proper initial segment of $\nu$.
- $P_n(\eta) \iff \eta$ has the length of $n$.

We refer $\mathcal{L}_S$ or $\mathcal{L}_1$-substructures of $\omega^{<\omega}$ by the word ‘trees’.

**Definition 2.2.11 (Indiscernible trees).** Let $B \subset M$.

1. Let $S$ be an $\mathcal{L}_S$-substructure of $\omega^{<\omega}$. For $a_S \subset M$, we say $a_S$ is an $\mathcal{L}_S$-indiscernible tree over $B$ if for all $S_0, S_1 \subset S$ such that $S_0 \equiv S_1$, it holds that $\text{tp}(a_{S_0}/B) = \text{tp}(a_{S_1}/B)$.

2. Let $S$ be an $\mathcal{L}_1$-substructure of $\omega^{<\omega}$. For $a_S \subset M$, we say $a_S$ is an $\mathcal{L}_1$-indiscernible tree over $B$ if for all $S_0, S_1 \subset S$ such that $S_0 \equiv S_1$, it holds that $\text{tp}(a_{S_0}/B) = \text{tp}(a_{S_1}/B)$.

Be careful the index set $S$ is not a subset of the big model $M$ and the $S$-indexed set $a_S$ is a subset of $M$.

**Example 2.2.12.** For $\eta, \nu \in \omega^{<\omega}$, we say $\eta$ is an ancestor or a descendant of $\nu$ if either of the nodes is a proper initial segment of the other, and we say $\eta$ and $\nu$ are siblings if $\eta$ and $\nu$ has the same length $n$ and the length of $\eta \cap \nu$ is $n - 1$.

Let $T$ be the theory of random graph in the language $\{ R(*, *) \}$. For distinct vertices $a_\omega^{<\omega}$ in the big model that satisfies for all $\eta, \nu \in \omega^{<\omega}$

$$\models R(a_\eta, a_\nu) \iff \text{"\eta is an ancestor or a descendant of } \nu\text{"}$$
form an $L_S$ and $L_1$-indiscernible tree.

Let $b_{\omega^{<\omega}}$ be the tree-indexed subset such that for all $\eta, \nu \in \omega^{<\omega}$,

$$\models R(a_\eta, a_\nu) \iff \text{"\eta is an ancestor or a descendant of \nu" or "\eta and \nu are siblings."}$$

Then, $b_\omega$ is an $L_S$-indiscernible tree but not an $L_1$-indiscernible tree. In fact,

$$\{ \emptyset, (0), (1) \} \subsetneq \{ \emptyset, (00), (10) \} \text{ but } \models R(b_{\langle 1 \rangle}, b_{\langle 1 \rangle}) \wedge \neg R(b_{\langle 0 \rangle}, b_{\langle 1 \rangle}).$$

Definition 2.2.13 (Subtree property [2], [3]). Let $B \subset M$.

1) Let $S$ be an $L_S$-substructure of $\omega^{<\omega}$. For a set of $L_B$-formulas $\Gamma(x_S)$, we say $\Gamma$ has $L_S$-subtree property if

$$\bigcup \{ \Gamma(x_{\sigma(S)}) \mid \sigma : I \rightarrow I \text{ is an } L_S\text{-embedding} \}$$

is consistent.

2) Let $S$ be an $L_1$-substructure of $\omega^{<\omega}$. For a set of $L_B$-formulas $\Gamma(x_S)$, we say $\Gamma$ has $L_S$-subtree property if

$$\bigcup \{ \Gamma(x_{\sigma(S)}) \mid \sigma : I \rightarrow I \text{ is an } L_1\text{-embedding} \}$$

is consistent.

Example 2.2.14. $\Gamma(x_{\omega^{<\omega}}) = \{ x_{\omega^{<\omega}} \text{ witnesses the $k$-tree property of } \varphi(x, y) \} \text{ has the } L_S\text{-subtree property (if } \Gamma \text{ is consistent).}$

$\Gamma$ can be concretely written as

$$\Gamma(y_\omega \times \omega) = \bigcup_{i \in \omega} \{ \exists x \left( \bigwedge_{j=0}^{k} \varphi(x, y_{\langle j \rangle}) \right) \mid j_0, \ldots, j_{k-1} \in \omega \} \cup \bigcup_{\nu \in \omega^\omega} \{ \exists x \left( \bigwedge_{i \in \omega} \varphi(x, y_{\nu|_i}) \right) \mid n \in \omega \}.$$

3 Existence of indiscernible trees

In this section, we prove that subtree property implies the existence of an indiscernible tree without Erdös-Rado theorem.

The existence of $L_S$-indiscernible trees is proved with the following theorem in [2], [3].

Theorem (Shelah, Theorem 2.6 of [5, p.662]). For all $k, n \in \omega$ and ordinal $\mu$, there exists an ordinal $\lambda$ such that for any $f : (\lambda^{<\omega})^k \rightarrow \mu$, there is an $L_S$-substructure $S \subset \lambda^{<\omega}$ with $S \simeq \omega^{<\omega}$ satisfying $f(X) = f(Y)$ for all $X, Y \in S^k$ with $X \simeq \omega^{<\omega}$. $S$

This is a variation of Erdös-Rado theorem regarding trees. We want to show the existence of indiscernible trees without this theorem.

3.1 $L_S$-indiscernible trees

Proposition 3.1.15 ([3]). Let $B$ be a set of parameters, and $\Gamma(x_{\omega^{<\omega}})$ be a set of $L_B$-formulas for $n \in \omega$. If $\Gamma(x_{\omega^{<\omega}})$ has the $L_S$-subtree property, then $\Gamma$ is realized by an $L_S$-indiscernible tree over $B$. 

Proof. We show $\Gamma(x_{\omega<n}) \cup \{x_{\omega<n}\}$ is an $\mathcal{L}_S$-indiscernible tree over $B^\omega$ is consistent, where

$$\{x_{\omega<n}\}$$

is an $\mathcal{L}_S$-indiscernible tree over $B^\omega$.

$$\{ \varphi(x_S) \leftrightarrow \varphi(x_T) \mid \varphi \in \mathcal{L}_B, S, T \subset \omega^{<n}, S \simeq_T \}.$$ We show this by induction on $n$. The case $n = 1$ is clear because $\omega^{<1} = \{\emptyset\}$.

Suppose the $n$ case holds. We write $k\omega^{<n}$ to denote the set $\{\sigma \in \omega^{<n+1} \mid \sigma(0) = k\}$ and $X_k$ to denote the set of variables $x_k\omega^{<n}$.

Claim A. $\Gamma(x_{\omega<n+1}) \cup \bigcup_{k \in \omega} \Sigma_k(x_{\omega<n+1})$ is consistent, where

$$\Sigma_k = \"X_k is an $\mathcal{L}_S$-indiscernible tree over $Bx_0 X_1 \ldots X_{k-1} X_{k+1} \ldots\"$$

$$= \left\{ \varphi(x_S) \leftrightarrow \varphi(x_T) \mid \varphi \in L(Bx_0 X_1 \ldots X_{k-1} X_{k+1} \ldots), S, T \subset k\omega^{<n}, S \simeq_T \right\}.$$ Proof of Claim A. Let $a_{\omega^{<n+1}} = a_0 A_1 A_2 \ldots \models \Gamma$, where $A_k = a_k\omega^{<n}$. First, observe that for any tree $S$ with $S \simeq_{\mathcal{L}_S} \omega^{<n}$, the tree $0 0 S \Gamma \omega^{<n}$ becomes an $\mathcal{L}_S$-subtree that is isomorphic to the whole $\omega^{<n+1}$. Therefore $\Gamma(a_0 X_0 A_1 A_2 \ldots)$ has $\mathcal{L}_S$-subtree property over $a_0 A_1 A_2 \ldots$ by the $\mathcal{L}_S$-subtree property of $\Gamma(x_{\omega<n})$. By induction hypothesis, $\Gamma(a_0 X_0 A_1 \ldots)$ is realized by $A'_0$, which is an $\mathcal{L}_S$-indiscernible tree over $a_0 A_1 A_2 \ldots$, $i.e.$ $\Gamma \cup \Sigma_0$ is consistent.

Similarly, $(\Gamma \cup \Sigma_0)(a_0 A'_0 X_1 A_2 \ldots)$ has subtree property over $a_0 A'_0 A_2 \ldots$. Again by induction hypothesis $((\Gamma \cup \Sigma_0)(a_0 A'_0 X_1 A_2 \ldots))$ is realized by $A'_1$, an $\mathcal{L}_S$-indiscernible tree over $a_0 A'_0 A_2 \ldots$. Notice $A'_0$ is still an $\mathcal{L}_S$-indiscernible tree over $a_0 A'_1 A_2 \ldots$, since especially $\Sigma_0(a_0 A'_0 A_1 A_2 \ldots)$ holds. Hence, $\Gamma \cup \Sigma_0 \cup \Sigma_1$ is consistent.

Iterating this procedure $m$ times, $\Gamma(x_{\omega<n+1}) \cup \bigcup_{k=0}^{m-1} \Sigma_k(x_{\omega<n+1})$ is consistent. By compactness, we have shown the claim. end of the proof of Claim A

Let $\Gamma'(x_{\omega<n+1}) = \Gamma(x_{\omega<n+1}) \cup \bigcup_{k \in \omega} \Sigma_k(x_{\omega<n+1})$.

Claim B. $\Gamma'(x_{\omega<n+1}) \cup \{x_0 X_1 \ldots \text{is an indiscernible sequence over $Bx_0^\omega$}\}$ is consistent, where

$$\Gamma'(x_{\omega<n+1}) \cup \{x_0 X_1 \ldots \text{is an indiscernible sequence over $Bx_0^\omega$}\}$$

$$= \{ \varphi(x_{i_0}, \ldots, X_{i_m}) \leftrightarrow \varphi(x_{j_0}, \ldots, X_{j_m}) \mid \varphi \in L(Bx_0^\omega), i_0 < \cdots < i_m, j_0 < \cdots < j_m \}.$$ Proof of Claim B. First, observe that for any subsequence $(i_k \omega^{<n})_{k \in \omega}$, the tree $x_0 i_0 \omega^{<n} i_1 \omega^{<n} i_2 \omega^{<n} \ldots$ is $\mathcal{L}_S$-isomorphic to the whole $x_{\omega^{<n}}$. Since $\Gamma'(x_{\omega^{<n+1}})$ has subtree property over $B$, $\Gamma'(x_0 X_0 X_1 \ldots)$ has subtree property over $Bx_0$. Therefore, there is a realization $a_{\omega^{<n+1}} = a_0 A_1 A_2 \ldots$ of $\Gamma'$, where $A_k = a_k\omega^{<n}$, such that $A_0 A_1 \ldots$ is an indiscernible sequence over $Bx_0$. This can be shown by an argument similar to the proof of Lemma 2.1.8. end of the proof of Claim B

Let $\Gamma''(x_{\omega<n+1}) = \Gamma'(x_{\omega<n+1}) \cup \{x_0 X_1 \ldots \text{is an indiscernible sequence over $Bx_0^\omega$}\}$.

Claim C. A realization of $\Gamma''(x_{\omega<n+1})$ is an $\mathcal{L}_S$-indiscernible tree realizing $\Gamma$.
Proof of Claim C. Let \( \varphi \in \mathcal{L}_B \), \( S, T \subset \omega^{<n+1} \) such that \( S \simeq T \), and \( \theta \equiv \varphi(x_S) \leftrightarrow \varphi(x_T) \). We show \( \Gamma'' \vdash \theta \). \( S, T \) have the form of

\[
S = \bigcup_{k=1}^{m} S_{i_k}, \quad S_{i_k} = \{ \nu \in S \mid \nu(0) = i_k \}, \quad i_0 < \cdots < i_m, \\
T = \bigcup_{k=1}^{m} T_{j_k}, \quad T_{j_k} = \{ \nu \in T \mid \nu(0) = j_k \}, \quad j_0 < \cdots < j_m.
\]

Let \( \sigma : \bigcup_{k=1}^{m} i_k \omega^{<n} \to \bigcup_{k=1}^{m} j_k \omega^{<n} \) be the natural isomorphism. Since \( \Gamma''(x_{\omega^{<n+1}}) \supset \text{"}X_0X_1\ldots\text{"} \) is an indiscernible sequence over \( Bx_\emptyset \),

\[
\Gamma''(x_{\omega^{<n+1}}) \vdash \varphi(x_\emptyset x_{S_{i_0}} \ldots x_{S_{i_m}}) \leftrightarrow \varphi(x_\emptyset x_{\sigma(S_{i_0})} \ldots x_{\sigma(S_{i_m})}).
\]

We have \( S \simeq T \) and so \( \sigma(S_{i_k}) \simeq T_{j_k} \) for each \( k = 1, \ldots, m \). Since \( \Gamma''(x_{\omega^{<n+1}}) \supset \text{"}X_k\text{"} \) is an \( \mathcal{L}_S \)-indiscernible tree over \( Bx_\emptyset X_0X_1\ldots X_{k-1}X_{k+1}\ldots \) for all \( k \in \omega \), it holds that

\[
\Gamma''(x_{\omega^{<n+1}}) \vdash \varphi(x_\emptyset x_{\sigma(S_{i_k})} \ldots x_{\sigma(S_{i_m})}) \leftrightarrow \varphi(x_\emptyset x_{T_{j_k}} \ldots x_{T_{j_m}}).
\]

Thus we have shown \( \Gamma''(x_{\omega^{<n+1}}) \vdash \theta \).

From the above argument, we have shown the \( n+1 \) case of proposition. \( \square \)

Theorem 3.1.16 ([3]). Let \( B \) be a set of parameters, and \( \Gamma(x_{\omega^\omega}) \) be a set of \( \mathcal{L}_B \)-formulas. If \( \Gamma(x_{\omega^\omega}) \) has the \( \mathcal{L}_S \)-subtree property, then \( \Gamma \) is realized by an \( \mathcal{L}_S \)-indiscernible tree over \( B \).

Proof. This is an immediate consequence from Proposition 3.1.15 and Compactness. \( \square \)

Example 3.1.17. \( \Gamma(x_{\omega^\omega}) = \text{"}x_{\omega^\omega}\text{"} \) witnesses the \( k \)-tree property of \( \varphi(x, y) \)" is realized by an \( \mathcal{L}_S \)-indiscernible tree (if \( \Gamma \) is consistent).

3.2 \( \mathcal{L}_1 \)-indiscernible trees

Definition 3.2.18 ([3]). Let \( X \) be a substructure of \( \omega^\omega \), i.e. \( X \) is closed under the binary function \( \cap \). We define \( \text{level}(X) \) by \( \text{level}(X) = \{ \text{dom}(\eta) \mid \eta \in X \} \).

Lemma 3.2.19 ([3]). Let \( n \in \omega \) and \( X, Y \) be \( n \)-element substructures of \( \omega^\omega \). \( X \simeq Y \) if and only if \( X \approx Y \) and \( \text{level}(X) = \text{level}(Y) \).

Proof. If we have \( X \simeq Y \), then \( X \approx Y \) and \( \text{level}(X) = \text{level}(Y) \) clearly holds.

Suppose \( X \simeq Y \) and \( \text{level}(X) = \text{level}(Y) \) holds. We put \( l = |\text{level}(X)| = |\text{level}(Y)| \) and fix the \( \mathcal{L}_1 \)-isomorphism \( \sigma : X \to Y \). Let \( \eta_i \in X \)\( \omega^x \) enumerates \( X \) and \( \nu_i = \sigma(\eta_i) \) for \( i < n \).

There are \( i_1, \ldots, i_l \) such that \( \eta_{i_1} <_{\text{fin}} \cdots <_{\text{fin}} \eta_{i_l} \) and so \( \nu_{i_1} <_{\text{fin}} \cdots <_{\text{fin}} \nu_{i_l} \). By the condition \( \text{level}(X) = \text{level}(Y) \), we have \( \text{dom}(\eta_{i_k}) = \text{dom}(\nu_{i_k}) \) for each \( 1 \leq k \leq l \). Since \( \mathcal{L}_1 \)-isomorphisms do not change the relation of having the same length, we have \( \text{dom}(\eta) = \text{dom}(\sigma(\eta)) \) thus \( P_m(\eta) \leftrightarrow P_m(\sigma(\eta)) \) for all \( \eta \in X \) and \( m \in \omega \). Hence \( \sigma \) is the \( \mathcal{L}_S \)-isomorphism between \( X \) and \( Y \). \( \square \)

Theorem 3.2.20 ([3]). Let \( B \) be a set of parameters, and \( \Gamma(x_{\omega^\omega}) \) be a set of \( \mathcal{L}_B \)-formulas. If \( \Gamma(x_{\omega^\omega}) \) has the \( \mathcal{L}_1 \)-subtree property, then \( \Gamma \) is realized by an \( \mathcal{L}_1 \)-indiscernible tree over \( B \).
Proof. We show the set of $L_B$-formulas

$$\Gamma(x_{\omega^{<\omega}}) = \Gamma \cup \left\{ \varphi(x_{X_1}) \leftrightarrow \varphi(x_{X_2}) \mid \varphi \text{ is an } L_B\text{-formula, } \right. \\
\left. X_1, X_2 \text{ are finite subsets of } \omega^{<\omega} \text{ with } X_1 \simeq_{\xi_i} X_2 \right\}$$

is consistent.

Claim. For a finite substructure $X$ of $\omega^{<\omega}$ and an $L_B$-formula $\varphi(x_X)$,

$$\Gamma_\varphi(x_{\omega^{<\omega}}) = \Gamma \cup \left\{ \varphi(x_{X_1}) \leftrightarrow \varphi(x_{X_2}) \mid X_1, X_2 \text{ are subsets of } \omega^{<\omega} \text{ with } X_1 \simeq_{\xi_i} X_2 \simeq_{\xi_1} X \right\}$$

is consistent.

Proof of Claim. We put $k = |\text{level}(X)|$. $\Gamma$ has $L_1$-subtree property so $L_S$-subtree property. By Proposition 3.1.16, $\Gamma$ has a realization $a_{\omega^{<\omega}}$ that is an $L_S$-indiscernible tree over $B$. We define the function $f: [\omega]^k \to \{0, 1\}$ by

$$f(\{n_1, \ldots, n_k\}) = \left\{ \begin{array}{ll} 1 & \text{if } \varphi(a_Y) \text{ holds for all } Y \simeq X \text{ with level}(Y) = \{n_1, \ldots, n_k\} \\
0 & \text{if } \neg\varphi(a_Y) \text{ holds for all } Y \simeq X \text{ with level}(Y) = \{n_1, \ldots, n_k\}. \end{array} \right.$$

This is well defined because $X \simeq Y$ and $\text{level}(X) = \text{level}(Y)$ imply $X \simeq Y$ and $a_{\omega^{<\omega}}$ is an $L_S$-indiscernible tree over $B$. By Ramsey’s theorem, there is an infinite $H \subset \omega$ such that $f$ is constant on $[H]^k$. Let $h_\omega$ enumerate the elements of $H$ in increasing order. For $\eta \in \omega^{<\omega}$ we define $\sigma_H: \omega^{<\omega} \to \omega^{<\omega}$ by $\text{dom}(\sigma_H(\eta)) = h_{\text{dom}(\eta)}$ and

$$\sigma_H(\eta)(n) = \left\{ \begin{array}{ll} 0 & \text{if } n \notin H \\
\eta(i) & \text{if } n = h_i \end{array} \right.$$

i.e. $\sigma_H(\eta) = \left( \begin{array}{c} 0 \ldots 0 \eta(0) 0 \ldots 0 \eta(1) 0 \ldots 0 \ldots \eta(d-1) 0 \ldots 0 \eta(d-1h_d-1^\check{h_d-1}0\ldots0) \end{array} \right)$, where $d = \text{dom}(\eta)$.

Observe that for $\eta, \mu, \nu \in \omega^{<\omega}$ if $\eta <_{\text{len}} \nu, \eta <_{\text{lex}} \nu, \eta <_{\text{lex}} \mu, \eta \cap \nu = \mu$ holds, then we have $\sigma_H(\eta) <_{\text{len}} \sigma_H(\nu), \sigma_H(\eta) <_{\text{lex}} \sigma_H(\nu), \sigma_H(\eta) \cap \sigma_H(\nu) = \sigma_H(\mu)$ respectively. Thus $\sigma_H$ is an $L_1$-embedding.

By the $L_1$-indiscernibility of $\Gamma$, $(a_{\sigma_H(\eta)}(\eta) \in \omega^{<\omega})$ is also a realization of $\Gamma$, and by the choice of $H$, $(a_{\sigma_H(\eta)}(\eta) \in \omega^{<\omega})$ satisfies $\Gamma$. Hence $\Gamma_\varphi$ is consistent. \hspace{1cm} \text{end of the proof of Claim}

Since for any $L_B$-formula $\varphi$ and $X \subset \omega^{<\omega}$, $\Gamma_\varphi$ in the above claim also has the $L_1$-subtree property, we can show the finite satisfiability of $\Gamma$ using the claim iteratively. \hspace{1cm} \Box

Example 3.2.21. Let $T$ be NTP$_2$ theory. If $\varphi(x, y)$ has the $k$-tree property, then there exists $k' \in \omega$ such that the set of formulas $\Gamma_{k'}(x_{\omega^{<\omega}}) = \{x_{\omega^{<\omega}} \text{ witnesses the } k'\text{-tree property of } \varphi(x, y)\}$ has the $L_1$-subtree property, hence $\Gamma_{k'}$ is realized by an $L_1$-indiscernible tree.

Here, we give a proof for this example.

Proof. Since the theory is NTP$_2$, there is $l \in \omega$ that satisfies the following condition: for all array of parameters $c_{i, \omega^{<\omega}}$, if $\{\varphi(x, c_{i, j}) \mid j \in \omega\}$ is $k$-inconsistent for all $i < l$, then there exists $\nu \in \omega'$ such that $\{\varphi(x, c_{i, \omega^{<\omega}}) \mid i < l\}$ is inconsistent. Let $k' = k \times l$, and for $N \in \omega$, let $\Gamma_N(y_{\omega^{<\omega}})$ be the set of formulas "$y_{\omega^{<\omega}}$ witnesses the $N$-tree property of $\varphi(x, y)$"
Claim. $\Gamma_k'$ has the $L_1$-subtree property.

Proof of Claim. We confirm the consistency of $\bigcup \{ \Gamma_k'(x_{\sigma(I)}) \mid \sigma : I \rightarrow I \text{ is an } L_1 \text{-embedding} \}$. Since $\Gamma_k$ has the $L_8$-subtree property, we can apply Theorem 3.1.16 to obtain an $L_8$-indiscernible tree $b_{\omega} < \omega$ which realizes $\Gamma_k$. Clearly, $b_{\omega} < \omega$ also realizes $\Gamma_k'$. We show $b_{\omega} < \omega$ is a realization of $\Gamma_k'(y_{\sigma(\omega_1 < \omega)})$ for all $L_1$-embedding $\sigma$. The condition $\{ \varphi(x, b_{\sigma(\nu_n)}) \mid n \in \omega \}$ is consistent for all $\nu \in \omega^\omega$ clearly holds because an $L_1$-embedding sends a path into a path and $b_{\omega} < \omega$ is a witness of the $k$-tree property of $\varphi$.

For the condition $\{ \varphi(x, b_{\sigma(\nu_1)}) \mid n \in \omega \}$ is $k'$-inconsistent for all $\eta \in \omega^{< \omega}$, since an $L_1$-embedding preserves the relation of having the same length, it suffices to show any subset $A \subset \omega^{< \omega}$ of $k'$ elements that have the same length, $\{ \varphi(x, b_{\eta}) \mid \eta \in A \}$ is inconsistent. Let $A$ be a subset of $k'$ elements in $\omega^{< \omega}$ each of which element has the same length, then either the case happens:

(1) There is $k$-element subset $A_1 \subset A$ that belongs to the same sequence of siblings.
(2) There is $l$-element subset $A_2 \subset A$ whose parents are pairwise distinct.

In the case (1), $\{ \varphi(x, b_{\eta}) \mid \eta \in A_1 \}$ is inconsistent, since all elements in $A_1$ are contained in a particular sequence of siblings and $b_{\omega} < \omega$ is a witness of the $k$-tree property of $\varphi$.

In the case (2), we put $A_2 = \{ \eta_1, \ldots, \eta_l \}$ and let $\theta^i \subset \omega^{< \omega}$ be the sequence of siblings that contains $\eta_i$ for $i = 1, \ldots, l$. Observe $\{ \varphi(x, b_{\mu}) \mid \mu \in \theta^i \}$ is $k$-inconsistent for each $i = 1, \ldots, l$. Because of the way we chose $l$, there is a path $\nu$ in the array $(b_{\theta^1}, \ldots, b_{\theta^l})$ such that $\{ \varphi(x, b_{\varphi(i)}) \mid i = 1, \ldots, l \}$ is inconsistent. By $L_8$-indiscernibility of $b_{\omega} < \omega$, it holds that $b_{\omega(1)}, \ldots, b_{\omega(l)} \equiv b_{\eta_1}, \ldots, b_{\eta_l}$, thus $\{ \varphi(x, b_{\eta_i}) \mid i = 1, \ldots, l \}$ is inconsistent. \[\text{end of the proof of Claim}\]

By the Theorem 3.2.20, we have $\Gamma_k'$ is realized by an $L_1$-indiscernible tree. \[\square\]

References