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Discrete Geometry on 3 Colored Point sets in the Plane

Mikio Kano
Department of Computer and Information Sciences, Ibaraki University, Hitachi, Ibaraki, Japan
kano@mx.ibaraki.ac.jp http://gorogoro.cis.ibaraki.ac.jp

1 3 colored point sets in the plane

Let $R$, $B$ and $G$ denote disjoint sets of red points, blue points and green points in the plane, respectively. If no three points of $R \cup B \cup G$ are collinear, we say that $R$, $B$ and $G$ are in general position in the plane. We always assume that given sets of colored points are in general position.

We begin with the following well-known theorem on two colored point sets in the plane. Notice that a geometric graph is a graph drawn in the plane whose edges are straight line segments, and every edge of an alternating matching joins two points with distinct colors.

Theorem 1 ([3]). If $|R| = |B|$, then there exists an alternating non-crossing geometric perfect matching on $R \cup B$ (see Figure 1).

![Figure 1: An alternating non-crossing geometric perfect matching on $R \cup B$.](image)

We generalize the above theorem by considering 3 colored point sets. The standard proof of the following theorem is basically similar to that of the above Theorem 1, but more difficult.
Corollary 2 (Kano, Suzuki, Uno [4]). If $|R \cup B \cup G| = 2n$, $|R| \leq n$, $|B| \leq n$ and $|G| \leq n$, then there exists an alternating non-crossing geometric perfect matching on $R \cup B \cup G$.

![Diagram of an alternating non-crossing geometric perfect matching on $R \cup B \cup G$.]

It is known as the discrete version of Ham-Sandwich theorem that if $|R| = 2m$ and $|B| = 2n$, then there exists a bisector line $l$ such that $|\text{left}(l) \cap R| = m$ and $|\text{left}(l) \cap B| = n$. It is easy to see that there exist configurations of 3 colored points in the plane such that there exists no line $l$ such that a half-plane determined by $l$ contains the same number of each colored points. Thus the condition in the next theorem is necessary. For a set $X$ of points in the plane, we denote the convex hull of $X$ by $\text{conv}(X)$.

Theorem 3 (Bereg and Kano [2]). Assume that $|R| = |B| = |G| = n$, where $n \geq 2$. If all the vertices of $\text{conv}(R \cup B \cup G)$ are red, then there exists a line $l$ such that $|\text{right}(l) \cap R| = |\text{right}(l) \cap B| = |\text{right}(l) \cap G| = k$ for some integer $1 \leq k \leq n - 1$ (see Figure 3).

We give one more result on three colored point sets in the plane, and explain a sketch of its proof.

Theorem 4 (Berege and etc. [1]). Assume that $n$ red points and $n$ blue points and $n$ green points lie on a circle in the plane. Then for every integer $1 \leq k \leq n - 1$, there exist two intervals $I$ and $J$ on the circle such that $I \cup J$ contains exactly $k$ red points, $k$ blue points and $k$ green points (see Figure 4).

We give a sketch of its proof.

Lemma 5. Let $n \geq 2$ be an integer. Then every integer $1 \leq k \leq n - 1$ can be obtained from $n$ by applying the following functions $f$ and $g$ some times.

\[ f(x) = \lfloor x/2 \rfloor \quad \text{and} \quad g(x) = n - x \]
Figure 3: All the vertices of $\text{conv}(R \cup B \cup G)$ are red; An line $l$ such that $\text{right}(l)$ contains exactly 3 red points, 3 blue points and 3 green points.

Figure 4: Two disjoint intervals $I$ and $J$ that contains exactly 3 red points, 3 blue points and 3 green points.
We show only one example, whose generalization gives us its proof. Suppose that \( n = 30 \) and \( k = 2 \). Then \( \lfloor n/2 \rfloor = 15 \). We construct the following series of intervals as follows: if an interval \([x, y]\) does not contain 15 and \( y < 15 \), then make an interval \([2x, 2y + 1]\). If \([x, y]\) does not contain 15 and \( 15 < x \), then make an interval \([30 - y, 30 - x]\). If an interval \([x, y]\) contains 15, then stop. Then we can obtain \( k = 2 \) from \( \lfloor n/2 \rfloor = 15 \) by applying the operations \( f(x) \) and \( g(x) \) as follows.

\[
\begin{align*}
  k &= 2 \rightarrow [4, 5] \rightarrow [8, 11] \rightarrow [16, 23] \rightarrow [7, 14] \rightarrow [14, 29] \ni 15 \\
  2 &\leftarrow 5 \leftarrow 11 \leftarrow 23 \leftarrow 7 \leftarrow 15
\end{align*}
\]

The next lemma follows immediately from Lemma 5.

**Lemma 6.** Let \( n \geq 2 \) be an integer, and let \( X \) be a subset of \( \{0, 1, 2, \ldots, n\} \). Define two functions \( f \) and \( g \) as follows:

\[
f(x) = \lfloor x/2 \rfloor \quad \text{and} \quad g(x) = n - x
\]

If \( X \) has the following properties, then \( X = \{0, 1, 2, \ldots, n\} \).

\[
n \in X; \text{ and if } k \in X, \text{ then } g(k) \in X \text{ and } f(k) \in X.
\]

**Sketch of the proof of Theorem 4.** Let us define

\[
X = \{1 \leq x \leq n : \text{ there exist two intervals } I \text{ and } J \text{ on the circle such that } I \cup J \text{ contains exactly } x \text{ red points, } x \text{ blue points and } x \text{ green points.}\}
\]

It is easy to see that \( n \in X \), and if \( k \in X \), then the complement \( I \cup J \) on the circle contains exactly \( n - k \) red points, \( n - k \) blue points and \( n - k \) green points, which implies \( g(k) = n - k \in X \). Moreover, we can show that if there exist intervals \( I \) and \( J \) on the circle such that \( I \cup J \) contains exactly \( k \) red points, \( k \) blue points and \( k \) green points, then there exist intervals \( I' \) and \( J' \) in \( I \cup J \) such that \( I' \cup J' \) contains exactly \( \lfloor k/2 \rfloor \) red points \( \lfloor k/2 \rfloor \) blue points and \( \lfloor k/2 \rfloor \) green points. Hence by Lemma 6, \( X = \{0, 1, 2, \ldots, n\} \), which implies that Theorem 4 holds.

**References**

