<table>
<thead>
<tr>
<th>Title</th>
<th>Equivariant definable homotopy extensions (Model theoretic aspects of the notion of independence and dimension)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kawakami, Tomohiro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2014), 1888: 1-8</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/195749">http://hdl.handle.net/2433/195749</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Equivariant definable homotopy extensions

Tomohiro Kawakami
Department of Mathematics, Wakayama University
Partially supported by Kakenhi(23540101).

Abstract

Let $G$ be a definably compact definable group, $X$ a definable $G$ set and $Y$ a definable closed $G$ subset of $X$. We prove that a pair $(X,Y)$ admits an equivariant definable homotopy extension.

1 Introduction

In this paper we consider equivariant definable homotopy extensions in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, \ldots)$ of a real closed field $R$. It is known that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers([8]).

Definable set and definable maps are studied in [3], [4], and see also [9]. Everything is considered in $\mathcal{N} = (R, +, \cdot, <, \ldots)$ and definable maps are assumed to be continuous unless otherwise stated.

2 Preliminaries

Let $R$ be a real closed field.

A structure $\mathcal{N}$ is given by the following data.

2010 Mathematics Subject Classification. 14P10, 57S99, 03C64.

Key Words and Phrases. O-minimal structures, real closed fields, Equivariant definable homotopy extensions.
1. A set $R$ is called the \textit{universe} or \textit{underlying set} of $\mathcal{N}$.

2. A collection of \textit{functions} $\{f_i | i \in I\}$, where $f_i : R^{n_i} \to R$ for some $n_i \geq 1$.

3. A collection of \textit{relations} $\{R_j | j \in J\}$, where $R_j \subset R^{m_j}$ for some $m_j \geq 1$.

4. A collection of distinguished elements $\{c_k | k \in K\} \subset R$, and each $c_k$ is called a \textit{constant}.

Any (or all) of the sets $I, J, K$ may be empty.

We say that $f$ (resp. $L$) is \textit{m}-place function (resp. \textit{m}-place relation) if $f : R^m \to R$ (resp. $L \subset R^m$).

A \textit{term} is a finite string of symbols obtained by repeated applications of the following three rules:

1. Constants are terms.

2. Variables are terms.

3. If $f$ is an \textit{m}-place function of $\mathcal{N}$ and $t_1, \ldots, t_m$ are terms, then the concatenated string $f(t_1, \ldots, t_m)$ is a term.

A \textit{formula} is a finite string of symbols $s_1 \ldots s_k$, where each $s_i$ is either a variable, a function, a relation, one of the logical symbols $=, \neg, \lor, \land, \exists, \forall$, one of the brackets $(, )$, or comma $.$. Arbitrary formulas are generated inductively by the following three rules:

1. For any two terms $t_1$ and $t_2$, $t_1 = t_2$ and $t_1 < t_2$ are formulas.

2. If $R$ is an \textit{m}-place relation and $t_1, \ldots, t_m$ are terms, then $R(t_1, \ldots, t_m)$ is a formula.

3. If $\phi$ and $\psi$ are formulas, then the negation $\neg \phi$, the disjunction $\phi \lor \psi$, and the conjunction $\phi \land \psi$ are formulas. If $\phi$ is a formula and $v$ is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

A subset $X$ of $R^n$ is \textit{definable} (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and elements $b_1, \ldots, b_m \in R$ such that $X = \{(a_1, \ldots, a_n) \in R^n | \phi(a_1, \ldots, a_n, b_1, \ldots, b_m) \text{ is true in } \mathcal{N}\}$. 
For any \(-\infty \leq a < b \leq \infty\), an open interval \((a, b)_R\) means \(\{x \in R|a < x < b\}\), for any \(a, b \in R\) with \(a < b\), a closed interval \([a, b]_R\) means \(\{x \in R|a \leq x \leq b\}\). We call \(\mathcal{N}\) o-minimal (order-minimal) if every definable subset of \(R\) is a finite union of points and open intervals.

A real closed field \((R, +, \cdot, <)\) is an o-minimal structure and every definable set is a semialgebraic set [10], and a definable map is a semialgebraic map [10]. In particular, the semialgebraic category is a special case of the definable one.

The topology of \(R\) is the interval topology and the topology of \(R^n\) is the product topology. Note that \(R^n\) is a Hausdorff space.

The field \(\mathbb{R}\) of real numbers, \(\mathbb{R}_{alg} = \{x \in \mathbb{R}|x\) is algebraic over \(\mathbb{Q}\}\) are Archimedean real closed fields.

The Puiseux series \(\mathbb{R}[X]^\wedge\), namely \(\sum_{i=k}^{\infty} a_i X^{\frac{i}{q}}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}\) is a non-Archimedean real closed field.

**Fact 2.1.** (1) The characteristic of a real closed field is 0.

(2) For any cardinality \(\kappa \geq \aleph_0\), there exist \(2^\kappa\) many non-isomorphic real closed fields whose cardinality is \(\kappa\).

(3) In a general real closed field, even for a \(C^\infty\) function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle's theorem, the mean value theorem do not hold. Even for a \(C^\infty\) function \(f\) in one variable, the result that \(f' > 0\) implies \(f\) is increasing does not hold.

**Definition 2.2.** Let \(X \subset R^n, Y \subset R^m\) be definable sets.

(1) A continuous map \(f : X \rightarrow Y\) is a definable map if the graph of \(f\) \((\subset R^n \times R^m)\) is definable.

(2) A definable map \(f : X \rightarrow Y\) is a definable homeomorphism if there exists a definable map \(f' : Y \rightarrow X\) such that \(f \circ f' = id_Y, f' \circ f = id_X\).

**Definition 2.3.** A group \(G\) is a definable group if \(G\) is definable and the group operations \(G \times G \rightarrow G, G \rightarrow G\) are definable.

As in the field \(\mathbb{R}\), for any real closed field \(R\), we can define the \(n\)-th general linear \(G(n, R)\), the \(n\)-th orthogonal group \(O(n)\).

Let \(G, G'\) be definable groups. A group homomorphism \(f : G \rightarrow G'\) is a definable group homomorphism if \(f\) is definable. A definable group homomorphism \(f : G \rightarrow GL(n, R)\) is called a definable \(G\) representation. A definable group homomorphism \(f : G \rightarrow O(n)\) is called a definable orthogonal
$G$ representation and $R^n$ with the orthogonal action induced from an orthogonal $G$ representation is called a \textit{definable orthogonal $G$ representation space}.

**Definition 2.4.** (1) A $G$ invariant definable subset of a definable orthogonal $G$ representation space is a \textit{definable $G$ set}.

Let $X, Y$ be definable $G$ sets.

(2) A definable map $f : X \to Y$ is a \textit{definable $G$ map} if for any $x \in X, g \in G$, $f(gx) = gf(x)$.

(3) A definable $G$ map $f : X \to Y$ is a \textit{definable $G$ homeomorphism} if there exists a definable $G$ map $h : Y \to X$ such that $f \circ h = id_Y$, $h \circ f = id_X$.

**Definition 2.5.** (1) A definable set $X \subset R^n$ is \textit{definably compact} if for any definable map $f : (a, b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in $X$.

(2) A definable set $X \subset R^n$ is \textit{definably connected} if there exist no definable open subsets $U, V$ of $X$ such that $X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset$.

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg}|0 \leq x \leq 1\}$ is definably compact and definably connected, but it is neither compact nor connected.

**Theorem 2.6 ([7]).** For a definable set $X \subset R^n$, $X$ is definably compact if and only if $X$ is closed and bounded.

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

**Proposition 2.7.** Let $X \subset R^n$, $Y \subset R^m$ be definable set, $f : X \to Y$ a definable map. If $X$ is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).

**Theorem 2.8.** (1) (The intermediate value theorem) For a definable function $f$ on a definably connected set $X$, if $a, b \in X$, $f(a) \neq f(b)$ then $f$ takes all values between $f(a)$ and $f(b)$.

(2) (Existence theorem of maximum and minimum) Every definable function on a definably compact set attains maximum and minimum.
(3) (Rolle's theorem) Let \( f: [a, b]_R \rightarrow R \) be a definable function such that \( f \) is differentiable on \((a, b)_R\) and \( f(a) = f(b) \). Then there exists \( c \) between \( a \) and \( c \) with \( f'(c) = 0 \).

(4) (The mean value theorem) Let \( f: [a, b]_R \rightarrow R \) be a definable function which is differentiable on \((a, b)_R\). Then there exists \( c \) between \( a \) and \( c \) with
\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

(5) Let \( f: (a, b)_R \rightarrow R \) be a differentiable definable function. If \( f' > 0 \) on \((a, b)_R\), then \( f \) is increasing.

Example 2.9. (1) Let \( \mathcal{N} \) be \((\mathbb{R}_{alg}, +, \cdot, <)\). Then \( f: \mathbb{R}_{alg} \rightarrow \mathbb{R}_{alg}, f(x) = 2^x \) is not defined over \( \mathbb{R}_{alg} \).

(2) Let \( \mathcal{N} \) be \((\mathbb{R}, +, \cdot, <)\). Then \( f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2^x \) is defined but not definable, and \( h: \mathbb{R} \rightarrow \mathbb{R}, h(x) = \sin x \) is defined but not definable.

3 Equivariant definable homotopy extensions

Let \( X, Y \) be definable set and \( f: X \rightarrow Y \) a definable map. We say that \( f \) is **definably proper** if for any definably compact subset \( C \) of \( Y \), \( f^{-1}(C) \) is a definably compact subset of \( X \).

Let \( A \subset R^n, S \subset R^m \) be definable sets, and let \( f: S \rightarrow A \) be a definable map. We say that \( f \) is **definably trivial** if there exist a definable set \( F \subset R^N \) for some \( N \in \mathbb{N} \), and a definable map \( h: S \rightarrow F \) such that \((f, h): S \rightarrow A \times F \) is a definable homeomorphism. In this case, each fiber \( f^{-1}(a) \) of \( f \) over \( a \) is definably homeomorphic to \( F \).

In o-minimal expansions of real closed fields, the following five theorems are known.

**Theorem 3.1.** (1) (Monotonicity theorem (e.g. 3.1.2, 3.1.6. [3])). Let \( f: (a, b)_R \rightarrow R \) be a function with the definable graph. Then there exist finitely many points \( a = a_0 < a_1 < \cdots < a_k = b \) such that on each subinterval \((a_j, a_{j+1})_R\), the function is either constant, or strictly monotone and continuous. Moreover, for any \( c \in (a, b)_R \), the limits \( \lim_{x \rightarrow c+0} f(x) \), \( \lim_{x \rightarrow c-0} f(x) \) exist in \( R \cup \{\infty\} \cup \{-\infty\} \).

(2) (Cell decomposition theorem (e.g. 3.2.11. [3])). For any definable subsets \( A_1, \ldots, A_k \) of \( R^n \), there exists a cell decomposition of \( R^n \) partitioning each \( A_1, \ldots, A_k \).
Let \( A \) be a definable subset of \( R^n \) and \( f : A \to R \) a function with the definable graph. Then there exists a cell decomposition \( D \) of \( R^n \) partitioning \( A \) such that each \( B \subset A, B \in D \), \( f|B : B \to R \) is continuous.

(3) (Triangulation theorem (e.g. 8.2.9. [3])). Let \( S \subset R^n \) be a definable set and let \( S_1, S_2, \ldots, S_k \) be definable subsets of \( S \). Then \( S \) has a triangulation in \( R^n \) compatible with \( S_1, \ldots, S_k \).

(4) (Piecewise trivialization theorem (e.g. 8.2.9. [3])). Let \( f : S \to A \) be a definable map between definable sets \( S \) and \( A \). Then there is a finite partition \( A_1, \ldots, A_k \) of \( A \) into definable sets \( A_i \) such that each \( f|f^{-1}(A_i) : f^{-1}(A_i) \to A_i \) is definably trivial.

(5) (Existence of definable quotients (e.g. 10.2.18 [3])). Let \( G \) be a definably compact definable group and \( X \) a definable \( G \) set. Then the orbit space \( X/G \) exists as a definable set and the orbit map \( \pi : X \to X/G \) is surjective, definable and definably proper.

**Question 3.2.** Let \( X, Y \) be definable sets and \( A \) a definable subset of \( X \).

(1) (Extensions of definable maps) Let \( f : A \to Y \) be a definable map. When does \( f : A \to Y \) extend a definable map \( F : X \to Y \)?

(2) (Definable homotopy extensions) Let \( f : X \to Y \) be a definable map and a definable homotopy \( F : A \times [0, 1]_R \to Y \) such that \( F(x, 0) = f(x) \) for any \( x \in A \). When does a definable homotopy \( H : X \times [0, 1]_R \to Y \) exist such that \( H(x, 0) = f(x) \) for any \( x \in X \) and \( H|A \times [0, 1]_R = F \)?

**Theorem 3.3** (Definable Tietze extension theorem [1]). Let \( X, Y \) be definable sets, \( A \) a definable closed subset of \( X \) and \( f : A \to R \) a definable function. Then there exists a definable function \( F : X \to R \) such that \( F|A = f \).

**Theorem 3.4** (Definable homotopy extension theorem [2]). Let \( X, Y \) be definable sets and \( A \) a definable closed subset of \( X \). For any definable map \( f : X \to Y \) and for any definable homotopy \( F : A \times [0, 1]_R \to Y \) such that \( F(x, 0) = f(x) \) for any \( x \in A \), there exists a definable homotopy \( H : X \times [0, 1]_R \to Y \) such that \( H(x, 0) = f(x) \) for any \( x \in X \) and \( H|A \times [0, 1]_R = F \).

To consider Question 3.2, we need to construct an obstruction theory in the definable category.

The following question is an equivariant version of Question 3.2.

**Question 3.5.** Let \( G \) be a definable group, \( X, Y \) a definable \( G \) sets and \( A \) a definable \( G \) subset of \( X \).
(1) (Extensions of definable $G$ maps) Let $f : A \to Y$ be a definable $G$ map. When does $f : A \to Y$ extend a definable $G$ map $F : X \to Y$?

(2) (Equivariant definable homotopy extensions) Let $f : X \to Y$ be a definable $G$ map and an equivariant definable homotopy $F : A \times [0, 1]_R \to Y$ such that $F(x, 0) = f(x)$ for any $x \in A$. When does an equivariant definable homotopy $H : X \times [0, 1]_R \to Y$ exist such that $H(x, 0) = f(x)$ for any $x \in X$ and $H|A \times [0, 1]_R = F$?

We have the following result.

**Theorem 3.6 ([6]).** Let $G$ be a definably compact definable group, $X$ a definable $G$ set and $A$ a definable closed $G$ subset of $X$. For any definable $G$ map $f : X \to Y$ and for any equivariant definable homotopy $F : A \times [0, 1]_R \to Y$ such that $F(x, 0) = f(x)$ for any $x \in A$, there exists an equivariant definable homotopy $H : X \times [0, 1]_R \to Y$ such that $H(x, 0) = f(x)$ for any $x \in X$ and $H|A \times [0, 1]_R = F$.

Theorem 3.6 is proved in the case where $R = \mathbb{R}$ ([5]).

To prove Theorem 3.6, we need the following results.

**Theorem 3.7 ([6]).** Let $G$ be a definably compact definable group and $Y$ a definable closed $G$ subset of a definable $G$ set $X$. Then there exists a $G$ invariant definable open neighborhood $U$ of $Y$ in $X$ such that $U$ is a definable strong $G$ deformation retract of both $U$ and of the closure $\text{cl} U$ of $U$ in $X$.

**Proposition 3.8 ([6]).** Let $G$ be a definably compact definable group and $A, B$ disjoint definable closed $G$ subsets of a definable $G$ set $X$. Then there exists a $G$ invariant definable map $f : X \to [0, 1]_R$ with $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

**References**


