Discrete Geometry on 3 Colored Point sets in the Plane

Mikio Kano
Department of Computer and Information Sciences,
Ibaraki University, Hitachi, Ibaraki, Japan
kano@mx.ibaraki.ac.jp http://gorogoro.cis.ibaraki.ac.jp

1 3 colored point sets in the plane

Let $R$, $B$ and $G$ denote disjoint sets of red points, blue points and green points in the plane, respectively. If no three points of $R \cup B \cup G$ are collinear, we say that $R$, $B$ and $G$ are in general position in the plane. We always assume that given sets of colored points are in general position.

We begin with the following well-known theorem on two colored point sets in the plane. Notice that a geometric graph is a graph drawn in the plane whose edges are straight line segments, and every edge of an alternating matching joins two points with distinct colors.

**Theorem 1** ([3]). If $|R| = |B|$, then there exists an alternating non-crossing geometric perfect matching on $R \cup B$ (see Figure 1).

![Figure 1: An alternating non-crossing geometric perfect matching on $R \cup B$.](image)

We generalize the above theorem by considering 3 colored point sets. The standard proof of the following theorem is basically similar to that of the above Theorem 1, but more difficult.
Corollary 2 (Kano, Suzuki, Uno [4]). If \(|R \cup B \cup G| = 2n, |R| \leq n, |B| \leq n \) and \(|G| \leq n\), then there exists an alternating non-crossing geometric perfect matching on \(R \cup B \cup G\).

![Diagram of colored points with matching]

Figure 2: An alternating non-crossing geometric perfect matching on \(R \cup B \cup G\).

It is known as the discrete version of Ham-Sandwich theorem that if \(|R| = 2m\) and \(|B| = 2n\), then there exists a bisector line \(l\) such that \(|left(l) \cap R| = m\) and \(|left(l) \cap B| = n\). It is easy to see that there exist configurations of 3 colored points in the plane such that there exists no line \(l\) such that a half-plane determined by \(l\) contains the same number of each colored points. Thus the condition in the next theorem is necessary. For a set \(X\) of points in the plane, we denote the convex hull of \(X\) by \(\text{conv}(X)\).

Theorem 3 (Bereg and Kano [2]). Assume that \(|R| = |B| = |G| = n\), where \(n \geq 2\). If all the vertices of \(\text{conv}(R \cup B \cup G)\) are red, then there exists a line \(l\) such that \(|right(l) \cap R| = |right(l) \cap B| = |right(l) \cap G| = k\) for some integer \(1 \leq k \leq n - 1\) (see Figure 3).

We give one more result on three colored point sets in the plane, and explain a sketch of its proof.

Theorem 4 (Berege and etc. [1]). Assume that \(n\) red points and \(n\) blue points and \(n\) green points lie on a circle in the plane. Then for every integer \(1 \leq k \leq n - 1\), there exist two intervals \(I\) and \(J\) on the circle such that \(I \cup J\) contains exactly \(k\) red points, \(k\) blue points and \(k\) green points (see Figure 4).

We give a sketch of its proof.

Lemma 5. Let \(n \geq 2\) be an integer. Then every integer \(1 \leq k \leq n - 1\) can be obtained from \(n\) by applying the following functions \(f\) and \(g\) some times.

\[
f(x) = \lfloor x/2 \rfloor \quad \text{and} \quad g(x) = n - x
\]
Figure 3: All the vertices of $\text{conv}(R \cup B \cup G)$ are red; An line $l$ such that $\text{right}(l)$ contains exactly 3 red points, 3 blue points and 3 green points.

Figure 4: Two disjoint intervals $I$ and $J$ that contains exactly 3 red points, 3 blue points and 3 green points.
We show only one example, whose generalization gives us its proof. Suppose that $n = 30$ and $k = 2$. Then $\lfloor n/2 \rfloor = 15$. We construct the following series of intervals as follows: if an interval $[x, y]$ does not contain 15 and $y < 15$, then make an interval $[2x, 2y + 1]$. If $[x, y]$ does not contain 15 and $15 < x$, then make an interval $[30 - y, 30 - x]$. If an interval $[x, y]$ contains 15, then stop. Then we can obtain $k = 2$ from $\lfloor n/2 \rfloor = 15$ by applying the operations $f(x)$ and $g(x)$ as follows.

\[
k = 2 \rightarrow [4, 5] \rightarrow [8, 11] \rightarrow [16, 23] \rightarrow [7, 14] \rightarrow [14, 29] \ni 15
\]

\[
2 \leftarrow 5 \leftarrow 11 \leftarrow 23 \leftarrow 7 \leftarrow 15
\]

The next lemma follows immediately from Lemma 5

**Lemma 6.** Let $n \geq 2$ be an integer, and let $X$ be a subset of $\{0, 1, 2, \ldots, n\}$. Define two functions $f$ and $g$ as follows:

\[
f(x) = \lfloor x/2 \rfloor \quad \text{and} \quad g(x) = n - x
\]

If $X$ has the following properties, then $X = \{0, 1, 2, \ldots, n\}$.

$\ n \in X; \ \text{and if} \ \ k \in X, \ \text{then} \ \ g(k) \in X \ \text{and} \ \ f(k) \in X.\n$

**Sketch of the proof of Theorem 4.** Let us define

\[
X = \{1 \leq x \leq n : \text{there exist two intervals } I \text{ and } J \text{ on the circle such that } I \cup J \text{ contains exactly } x \text{ red points, } x \text{ blue points and } x \text{ green points.}\}
\]

It is easy to see that $n \in X$, and if $k \in X$, then the complement $I \cup J$ on the circle contains exactly $n - k$ red points, $n - k$ blue points and $n - k$ green points, which implies $g(k) = n - k \in X$. Moreover, we can show that if there exist intervals $I$ and $J$ on the circle such that $I \cup J$ contains exactly $k$ red points, $k$ blue points and $k$ green points, then there exist intervals $I'$ and $J'$ in $I \cup J$ such that $I' \cup J'$ contains exactly $\lfloor k/2 \rfloor$ red points $\lfloor k/2 \rfloor$ blue points and $\lfloor k/2 \rfloor$ green points. Hence by Lemma 6, $X = \{0, 1, 2, \ldots, n\}$, which implies that Theorem 4 holds.

**References**

