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EQUIANGULAR LINES AND SEIDEL MATRICES WITH THREE EIGENVALUES II

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ABSTRACT. We discuss some preliminary results on equiangular lines in $\mathbb{R}^d$ whose Seidel matrix has three different eigenvalues.

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1. INTRODUCTION AND MAIN RESULTS

This paper is based on a talk given at RIMS, and describes some preliminary results from a forthcoming paper jointly with Gary Greaves and Akihiro Munemasa [3].

A set of $n \geq 1$ lines, represented by the unit vectors $v_1, \ldots, v_n \in \mathbb{R}^d$, is called equiangular if there exists a constant $\alpha > 0$ for which $|\langle v_i, v_j \rangle| = \pm\alpha$ holds for every $1 \leq i < j \leq n$. Such lines arise in many applications [5]. The fundamental problem in this area is the determination of the *maximum number of equiangular lines* $N(d)$ in $\mathbb{R}^d$. In the following table, which is essentially the same as in [6], we display lower and upper bounds on $N(d)$ for the first few values of d.

d	2	3	4	5	6	7-13	14	15	16	17	18	19	20	21	22	23
$N(d)$	3	6	6	10	16	28	28-30	36	40-42	48-51	48-61	72-76	90-96	126	176	276
$1/\alpha$	2	$\sqrt{5}$	$\sqrt{5}, 3$	3	3	3	$3, 5$	5	5	5	5	5	5	5	5	5

TABLE 1. The maximum number of equiangular lines for $d \leq 23$.

We remark that there exist several *incorrectly* revised tables in the current literature (e.g. the one in [1, p. 884]) which might suggest to the uninitiated that $N(d)$ is known for small d. This is, however, not the case as $d = 14$ is already undecided. Table 1 shows that despite of considerable amount of research in the past 40 years, determining $N(d)$ even for relatively small values of d is still out of reach. Methods, obtaining configurations with the above indicated number of lines are fairly standard and are discussed in details throughout the scattered literature [1], [6], [11], [12] and [13].

Remark 1.1. Seidel seems to claim in [1, p. 884] that the lower bounds indicated above cannot be improved unless $d = 19$ or 20. We tend to believe that this is not the case, but it is unclear whether or not his statement follows implicitly from the cited literature.

The Gram matrix of the equiangular line system $[G]_{i,j} := \langle v_i, v_j \rangle$, $1 \leq i, j \leq n$, is of fundamental interest, since it contains all the relevant information and thus study of equiangular

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lines via matrix theoretical and linear algebraic tools is possible. It is, however, more convenient to consider the *Seidel matrix* $S := (G-I)/\alpha$ instead, which is a symmetric matrix with zero diagonal and ± 1 entries otherwise. The algebraic multiplicity of the smallest eigenvalue λ_0 of S describes the smallest possible dimension d where the line system fits in with common angle $\alpha = -1/\lambda_0$. Seidel matrices are the central objects of this work. We remark here that there is an ambient graph $\Gamma(S)$ associated with each Seidel matrix, whose adjacency matrix A can be obtained from the formula $A := (J - S - I)/2$.
In the following we review equiangular line systems with common angle $1/3$ and $1/5$ in $\mathbb{R}^d$, or, equivalently, Seidel matrices with smallest eigenvalue -3 and -5. The goal is to get some insight into the size of maximal equiangular line systems with prescribed common angle α in $\mathbb{R}^d$, which we denote by $N_{1/\alpha}(d)$. The case $N_3(d)$ is completely understood and was determined by Lemmens and Seidel in [6].

Theorem 1.2 (See [4], [6], [7]). *The maximum number of equiangular lines in* $\mathbb{R}^d$ *with common angle* $\alpha = 1/3$ *is described in Table 2 below.*

d	2	3	4	5	6	7-15	16-
$N_3(d)$	2	4	6	10	16	28	$2(d-1)$

TABLE 2. The maximum number of equiangular lines with $\alpha = 1/3$.

It is easy to see that the Seidel matrix of any maximal set of equiangular lines with common angle $1/3$ must contain an $I_4 - J_4$ principal submatrix. This leads to the concept of *pillars*, which is a nontrivial geometric interpretation of equiangular line systems [6]. If the same principle, that is, the existence of a $I_6 - J_6$ principal submatrix, would hold for $\alpha = 1/5$ as well, then the following result would describe the size of such maximal equiangular line systems.

Theorem 1.3 (See [6]). *Any set of unit vectors with common angle* $1/5$ *in* $\mathbb{R}^d$*, which contains a* $I_6 - J_6$ *principal submatrix, has maximum cardinality* 276 *for* $23 \leq d \leq 185$*, and* $\lfloor 3(d-1)/2 \rfloor$ *for* $d \geq 186$.

We remark here that the Seidel matrix with spectrum $\{[-5]^3, [-1]^2, [1]^3, [3]^3, [5]^1\}$ describing one of the four maximal equiangular line systems in $\mathbb{R}^9$ with common angle $1/5$ does not contain (up to switching) any $I_6 - J_6$ principal submatrices.
Theorem 1.3 was subsequently improved by Neumaier, who determined the maximum number of equiangular lines with common angle $1/5$ in $\mathbb{R}^d$ for large d. It turns out that the Seidel matrix of all such line systems is switching equivalent to one whose ambient graph has largest eigenvalue at most 2. Such graphs are called *Dynkin graphs.*

Theorem 1.4 (Neumaier, [8]). *Assume that* S *is a Seidel matrix of order* $n \geq 45374$ *with smallest eigenvalue* -5. *Then* S *is switching equivalent to some Seidel matrix* S' *such that the ambient graph* $\Gamma' = (J - S' - I)/2$ *is a Dynkin graph.*

Corollary 1.5. *Assume that* $d \geq 30251$. *Then* $N_5(d) = \lfloor (3d-1)/2 \rfloor$.

Proof. See [3]. □

The main contribution of this manuscript is the description of the analogue of Table 2 corresponding to common angle $\alpha = 1/5$. This case is still far from being completely understood.

Proposition 1.6. *Bounds on $N_5(d)$:*

d	2-4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$N_5(d)$	d	6	7	9	10	12	16	18	20-22	26	28-30	36	40-42	48-51	48-61	72-76

d	20	21	22	23	24-185	186-30250	30251-
$N_5(d)$	90-96	126	176	276	276-$d(d+1)/2$	$\lfloor 3(d-1)/2 \rfloor$-$d(d+1)/2$	$\lfloor 3(d-1)/2 \rfloor$

TABLE 3. The maximum number of equiangular lines with $\alpha = 1/5$.

Proof. The lower bounds come from direct constructions, while the upper bounds are available from the literature. The case $d > 30250$ follows from Neumaier's result. See [3] for more details. □

Some of the lower bounds in Table 3 above correspond to Seidel matrices with three eigenvalues. The known maximal set of equiangular lines in dimensions 19-23 all come from the Witt-design. The examples in dimension 21, 22 and 23 are regular two-graphs; Taylor's example in dimension 20 has four distinct eigenvalues [12]; while the following construction, discovered by Asche, leads to a Seidel matrix with three distinct eigenvalues in dimension 19.

Example 1.7 (See [12, p. 124]). *Let $\mathcal{B}$ be the set of 759 blocks of the Witt-design, the "octads", defined on the ground set $X = \{1, 2, \dots, 24\}$, let e_i denote the standard basis in $\mathbb{R}^{24}$ for $1 \leq i \leq 24$ and for a subset $T \subseteq X$ let us denote $e_T := \sum_{i \in T} e_i$. Let $B_1, B_2 \in \mathcal{B}$ such that $1 \notin B_1, B_2$ and $B_1 \cap B_2 = \{2, 3\}$. The vectors $v_B := (4e_B - 4e_1 - e_X)/\sqrt{80}$ for which $1 \in B \in \mathcal{B}$ are all orthogonal to $4e_1 + e_X$. Those, which in addition are orthogonal to all of $e_1 - e_2$, $e_1 - e_3$, v_{B_1} and v_{B_2} form an equiangular line system of 72 lines in $\mathbb{R}^{19}$. Moreover, the corresponding Seidel matrix has spectrum $\{[-5]^{53}, [13]^{16}, [19]^{3}\}$.*

Consult Appendix A for the parameter sets of some additional hypothetical Seidel matrices of small orders. We remark here that for large d maximal equiangular line systems can be obtained in the cases $\alpha = 1/3$ and $\alpha = 1/5$ by the following easy construction.

Lemma 1.8. *There exists mn equiangular lines in $\mathbb{R}^{mn-m+1}$ with common angle $\alpha = 1/(2n-1)$ for every $m, n \geq 2$. Moreover we can assume that the corresponding Seidel matrix has spectrum $\{[1-2n]^{m-1}, [1]^{m(n-1)}, [n(m-2)+1]^{1}\}$.*

Proof. It is easy to see, by using properties of the Kronecker product, that the $mn \times mn$ matrix $S := J_n \otimes (J_m - 2I_m) + I_{mn}$ has the desired spectrum. Under the assumptions on m and n its smallest eigenvalue is $1 - 2n$, hence the result follows. □

It would be nice to see a combinatorial interpretation of Seidel matrices with three distinct eigenvalues. Such new perspective might shed some light on the existence of the hypothetical Seidel matrices highlighted in the appendix. This will hopefully lead to improvements upon the best known lower bounds on the number of equiangular lines in small dimensions.

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APPENDIX A. A SUPPLEMENTARY TABLE

Here we display a list of feasible spectrum of Seidel matrices whose existence might reach, or improve upon the maximum number of equiangular lines in dimensions 14, 16-20.

n	d	λ	μ	ν	Exist?	Remark
28	14	$[-5]^{14}$	$[3]^7$	$[7]^7$	Y	[13]
30	14	$[-5]^{16}$	$[5]^9$	$[7]^5$	?	
40	16	$[-5]^{24}$	$[5]^6$	$[9]^{10}$	?	
40	16	$[-5]^{24}$	$[7]^{15}$	$[15]^1$	Y	[10]
42	16	$[-5]^{26}$	$[7]^7$	$[9]^9$	?	
48	17	$[-5]^{31}$	$[7]^8$	$[11]^9$	Y	[6]
49	17	$[-5]^{32}$	$[9]^{16}$	$[16]^1$	?	
48	18	$[-5]^{30}$	$[3]^6$	$[11]^{12}$	?	
48	18	$[-5]^{30}$	$[7]^{16}$	$[19]^2$	?	
54	18	$[-5]^{36}$	$[7]^9$	$[13]^9$	?	
60	18	$[-5]^{42}$	$[11]^{15}$	$[15]^3$	?	
72	19	$[-5]^{53}$	$[13]^{16}$	$[19]^3$	Y	Example 1.7
75	19	$[-5]^{56}$	$[10]^1$	$[15]^{18}$	?	
90	20	$[-5]^{70}$	$[13]^5$	$[19]^{15}$	?	
95	20	$[-5]^{75}$	$[14]^1$	$[19]^{19}$	?	

TABLE 4. Feasible parameter sets of Seidel matrices with 3 distinct eigenvalues.

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