数理解析研究所講究録
第 1891 巻 2014 年 19-36

Quantum scattering in crossed constant magnetic and time-dependent electric fields

神戸大学大学院理学研究科　　足立　匡義 (Tadayoshi ADACHI)
Graduate School of Science, Kobe University

1 Introduction

In this article, we would like to mention the results of our paper [1], which is concerned with the study of the quantum dynamics of a charged particle in the presence of crossed constant magnetic and time-dependent electric fields.

We consider a quantum system of a charged particle moving in the plane $\boldsymbol{R}^2$ in the presence of the constant magnetic field $\boldsymbol{B}$ which is perpendicular to the plane, and the time-dependent electric field $\boldsymbol{E}(t)$ which always lies in the plane. For the sake of simplicity, we write $\boldsymbol{B}$ as $(0,0,B)$ with $B>0$, and $\boldsymbol{E}(t)=(E_1(t),E_2(t),0)$. Then the free Hamiltonian under consideration is defined by

$$H_0(t)=H_{0,L}-qE(t)\cdot x, \quad H_{0,L}=(p-qA(x))^2/(2m), \tag{1.1}$$

where $m>0$, $q\in\boldsymbol{R}\setminus\{0\}$, $x=(x_1,x_2)$ and $p=(p_1,p_2)=(-i\partial_1,-i\partial_2)$ are the mass, the charge, the position, and the usual momentum of the charged particle, respectively, and

$$A(x)=(-Bx_2/2,Bx_1/2)$$

is the vector potential in the symmetric gauge. Here we put $E(t)=(E_1(t),E_2(t))$. $H_{0,L}$ is called the free Landau Hamiltonian. It is well known that

$$\sigma(H_{0,L})=\sigma_{\mathrm{pp}}(H_{0,L})=\left\{|\omega|(n+1/2)\;\middle|\;n\in\boldsymbol{N}\cup\{0\}\right\}$$

holds, where $\omega=qB/m$. $|\omega|$ is called the Larmor frequency. Each eigenvalue of $H_{0,L}$, which is called a Landau level, is of infinite multiplicity (see e.g. Avron-Herbst-Simon [5]). In fact, this can be seen as follows: First of all, we introduce the momentum D and the pseudomomentum k of the charged particle in the presence of $\boldsymbol{B}$ as

$$D=p-qA(x), \quad k=p+qA(x).$$

Writing D and k as (D_1,D_2) and (k_1,k_2), respectively, we have

$$(D_1,D_2)=(p_1+qBx_2/2,p_2-qBx_1/2), \quad (k_1,k_2)=(p_1-qBx_2/2,p_2+qBx_1/2).$$

One of the basic properties of D and k is that

$$i[D_1,D_2]=-qB, \quad i[k_1,k_2]=qB, \quad i[D_j,k_l]=0 \quad (j,l\in\{1,2\}). \tag{1.2}$$

Putting

$$\tilde{U} = e^{iqBx_1x_2/2}e^{ip_1p_2/(qB)},$$

we have

$$\tilde{U}^*D_1\tilde{U} = qBx_2, \quad \tilde{U}^*D_2\tilde{U} = p_2,$$
$$\tilde{U}^*k_1\tilde{U} = p_1, \qquad \tilde{U}^*k_2\tilde{U} = qBx_1$$

(see e.g. Skibsted [22]). In particular, we have

$$\tilde{U}^*H_{0,L}\tilde{U} = \mathrm{Id} \otimes \{p_2^2/(2m) + m\omega^2x_2^2/2\}$$

on $\tilde{U}^*L^2(\boldsymbol{R}^2) = L^2(\boldsymbol{R}_{x_1}) \otimes L^2(\boldsymbol{R}_{x_2})$, which implies the infinite multiplicity of each Landau level. In order to deal with the one dimensional harmonic oscillator $p_2^2/(2m) + m\omega^2x_2^2/2$, we introduce the annihilation operator $\tilde{a}$ and the creation operator $\tilde{a}^*$ as

$$\tilde{a} = (|q|Bx_2 + ip_2)/\sqrt{2|q|B}, \quad \tilde{a}^* = (|q|Bx_2 - ip_2)/\sqrt{2|q|B}.$$

Then we have

$$p_2^2/(2m) + m\omega^2x_2^2/2 = |\omega|(\tilde{a}^*\tilde{a} + 1/2).$$

We also put

$$\tilde{b} = (|q|Bx_1 + ip_1)/\sqrt{2|q|B}, \quad \tilde{b}^* = (|q|Bx_1 - ip_1)/\sqrt{2|q|B},$$

and introduce a, a^*, b and b^* as

$$a = \tilde{U}\tilde{a}\tilde{U}^* = (qD_1/|q| + iD_2)/\sqrt{2|q|B}, \quad a^* = \tilde{U}\tilde{a}^*\tilde{U}^* = (qD_1/|q| - iD_2)/\sqrt{2|q|B},$$
$$b = \tilde{U}\tilde{b}\tilde{U}^* = (ik_1 + qk_2/|q|)/\sqrt{2|q|B}, \quad b^* = \tilde{U}\tilde{b}^*\tilde{U}^* = (-ik_1 + qk_2/|q|)/\sqrt{2|q|B}.$$

Then we obtain an complete orthonormal system $\{(b^*)^l(a^*)^n\phi_0/\sqrt{l!n!}\}_{(l,n)\in(\boldsymbol{N}\cup\{0\})^2}$ of $L^2(\boldsymbol{R}^2)$, which consists of eigenfunctions of $H_{0,L}$, where $\phi_0(x) = \sqrt{|q|B/(2\pi)}e^{-|q|Bx^2/4}$. In fact, $(b^*)^l(a^*)^n\phi_0/\sqrt{l!n!}$ is an eigenfunction of $H_{0,L}$ belonging to the Landau level $|\omega|(n+1/2)$.

We see that $H_0(t)$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^2)$ for any $t \in \boldsymbol{R}$, by virtue of Kato's inequality associated with $H_{0,L}$ and Nelson's commutator theorem (see e.g. Reed-Simon [19] and Gérard-Łaba [15]). Its closure is also denoted by $H_0(t)$. Then $H_0(t)$ can be written as

$$\begin{aligned} H_0(t) &= D^2/(2m) - q(-qB^2/2)^{-1}E(t)\cdot A(k-D) \\ &= D^2/(2m) - \alpha(t)\cdot D + \alpha(t)\cdot k \\ &= (D - m\alpha(t))^2/(2m) + \alpha(t)\cdot k - m\alpha(t)^2/2 \end{aligned} \tag{1.3}$$

where

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) = (E_2(t)/B, -E_1(t)/B) = -2A(E(t))/B^2$$

is the instantaneous drift velocity of the charged particle. Here we used

$$k - D = 2qA(x), \quad A(A(x)) = -(B/2)^2x, \quad y\cdot A(x) = -A(y)\cdot x.$$

We note that

$$(\alpha(t), 0) = \boldsymbol{E}(t) \times \boldsymbol{B}/B^2,$$

and that $\alpha(t)$ is independent of the charge $q \in \boldsymbol{R} \setminus \{0\}$. We also see that when $\alpha(t) \neq 0$, $\sigma(H_0(t))$ is purely absolutely continuous and

$$\sigma(H_0(t)) = \boldsymbol{R},$$

by virtue of (1.3).

When $E(t) \equiv (E_1, E_2)$, that is, $E(t)$ is independent of t, Skibsted [22] essentially obtained the following factorization of the unitary propagator $U_0(t,s)$ generated by $H_0(t)$:

$$U_0(t,0) = U_1(t)e^{-itH_{0,L}}U_1(0)^*, \quad U_1(t) = e^{itm\alpha^2/2}e^{-it\alpha\cdot p}e^{i(tqA(\alpha)+m\alpha)\cdot x}, \tag{1.4}$$

where

$$\alpha = (\alpha_1, \alpha_2) = (E_2/B, -E_1/B) = -2A(E)/B^2$$

is the drift velocity of the charged particle, where $E = (E_1, E_2)$. Since $H_0(t)$ is independent of t in this case, $U_0(t,s)$ can be represented as $e^{-i(t-s)H_0}$ by writing this time-independent Hamiltonian $H_0(t)$ as $H_0 = H_{0,L} - qE \cdot x$. $U_1(t)$ is a version of the Galilei transform which reflects the effect of the magnetic field $\boldsymbol{B}$. We note that $U_1(0) = e^{im\alpha\cdot x} \neq \mathrm{Id}$ because of $\alpha \neq 0$.

After that, for a general time-dependent electric field $E(t)$, Chee [6] proposed the following factorization of $U_0(t,s)$:

$$\begin{aligned} U_0(t,0) &= M(R(t))e^{-itH_{0,L}}J(u(t))^*, \\ M(R(t)) &= e^{i\int_0^t R(s)\cdot qA(\dot{R}(s))\,ds}e^{-iR(t)\cdot qA(x)}e^{-iR(t)\cdot p}, \\ J(u(t)) &= e^{i\int_0^t u(s)\cdot qA(\dot{u}(s))\,ds}e^{iu(t)\cdot qA(x)}e^{-iu(t)\cdot p}, \end{aligned} \tag{1.5}$$

where $R(t) = (R_1(t), R_2(t))$ and $u(t) = (u_1(t), u_2(t))$ are given by

$$R(t) = \int_0^t \alpha(s)\,ds, \quad \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \int_0^t \begin{pmatrix} \cos\omega s & -\sin\omega s \\ \sin\omega s & \cos\omega s \end{pmatrix} \begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \end{pmatrix} ds, \tag{1.6}$$

with $\dot{R}(t) = dR(t)/dt$ and $\dot{u}(t) = du(t)/dt$. Here we note that $\dot{R}(t) = \alpha(t)$, and that one has $R(t) = t\alpha$ when $E(t) \equiv E$. What we emphasize here is that $e^{-i\int_0^t R(s)\cdot qA(\dot{R}(s))\,ds}M(R(t))$ and $e^{-i\int_0^t u(s)\cdot qA(\dot{u}(s))\,ds}J(u(t))$ are just the magnetic translations $T(R(t))$ and $S(u(t))$ generated by k and D, respectively, where

$$T(y) = e^{-iy\cdot qA(x)}e^{-iy\cdot p} = e^{-iy\cdot k}, \quad S(y) = e^{iy\cdot qA(x)}e^{-iy\cdot p} = e^{-iy\cdot D}$$

for $y \in \boldsymbol{R}^2$ (see e.g. [5] and [15]). For reference, we state one of the features which distinguish between the Galilei transform $U_1(t)$ and the magnetic translation $T(t\alpha)$, where α is the drift velocity:

$$U_1(t)^* x U_1(t) = x + t\alpha, \quad U_1(t)^* D U_1(t) = D + m\alpha,$$
$$T(t\alpha)^* x T(t\alpha) = x + t\alpha, \quad T(t\alpha)^* D T(t\alpha) = D.$$

On the other hand, in the absence of the magnetic field $\boldsymbol{B}$, it is well known that the following factorization of $U_0(t, s)$, which is called the Avron-Herbst formula, holds (see e.g. Cycon-Froese-Kirsch-Simon [7]):

$$U_0(t, 0) = e^{-ia^0(t)} e^{ib^0(t)\cdot x} e^{-ic^0(t)\cdot p} e^{-itK_0}, \tag{1.7}$$

where $K_0 = p^2/(2m)$, and

$$b^0(t) = \int_0^t qE(s)\, ds, \; c^0(t) = \int_0^t b^0(s)/m\, ds, \; a^0(t) = \int_0^t b^0(s)^2/(2m)\, ds. \tag{1.8}$$

Inspired by these two formulas (1.5) and (1.7), we have derived an Avron-Herbst type formula for $U_0(t, s)$:

Theorem 1.1 (Adachi-Kawamoto [1]). *The following Avron-Herbst type formula for $U_0(t, 0)$*

$$U_0(t, 0) = e^{-ia(t)} e^{ib(t)\cdot x} T(c(t)) e^{-itH_{0,L}}, \quad T(c(t)) = e^{-ic(t)\cdot qA(x)} e^{-ic(t)\cdot p} \tag{1.9}$$

holds, where $b(t) = (b_1(t), b_2(t))$, $c(t) = (c_1(t), c_2(t))$ and $a(t)$ are given by

$$\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} = \int_0^t \begin{pmatrix} \cos\omega(t-s) & \sin\omega(t-s) \\ -\sin\omega(t-s) & \cos\omega(t-s) \end{pmatrix} \begin{pmatrix} qE_1(s) \\ qE_2(s) \end{pmatrix} ds,$$
$$c(t) = \int_0^t b(s)/m\, ds, \quad a(t) = \int_0^t \{b(s)^2/(2m) + b(s)\cdot qA(c(s))/m\}\, ds. \tag{1.10}$$

Here we note that by taking B as 0 formally in (1.9) and (1.10), one can obtain the Avron-Herbst formula (1.7) in the absence of the magnetic field $\boldsymbol{B}$ because $\omega = 0$ and $A(x) \equiv 0$. Hence we have obtained a natural extension of the Avron-Herbst formula to the case of the presence of the magnetic field $\boldsymbol{B}$, by virtue of the magnetic translation $T(c(t))$.

From now on, we will discuss a scattering problem for the free Hamiltonian $H_0(t)$ and the perturbed Hamiltonian $H(t) = H_0(t) + V(x)$, where the time-independent potential $V(x)$ satisfies that $|V(x)| \to 0$ as $|x| \to \infty$.

Now we explain an advantage of the Avron-Herbst type formula (1.9) from the point of view of the scattering theory: Put

$$E_{\nu,\theta}(t) = E_0(\cos(\nu t + \theta), \sin(\nu t + \theta))$$

for $E_0 > 0$, $\nu \in \boldsymbol{R}$ and $\theta \in [0, 2\pi)$. We note that $|E_{\nu,\theta}(t)| \equiv E_0$. Now we consider the case where $E(t) = E_{\nu,\theta}(t)$. By a straightforward calculation, we have

$$R(t) = \begin{cases} -E_0((\delta\cos)(\nu t), (\delta\sin)(\nu t))/(\nu B), & \nu \neq 0, \\ E_0(t\sin\theta, -t\cos\theta)/B, & \nu = 0, \end{cases}$$

$$u(t) = \begin{cases} -E_0((\delta\cos)(\tilde\nu t), (\delta\sin)(\tilde\nu t))/(\tilde\nu B), & \tilde\nu \neq 0, \\ E_0(t\sin\theta, -t\cos\theta)/B, & \tilde\nu = 0, \end{cases}$$

where we put $\tilde\nu = \nu + \omega$, $(\delta\cos)(s) = \cos(s+\theta) - \cos\theta$ and $(\delta\sin)(s) = \sin(s+\theta) - \sin\theta$ for the sake of brevity. Hence we see that $R(t)$ is growing of order $|t|$ when $\nu = 0$ because of $|R(t)| = E_0|t|/B$ although $R(t)$ is bounded in t when $\nu \neq 0$, and that $u(t)$ is growing of order $|t|$ when $\tilde\nu = 0$ because of $|u(t)| = E_0|t|/B$ although $u(t)$ is bounded in t when $\tilde\nu \neq 0$. In consequence of (1.5) and the growth of $R(t)$ or $u(t)$, the possibility of the existence of scattering states for the system under consideration in the case where $\nu\tilde\nu = 0$ is suggested: In fact, it follows from

$$\begin{pmatrix} e^{-itH_{0,L}} D_1 e^{itH_{0,L}} \\ e^{-itH_{0,L}} D_2 e^{itH_{0,L}} \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},$$

which can be obtained by (1.2), that

$$e^{-itH_{0,L}} S(u(t))^* = e^{-itH_{0,L}} e^{iu(t)\cdot D} = e^{i\tilde u(t)\cdot D} e^{-itH_{0,L}} = S(\tilde u(t))^* e^{-itH_{0,L}}$$

holds, where $\tilde u(t) = (\tilde u_1(t), \tilde u_2(t))$ with

$$\begin{pmatrix} \tilde u_1(t) \\ \tilde u_2(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$
$$= \int_0^t \begin{pmatrix} \cos\omega(t-s) & \sin\omega(t-s) \\ -\sin\omega(t-s) & \cos\omega(t-s) \end{pmatrix} \begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \end{pmatrix} ds.$$

Hence we obtain

$$U_0(t,0) = e^{i\int_0^t R(s)\cdot qA(\dot R(s))\,ds} e^{-i\int_0^t u(s)\cdot qA(\dot u(s))\,ds} T(R(t)) S(\tilde u(t))^* e^{-itH_{0,L}} \tag{1.11}$$

from (1.5) by a straightforward calculation. Let ϕ be an eigenfunction of $H_{0,L}$ belonging to some Landau level λ. Here we note that

$$\|F(|x| \le Ct) U_0(t,0)\phi\|_{L^2(\boldsymbol{R}^2)} = \|F(|x + R(t) - \tilde u(t)| \le Ct)\phi\|_{L^2(\boldsymbol{R}^2)}$$

for $t > 0$, and $|\tilde u(t)| = |u(t)|$, where $F(|x| \le Ct)$ stands for the characteristic function of the set $\{x \in \boldsymbol{R}^2 \mid |x| \le Ct\}$. In the case where $\nu\tilde\nu = 0$, $|R(t) - \tilde u(t)| \ge 3E_0 t/(4B)$ holds for sufficiently large $t > 0$. Then, by taking C as $E_0/(2B)$, we obtain

$$\|F(|x| \le E_0 t/(2B)) U_0(t,0)\phi\|_{L^2(\boldsymbol{R}^2)} \le \|F(|x| \ge E_0 t/(4B))\phi\|_{L^2(\boldsymbol{R}^2)} \to 0$$

as $t \to \infty$, by virtue of the triangle inequality. This suggests the possibility of the existence of scattering states in the case where $\nu\tilde{\nu} = 0$. As is well known, the case where $\tilde{\nu} = 0$, that is, $\nu = -\omega$, is closely related with the phenomenon of the cyclotron resonance. The formula (1.11) can be also obtained by the idea of Enss-Veselić [12]: We first introduce

$$\hat{H}_0(t) = \hat{H}_{0,\hat{\omega}} - \hat{f}(t)z + \hat{g}(t)p_z, \quad \hat{H}_{0,\hat{\omega}} = p_z^2/(2m) + m\hat{\omega}^2 z^2/2$$

acting on $L^2(\boldsymbol{R}_z)$, where $z \in \boldsymbol{R}$ and $p_z = -id/dz$. Then one can obtain a factorization of the propagator $\hat{U}_0(t,s)$ generated by $\hat{H}_0(t)$:

$$\hat{U}_0(t,0) = e^{-i\hat{a}(t)}e^{i\hat{b}(t)z}e^{-i\hat{c}(t)p_z}e^{-it\hat{H}_{0,\hat{\omega}}}.$$

In fact, the differential equations which $\hat{a}(t)$, $\hat{b}(t)$ and $\hat{c}(t)$ should obey are as follows:

$$\begin{cases} \begin{pmatrix} \dot{\hat{b}}(t) \\ \dot{\hat{c}}(t) \end{pmatrix} = \begin{pmatrix} 0 & -m\hat{\omega}^2 \\ 1/m & 0 \end{pmatrix} \begin{pmatrix} \hat{b}(t) \\ \hat{c}(t) \end{pmatrix} + \begin{pmatrix} \hat{f}(t) \\ \hat{g}(t) \end{pmatrix}, \\ \dot{\hat{a}}(t) = \hat{b}(t)\dot{\hat{c}}(t) - \hat{b}(t)^2/(2m) - m\hat{\omega}^2\hat{c}(t)^2/2 \end{cases}$$

with $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = 0$. Then one can obtain

$$\begin{pmatrix} \hat{b}(t) \\ \hat{c}(t) \end{pmatrix} = \int_0^t \begin{pmatrix} \cos\hat{\omega}(t-s) & -m\hat{\omega}\sin\hat{\omega}(t-s) \\ \sin\hat{\omega}(t-s)/(m\hat{\omega}) & \cos\hat{\omega}(t-s) \end{pmatrix} \begin{pmatrix} \hat{f}(s) \\ \hat{g}(s) \end{pmatrix} ds \tag{1.12}$$

by a straightforward calculation. Here we note that $H_0(t) = H_{0,L} - \alpha(t)\cdot D + \alpha(t)\cdot k$ holds (see (1.3)). Using $\tilde{U}^* H_0(t)\tilde{U} = \hat{H}_{0,\omega} - \alpha(t)\cdot\tilde{D} + \alpha(t)\cdot\tilde{k}$ with $z = x_2$, $\tilde{D} = (qBx_2, p_2)$ and $\tilde{k} = (p_1, qBx_1)$, we obtain

$$\tilde{U}^* U_0(t,0)\tilde{U} = \check{T}(t,0)e^{-i\hat{a}(t)}e^{i\hat{b}(t)x_2}e^{-i\hat{c}(t)p_2}e^{-it\hat{H}_{0,\omega}}$$

with $\hat{f}(t) = qB\alpha_1(t) = qE_2(t)$, $\hat{g}(t) = -\alpha_2(t) = E_1(t)/B$ and $R(t) = \int_0^t \alpha(s)\,ds$, where $\check{T}(t,s)$ is the propagator generated by $\alpha(t)\cdot\tilde{k} = \alpha_1(t)p_1 + qB\alpha_2(t)x_1$. In the same way as above, we obtain the following representation of $\check{T}(t,0)$:

$$\begin{aligned} &\check{T}(t,0) = e^{-i\check{a}(t)}e^{-i\check{b}(t)x_1}e^{-i\check{c}(t)p_1}, \\ &\check{b}(t) = qBR_2(t), \quad \check{c}(t) = R_1(t), \quad \check{a}(t) = -\int_0^t qBR_2(s)\alpha_1(s)\,ds. \end{aligned}$$

Noting that $qB = m\omega$ and using the Baker-Campbell-Hausdorff formula, we have

$$\begin{aligned} U_0(t,0) &= e^{-i\check{a}(t)}e^{-i\check{b}(t)k_2/(qB)}e^{-i\check{c}(t)k_1}e^{-i\hat{a}(t)}e^{i\hat{b}(t)D_1/(qB)}e^{-i\hat{c}(t)D_2}e^{-itH_{0,L}} \\ &= e^{-i(\check{a}(t)+\check{b}(t)\check{c}(t)/2)}e^{-i(\hat{a}(t)-\hat{b}(t)\hat{c}(t)/2)}T(R(t))S(\tilde{u}(t))e^{-itH_{0,L}} \end{aligned}$$

with

$$\begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{pmatrix} = \begin{pmatrix} \hat{b}(t)/(qB) \\ -\hat{c}(t) \end{pmatrix} = \int_0^t \begin{pmatrix} \cos\omega(t-s) & \sin\omega(t-s) \\ -\sin\omega(t-s) & \cos\omega(t-s) \end{pmatrix} \begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \end{pmatrix} ds.$$

By a straightforward calculation, we also have

$$-\frac{d}{dt}(\check{a}(t) + \check{b}(t)\check{c}(t)/2) = R(t) \cdot qA(\dot{R}(t)), \quad \frac{d}{dt}(\hat{a}(t) - \hat{b}(t)\hat{c}(t)/2) = u(t) \cdot qA(\dot{u}(t)),$$

which yields (1.11).

Now we will make a similar calculation on $c(t)$. In fact, we have

$$b_1(t) = \begin{cases} qE_0\{\sin(\nu t + \theta) - \sin(-\omega t + \theta)\}/\tilde{\nu}, & \tilde{\nu} \neq 0, \\ qE_0 t\cos(-\omega t + \theta), & \tilde{\nu} = 0, \end{cases}$$

$$b_2(t) = \begin{cases} -qE_0\{\cos(\nu t + \theta) - \cos(-\omega t + \theta)\}/\tilde{\nu}, & \tilde{\nu} \neq 0, \\ qE_0 t\sin(-\omega t + \theta), & \tilde{\nu} = 0, \end{cases}$$

as for $b(t)$. Here we used $\tilde{\nu} - \omega = \nu$. Hence we have

$$c_1(t) = \begin{cases} -(\omega/\tilde{\nu})E_0\{(\delta\cos)(\nu t)/\nu + (\delta\cos)(-\omega t)/\omega\}/B, & \nu\tilde{\nu} \neq 0, \\ E_0\{t\sin\theta - (\delta\cos)(-\omega t)/\omega\}/B, & \nu = 0, \\ E_0\{-t\sin(-\omega t + \theta) + (\delta\cos)(-\omega t)/\omega\}/B, & \tilde{\nu} = 0, \end{cases}$$

$$c_2(t) = \begin{cases} -(\omega/\tilde{\nu})E_0\{(\delta\sin)(\nu t)/\nu + (\delta\sin)(-\omega t)/\omega\}/B, & \nu\tilde{\nu} \neq 0, \\ E_0\{-t\cos\theta - (\delta\sin)(-\omega t)/\omega\}/B, & \nu = 0, \\ E_0\{t\cos(-\omega t + \theta) + (\delta\sin)(-\omega t)/\omega\}/B, & \tilde{\nu} = 0, \end{cases}$$

where we used $\omega = qB/m$. Hence we see that $c(t)$ is growing of order $|t|$ when $\nu\tilde{\nu} = 0$, although $c(t)$ is bounded in t when $\nu\tilde{\nu} \neq 0$. We note that when $\nu = 0$,

$$c(t) - E_0(-(\delta\cos)(-\omega t), -(\delta\sin)(-\omega t))/(\omega B) = t\alpha \tag{1.13}$$

holds by $(E_1, E_2) = E_0(\cos\theta, \sin\theta)$, and that when $\tilde{\nu} = 0$, that is, $\nu = -\omega$,

$$c(t) - E_0((\delta\cos)(-\omega t), (\delta\sin)(-\omega t))/(\omega B) = -t\alpha(t) \tag{1.14}$$

holds. In consequence of (1.9), the possibility of the existence of scattering states for the system under consideration in the case where $\nu\tilde{\nu} = 0$ is suggested by the growth of $c(t)$ only: In fact,

$$\|F(|x| \leq Ct)U_0(t,0)\phi\|_{L^2(\boldsymbol{R}^2)} = \|F(|x + c(t)| \leq Ct)\phi\|_{L^2(\boldsymbol{R}^2)}$$

holds for some eigenfunction ϕ of $H_{0,L}$, and in the case where $\nu\tilde{\nu} = 0$, $|c(t)| \geq 3E_0 t/(4B)$ holds for sufficiently large $t > 0$. Thus, by the same argument as above, we see that $\|F(|x| \leq E_0 t/(2B))U_0(t,0)\phi\|_{L^2(\boldsymbol{R}^2)} \to 0$ as $t \to \infty$ in the case where $\nu\tilde{\nu} = 0$. Here we note that

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} R_1(t) \\ R_2(t) \end{pmatrix} - \begin{pmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \end{pmatrix} = \int_0^t \begin{pmatrix} 1-\cos\omega(t-s) & -\sin\omega(t-s) \\ \sin\omega(t-s) & 1-\cos\omega(t-s) \end{pmatrix} \begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \end{pmatrix} ds \tag{1.15}$$

can be verified by a straightforward calculation. Moreover, it follows from (1.9) that

$$U_0(t,s) = \mathscr{T}(t) e^{-i(t-s)H_{0,L}} \mathscr{T}(s)^*, \quad \mathscr{T}(t) = e^{-ia(t)} e^{ib(t)\cdot x} T(c(t)), \tag{1.16}$$

holds, although such a formula cannot be obtained from (1.5) easily. We note that $\mathscr{T}(0) = \mathrm{Id}$ by definition. These show an advantage of the Avron-Herbst type formula (1.9).

The existence of scattering states is equivalent to the existence of (modified) wave operators, as is well known. In this article, we consider the case where $E(t) = E_{\nu,\theta}(t)$ with $\nu \in \{0, -\omega\}$ and $\theta \in [0, 2\pi)$ only, give a short-range condition on the potential V, which implies the existence of usual wave operators, and propose a rather simple modifier by which the modified wave operators can be defined for some long-range potentials. Now we pose the following assumption $(V1)$ on V:

$(V1)$ V is written as the sum of real-valued functions V^{sing}, V^{s} and V^{l}, and that V^{sing}, V^{s} and V^{l} satisfy the following conditions: V^{sing} is compactly supported, belongs to $L^p(\boldsymbol{R}^2)$ with $2 \leq p < \infty$, and satisfies $|\nabla V^{\mathrm{sing}}| \in L^{2p/(p+1)}(\boldsymbol{R}^2)$. V^{s} belongs to $C^1(\boldsymbol{R}^2)$, and satisfies

$$|V^{\mathrm{s}}(x)| \leq C_0 \langle x \rangle^{-\rho_{\mathrm{s},0}}, \quad |(\nabla V^{\mathrm{s}})(x)| \leq C_1 \langle x \rangle^{-\rho_{\mathrm{s},1}} \tag{1.17}$$

for some $\rho_{\mathrm{s},0} > 1$ and $\rho_{\mathrm{s},1} > 0$, where C_0 and C_1 are non-negative constants. V^{l} belongs to $C^1(\boldsymbol{R}^2)$, and satisfies

$$|V^{\mathrm{l}}(x)| \leq \tilde{C}_0 \langle x \rangle^{-\rho_{\mathrm{l}}}, \quad |(\nabla V^{\mathrm{l}})(x)| \leq \tilde{C}_1 \langle x \rangle^{-1-\rho_{\mathrm{l}}} \tag{1.18}$$

for some $0 < \rho_{\mathrm{l}} \leq 1$, where $\tilde{C}_0$ and $\tilde{C}_1$ are non-negative constants.

Under this assumption $(V1)$, we see that the propagator $U(t,s)$ generated by

$$H(t) = H_0(t) + V \tag{1.19}$$

exists uniquely, by virtue of the results of Yajima [23] and $\mathscr{T}(t)$ in (1.16). If the local singularity of V^{sing} is like $|x|^{-\gamma}$, and that of $|\nabla V^{\mathrm{sing}}|$ is like $|x|^{-1-\gamma}$, then γ should satisfy $0 < \gamma < 1/2$.

Then we obtain the following result about the existence of (modified) wave operators:

Theorem 1.2 (Adachi-Kawamoto [1]). *Suppose that* $(V1)$ *is satisfied, and that* $E(t) = E_{\nu,\theta}(t)$ *with* $\nu \in \{0, -\omega\}$ *and* $\theta \in [0, 2\pi)$. *If* $V^{\mathrm{l}} = 0$, *then the wave operators*

$$W^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) \tag{1.20}$$

exist. If $V^{\mathrm{l}} \neq 0$, *then the modified wave operators*

$$W_G^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) e^{-i\int_0^t V^{\mathrm{l}}(c(s))\,ds} \tag{1.21}$$

exist.

Next we will consider the problem of the asymptotic completeness of wave operators when $\nu = 0$, that is, $E(t)$ is independent of t. Since the Hamiltonians under consideration are independent of t when $\nu = 0$, we write $H_0(t)$ and $H(t)$ as H_0 and H, respectively. Then $U_0(t,s)$ and $U(t,s)$ are represented as $e^{-i(t-s)H_0}$ and $e^{-i(t-s)H}$, respectively. We need the following assumption $(V2)$ on V, which is stronger than $(V1)$ in terms of the regularity of V:

$(V2)$ V is written as the sum of real-valued functions V^{s} and V^{l}, and that V^{s} and V^{l} satisfy the following conditions: V^{s} belongs to $C^2(\boldsymbol{R}^2)$, and satisfies $|\partial^\alpha V^{\mathrm{s}}(x)| \le C_2$ with $|\alpha| = 2$ in addition to (1.17), where C_2 is a non-negative constant. V^{l} belongs to $C^2(\boldsymbol{R}^2)$, and satisfies $|\partial^\alpha V^{\mathrm{l}}(x)| \le \tilde{C}_2$ with $|\alpha| = 2$ in addition to (1.18), where $\tilde{C}_2$ is a non-negative constant.

The result of the asymptotic completeness obtained in this article is as follows:

Theorem 1.3 (Adachi-Kawamoto [1]). *Suppose that* $(V2)$ *is satisfied, and that* $E(t)$ *is written as* $E_{0,\theta}(t) \equiv E_0(\cos\theta, \sin\theta)$ *with* $\theta \in [0, 2\pi)$. *Assume further the short-range condition* $V^{\mathrm{l}} = 0$. *Then* $W^{\pm}$ *are asymptotically complete, that is,*

$$\operatorname{Ran} W^{\pm} = L^2_{\mathrm{c}}(H), \tag{1.22}$$

where $L^2_{\mathrm{c}}(H)$ *is the continuous spectral subspace of the Hamiltonian* H.

Unfortunately the long-range case cannot be dealt with by our analysis. The propagation estimates obtained in this article (see e.g. Proposition 4.4) are not sufficient for the study of the long-range case.

In considering the case where $\nu = -\omega$, the rotating frame is useful: For $x = (x_1, x_2) \in \boldsymbol{R}^2$, we define $\hat{R}(\omega t)^{-1}x = ((\hat{R}(\omega t)^{-1}x)_1, (\hat{R}(\omega t)^{-1}x)_2)$ by

$$\begin{pmatrix} (\hat{R}(\omega t)^{-1}x)_1 \\ (\hat{R}(\omega t)^{-1}x)_2 \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and put $L = x_1 p_2 - x_2 p_1$, which is called the angular momentum. Then $e^{-i\omega t L}$ can be represented as

$$(e^{-i\omega t L}\phi)(x) = \phi(\hat{R}(\omega t)^{-1}x)$$

(see e.g. Enss-Kostrykin-Schrader [11]). Let $\Psi(t,x)$ be a solution of the Schrödinger equation

$$i\partial_t\Psi(t) = H(t)\Psi(t), \quad H(t) = H_{0,L} - qE_{-\omega,\theta}(t)\cdot x + V(x).$$

For such a $\Psi(t,x)$, put

$$\Phi(t,x) = (e^{-i\omega tL}\Psi(t))(x) = \Psi(t,\hat{R}(\omega t)^{-1}x).$$

Then one can see that $\Phi(t,x)$ satisfies the Schrödinger equation

$$i\partial_t\Phi(t) = \hat{H}(t)\Phi(t), \quad \hat{H}(t) = \omega L + e^{-i\omega tL}H(t)e^{i\omega tL}.$$

By a straightforward calculation, we have

$$\begin{aligned}\hat{H}(t) &= \omega L + H_{0,L} - qE_{-\omega,\theta}(t)\cdot(\hat{R}(\omega t)^{-1}x) + V(\hat{R}(\omega t)^{-1}x)\\ &= (p+qA(x))^2/(2m) - qE_{0,\theta}(t)\cdot x + V(\hat{R}(\omega t)^{-1}x)\\ &= (p+qA(x))^2/(2m) - qE_0(\cos\theta,\sin\theta)\cdot x + V(\hat{R}(\omega t)^{-1}x)\\ &= \hat{H}_0 + V(\hat{R}(\omega t)^{-1}x).\end{aligned}$$

Here we used

$$H_{0,L} = p^2/(2m) + m\omega^2x^2/8 - \omega L/2.$$

Hence we see that the problem under consideration can be reduced to the one in the case where $\nu = 0$, the magnetic field is given by $-\boldsymbol{B}$, and the potential is given as the rotating potential $V(\hat{R}(\omega t)^{-1}x)$, which is periodic in time. In particular, in the case where the regular short-range potential V is radial, that is, V depends on $|x|$ only, the asymptotic completeness can be guaranteed by virtue of Theorem 1.3, because $V(\hat{R}(\omega t)^{-1}x) \equiv V(x)$.

In the same way as above, the scattering problems for the Hamiltonian perturbed by the rotating potential $V(\hat{R}(\omega t)x)$

$$\tilde{H}(t) = H_{0,L} - qE_{-\omega,\theta}(t)\cdot x + V(\hat{R}(\omega t)x)$$

can be reduced to the ones for the time-independent Hamiltonian

$$\hat{H} = \hat{H}_0 + V(x).$$

Then the asymptotic completeness can be guaranteed by virtue of Theorem 1.3, even if the regular short-range potential V is not radial.

2 Avron-Herbst type formula

We first give the differential equations which $a(t)$, $b(t)$ and $c(t)$ in (1.9) should satisfy with the initial conditions $a(0) = 0$ and $b(0) = c(0) = 0$, by formal observation: Suppose that (1.9) holds. By differentiating (1.9) in t formally, one can obtain

$$i\dot{U}_0(t,0) = e^{-ia(t)}e^{ib(t)\cdot x}T(c(t))H_{0,L}e^{-itH_{0,L}}$$

$$+\, e^{-ia(t)}e^{ib(t)\cdot x}e^{-ic(t)\cdot qA(x)}(\dot{c}(t)\cdot p)e^{-ic(t)\cdot p}e^{-itH_{0,L}}$$
$$+\,(\dot{a}(t)-\dot{b}(t)\cdot x+\dot{c}(t)\cdot qA(x))U_0(t,0).$$

Here we note that $H_{0,L} = D^2/(2m)$ commutes with $T(c(t))$ since the magnetic translation $T(c(t))$ is generated by the pseudomomentum k which commutes with D as mentioned before, and that $e^{-ic(t)\cdot qA(x)}pe^{ic(t)\cdot qA(x)} = p - qA(c(t))$ since $c(t)\cdot qA(x) = -qA(c(t))\cdot x$. Thus one has

$$\begin{aligned}
H_0(t) &= (p-b(t)-qA(x))^2/(2m) + \dot{c}(t)\cdot(p-b(t)-qA(c(t)))\\
&\quad + \dot{a}(t) - \dot{b}(t)\cdot x + \dot{c}(t)\cdot qA(x)\\
&= H_{0,L} + (-b(t)/m + \dot{c}(t))\cdot(p-qA(x)) - (\dot{b}(t)+2qA(\dot{c}(t)))\cdot x\\
&\quad + \dot{a}(t) - \dot{c}(t)\cdot(b(t)+qA(c(t))) + b(t)^2/(2m)
\end{aligned}$$

since $i\dot{U}_0(t,0) = H_0(t)U_0(t,0)$ and $\dot{c}(t)\cdot qA(x) = -qA(\dot{c}(t))\cdot x$. It follows from this that

$$\begin{aligned}
&-b(t)/m + \dot{c}(t) = 0, \quad \dot{b}(t) + 2qA(\dot{c}(t)) = qE(t),\\
&\dot{a}(t) - \dot{c}(t)\cdot(b(t)+qA(c(t))) + b(t)^2/(2m) = 0.
\end{aligned}$$

Thus one obtain the differential equations

$$\begin{cases}
\dot{b}(t) + 2qA(b(t))/m = qE(t),\\
\dot{c}(t) = b(t)/m,\\
\dot{a}(t) = b(t)^2/(2m) + b(t)\cdot qA(c(t))/m,
\end{cases} \tag{2.1}$$

for $a(t)$, $b(t)$ and $c(t)$. The first equation of (2.1) is written as

$$\frac{d}{dt}\begin{pmatrix} b_1(t)\\ b_2(t)\end{pmatrix} + \begin{pmatrix} 0 & -\omega\\ \omega & 0\end{pmatrix}\begin{pmatrix} b_1(t)\\ b_2(t)\end{pmatrix} = \begin{pmatrix} qE_1(t)\\ qE_2(t)\end{pmatrix} \tag{2.2}$$

with $\omega = qB/m$. Thus, by putting

$$\begin{pmatrix} \tilde{b}_1(t)\\ \tilde{b}_2(t)\end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix} b_1(t)\\ b_2(t)\end{pmatrix},$$

the equation (2.2) can be reduced to

$$\frac{d}{dt}\begin{pmatrix} \tilde{b}_1(t)\\ \tilde{b}_2(t)\end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix}\begin{pmatrix} qE_1(t)\\ qE_2(t)\end{pmatrix} \tag{2.3}$$

as is well known. Therefore the solution of (2.1) with the initial conditions $a(0) = 0$ and $b(0) = c(0) = 0$ is given by (1.10). This fact yields Theorem 1.1. As for the detailed proof, see [1].

Remark 2.1. Recently Asai [2] has used the Avron-Herbst type formula in Theorem 1.1 in the study of the existence of the wave operators in the case where $E(t)$ is given by

$$E(t) = E_0(1+|t|)^{-\mu}(\cos(\nu t+\theta), \sin(\nu t+\theta)) + \bar{E}(t),$$

where $0 < \mu < 1$, $\nu \in \{0, -\omega\}$, and $\bar{E}(t) = (\bar{E}_1(t), \bar{E}_2(t))$ satisfies

$$\left| \int_0^t \begin{pmatrix} 1-\cos\omega(t-s) & -\sin\omega(t-s) \\ \sin\omega(t-s) & 1-\cos\omega(t-s) \end{pmatrix} \begin{pmatrix} \bar{\alpha}_1(s) \\ \bar{\alpha}_2(s) \end{pmatrix} ds \right| \le C_{\bar{E}} \min\{|t|, |t|^{1-\mu_1}\} \tag{2.4}$$

with some μ_1 such that $\mu < \mu_1 \le 1$, where $\bar{\alpha}(t) = (\bar{\alpha}_1(t), \bar{\alpha}_2(t)) = (\bar{E}_2(t)/B, -\bar{E}_1(t)/B)$. Then, by virtue of (1.15), one can see that $|c(t)|$ is growing of order $|t|^{1-\mu}$, which implies that the potential $V(x)$ satisfying $|V(x)| \le C\langle x\rangle^{-\rho}$ with $\rho > 1/(1-\mu)$ is of short-range. One of the typical examples of such $\bar{E}(t)$'s is the one satisfying $|\bar{E}(t)| \le C(1+|t|)^{-\mu_2}$ with $\mu_2 > \mu$. However, $\bar{E}(t) = E_{\nu,\theta}(t)$ with $\nu \in \boldsymbol{R} \setminus \{0, -\omega\}$ also satisfies (2.4) with $\mu_1 = 1$ as is seen above, which implies that the "perturbation" term $\bar{E}(t)$ is not necessarily decaying faster than the "leading" term $E_0(1+|t|)^{-\mu}(\cos(\nu t+\theta), \sin(\nu t+\theta))$ of $E(t)$.

3 Existence of wave operators

In the present and next sections, we sometimes use the following convention for smooth cut-off functions F_δ with $0 \le F_\delta \le 1$ for sufficiently small $\delta > 0$: We define

$$\begin{aligned} F_\delta(s \le d) &= 1 \quad \text{for} \quad s \le d-\delta, \quad &= 0 \quad \text{for} \quad s \ge d, \\ F_\delta(s \ge d) &= 1 \quad \text{for} \quad s \ge d+\delta, \quad &= 0 \quad \text{for} \quad s \le d, \end{aligned}$$

and $F_\delta(d_1 \le s \le d_2) = F_\delta(s \ge d_1)\, F_\delta(s \le d_2)$.

Throughout this section, we suppose that $(V1)$ is satisfied, and that $E(t) = E_{\nu,\theta}(t) = E_0(\cos(\nu t+\theta), \sin(\nu t+\theta))$ with $\nu \in \{0, -\omega\}$ and $\theta \in [0, 2\pi)$. Then it follows from (1.13) and (1.14) that

$$|c(t)| \ge 9E_0|t|/(10B)$$

for $|t| \ge 20/|\omega|$, because

$$|E_0((\delta\cos)(-\omega t), (\delta\sin)(-\omega t))/(\omega B)| = 2E_0|\sin(-\omega t/2)|/(|\omega|B) \le 2E_0/(|\omega|B)$$

and $|\alpha| = E_0/B$.

The following propagation estimate for $U_0(t,0)$ is useful for the proof of Theorem 1.2.

Proposition 3.1. *Let $\phi \in \mathscr{D}((p^2+x^2)^N)$ with $N \in \boldsymbol{N}$, $\epsilon > 0$ and $\sigma > 0$. Then*

$$\|F_\epsilon(t^{-\sigma}|x-c(t)| \ge \epsilon)U_0(t,0)\phi\|_{L^2(\boldsymbol{R}^2)} = O(t^{-2N\sigma}) \tag{3.1}$$

holds as $t \to \infty$.

In the proof, we have only to use

$$U_0(t,0)^* F_\epsilon(t^{-\sigma}|x-c(t)| \geq \epsilon) U_0(t,0) = e^{itH_{0,L}} F_\epsilon(t^{-\sigma}|x| \geq \epsilon) e^{-itH_{0,L}}$$

by virtue of the Avron-Herbst type formula (1.9). As for the detailed proof, see [1].

Now we state the outline of the proof of Theorem 1.2. We first consider the case where $V^1 = 0$. By density argument, one has only to prove the existence of $W^+\phi$ for $\phi \in \mathscr{S}(\boldsymbol{R}^2)$. Let $f \in C_0^\infty(\boldsymbol{R}^2)$ be such that $0 \leq f \leq 1$, $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ for $|x| \geq 2$, and σ be such that $0 < \sigma < 1$. Put $g = 1 - f$. Then we see that

$$\lim_{t\to\infty} U(t,0)^* g(t^{-\sigma}(x-c(t))) U_0(t,0)\phi = 0 \tag{3.2}$$

by virtue of Proposition 3.1. Thus we have only to prove the existence of

$$\lim_{t\to\infty} U(t,0)^* f(t^{-\sigma}(x-c(t))) U_0(t,0)\phi. \tag{3.3}$$

Here we note that on the support of $f(t^{-\sigma}(x-c(t)))$,

$$|x| \geq |c(t)| - |x-c(t)| \geq |c(t)| - 2t^\sigma$$

holds, and that $|c(t)| \geq 9E_0 t/(10B)$ for $t \geq 20/|\omega|$ as mentioned above. Thus we see that

$$Vf(t^{-\sigma}(x-c(t))) = O(t^{-\rho_{s,0}})$$

as $t \to \infty$ by the assumption on V and $\sigma < 1$. By virtue of this and Proposition 3.1, one can obtain

$$\frac{d}{dt}(U(t,0)^* f(t^{-\sigma}(x-c(t))) U_0(t,0)\phi) = O(t^{-\rho_{s,0}}) + O(t^{-(2N+1)\sigma}).$$

By taking $N \in \boldsymbol{N}$ so large that $(2N+1)\sigma > 1$, one can show the existence of (3.3) because of $\rho_{s,0} > 1$ and $(2N+1)\sigma > 1$, by virtue of the Cook-Kuroda method.

We next consider the case where $V^1 \neq 0$. By density argument, one has only to prove the existence of $W_G^+\phi$ for $\phi \in \mathscr{S}(\boldsymbol{R}^2)$. Let σ be such that $0 < \sigma < \rho_l \leq 1$. In the same way as in the case where $V^1 = 0$, we see that

$$\lim_{t\to\infty} U(t,0)^* g(t^{-\sigma}(x-c(t))) U_0(t,0) e^{-i\int_0^t V^1(c(s))\,ds}\phi = 0 \tag{3.4}$$

by virtue of Proposition 3.1. Here we note that the modifier $e^{-i\int_0^t V^1(c(s))\,ds}$ commutes with $U_0(t,0)$. Thus we have only to prove the existence of

$$\lim_{t\to\infty} U(t,0)^* f(t^{-\sigma}(x-c(t))) U_0(t,0) e^{-i\int_0^t V^1(c(s))\,ds}\phi. \tag{3.5}$$

To this end, we will estimate $(V^1(x) - V^1(c(t))) f(t^{-\sigma}(x-c(t))) U_0(t,0) e^{-i\int_0^t V^1(c(s))\,ds}\phi$. We put $V_1(t,x) = V^1(x) g(5Bx/(2E_0 t))$. Then

$$(V^1(x) - V^1(c(t))) f(t^{-\sigma}(x-c(t))) = (V_1(t,x) - V_1(t,c(t))) f(t^{-\sigma}(x-c(t)))$$

holds for $t \geq \max\{20/|\omega|, (20B/E_0)^{1/(1-\sigma)}\}$, since $g(5Bx/(2E_0t)) = 1$ for $|x| \geq 4E_0t/(5B)$, and $|c(t)| \geq 9E_0t/(10B)$ for $t \geq 20/|\omega|$ as mentioned above. By rewriting $V_1(t,x) - V_1(t,c(t))$ as

$$V_1(t,x) - V_1(t,c(t)) = \int_0^1 (\nabla V_1)(t, c(t) + \tau(x - c(t))) \cdot (x - c(t))\, d\tau$$

and taking account of $\sup_{y \in \boldsymbol{R}^2} |(\nabla V_1)(t,y)| = O(t^{-1-\rho_1})$ by the definition of V_1 and the assumption on V^1, we have

$$(V^1(x) - V^1(c(t))) f(t^{-\sigma}(x - c(t))) U_0(t,0) e^{-i\int_0^t V^1(c(s))\,ds} \phi = O(t^{-1-\rho_1+\sigma}).$$

Therefore, in the same way as in the case where $V^1 = 0$, we obtain

$$\begin{aligned} &\frac{d}{dt}(U(t,0)^* f(t^{-\sigma}(x - c(t))) U_0(t,0) e^{-i\int_0^t V^1(c(s))\,ds} \phi) \\ &= O(t^{-\rho_{s,0}}) + O(t^{-(2N+1)\sigma}) + O(t^{-(1+\rho_1-\sigma)}) \end{aligned}$$

for any $N \in \boldsymbol{N}$. By taking $N \in \boldsymbol{N}$ so large that $(2N+1)\sigma > 1$, one can show the existence of (3.5) because of $\rho_{s,0} > 1$, $(2N+1)\sigma > 1$ and $1 + \rho_1 - \sigma > 1$, by virtue of the Cook-Kuroda method. As for the detailed proof of Theorem 1.2, see [1].

4 Asymptotic completeness

Throughout this section, we suppose that $E(t) = E_{0,\theta}(t) \equiv E_0(\cos\theta, \sin\theta)$. Then we write $E(t)$, $H_0(t)$ and $H(t)$ as

$$E = (E_1, E_2), \quad H_0 = H_{0,L} - qE \cdot x, \quad H = H_0 + V,$$

respectively, because $E(t)$, $H_0(t)$ and $H(t)$ are independent of t in this case. Since $H_0 = (D - m\alpha)^2/(2m) + \alpha \cdot k - m\alpha^2/2$ (see (1.3)) and V is H_0-compact under the assumption $(V1)$, we see that

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = \boldsymbol{R}, \quad \sigma(H) = \sigma_{\mathrm{ess}}(H) = \boldsymbol{R}$$

because of $\alpha \neq 0$, by virtue of the Weyl theorem. The following result can be obtained by virtue of the Mourre theory:

Proposition 4.1. *Suppose that $(V1)$ is satisfied. Then the pure point spectrum $\sigma_{\mathrm{pp}}(H)$ of H is at most countable, and has no accumulation point. Each eigenvalue of H has at most finite multiplicity.*

In fact, putting $\tilde{A} = qE \cdot k$, we have the Mourre estimate

$$f(H)i[H, \tilde{A}]f(H) = q^2|E|^2 f(H)^2 + K_f, \tag{4.1}$$

where $f \in C_0^\infty(\boldsymbol{R}; \boldsymbol{R})$ and $K_f = -f(H)qE \cdot (\nabla V)f(H)$, which is compact on $L^2(\boldsymbol{R}^2)$.

In obtaining some useful propagation estimates for e^{-itH}, we need the assumption $(V2)$. Here we note that $[H, \tilde{A}]$ and $[[H, \tilde{A}], \tilde{A}]$ are bounded under the assumption $(V2)$:

Proposition 4.2. *Suppose that $(V2)$ is satisfied. Let $c_0, c_1 \in \boldsymbol{R}$ be such that $c_0 < c_1 < q^2|E|^2$, and let $\epsilon > 0$. Then for any real-valued $f \in C_0^\infty(\boldsymbol{R} \setminus \sigma_{\rm pp}(H))$, there exists $C > 0$ such that*

$$\int_1^\infty \|F_\epsilon(c_0 \le \tilde{A}/t \le c_1)f(H)e^{-itH}\psi\|^2_{L^2(\boldsymbol{R}^2)} \frac{dt}{t} \le C\|\psi\|^2_{L^2(\boldsymbol{R}^2)} \tag{4.2}$$

for any $\psi \in L^2(\boldsymbol{R}^2)$. Moreover,

$$\int_1^\infty \|F_\epsilon(\tilde{A}/t \le c_1)f(H)e^{-itH}\psi\|^2_{L^2(\boldsymbol{R}^2)} \frac{dt}{t} < \infty \tag{4.3}$$

for any $\psi \in \mathscr{D}(\langle \tilde{A} \rangle^{1/2})$.

Proposition 4.3. *Suppose that $(V2)$ is satisfied. Let $c_1 \in \boldsymbol{R}$ be such that $c_1 < q^2|E|^2$, and let $\epsilon > 0$. Then for any real-valued $f \in C_0^\infty(\boldsymbol{R} \setminus \sigma_{\rm pp}(H))$,*

$$\operatorname*{s-lim}_{t\to\infty} F_\epsilon(\tilde{A}/t \le c_1)f(H)e^{-itH} = 0 \tag{4.4}$$

holds.

These can be shown in the same way as in Sigal-Soffer [20].

Taking account of

$$qE \cdot (k - D) = 2q^2 E \cdot A(x) = -2q^2 A(E) \cdot x = q^2 B^2 \alpha \cdot x,$$

we have

$$\begin{aligned} &\{F_\epsilon(c_0 \le \tilde{A}/t \le c_1) - F_\epsilon(c_0 \le q^2B^2\alpha \cdot x/t \le c_1)\}f(H) = O(t^{-1}), \\ &\{F_\epsilon(\tilde{A}/t \le c_1) - F_\epsilon(q^2B^2\alpha \cdot x/t \le c_1)\}f(H) = O(t^{-1}). \end{aligned} \tag{4.5}$$

Hence the next proposition follows from (4.5), Propositions 4.2 and 4.3 immediately:

Proposition 4.4. *Suppose that $(V2)$ is satisfied. Let $c_0, c_1 \in \boldsymbol{R}$ be such that $c_0 < c_1 < q^2|E|^2$, and let $\epsilon > 0$. Then for any real-valued $f \in C_0^\infty(\boldsymbol{R} \setminus \sigma_{\rm pp}(H))$, there exists $C > 0$ such that*

$$\int_1^\infty \|F_\epsilon(c_0 \le q^2B^2\alpha \cdot x/t \le c_1)f(H)e^{-itH}\psi\|^2_{L^2(\boldsymbol{R}^2)} \frac{dt}{t} \le C\|\psi\|^2_{L^2(\boldsymbol{R}^2)} \tag{4.6}$$

for any $\psi \in L^2(\boldsymbol{R}^2)$. Moreover,

$$\operatorname*{s-lim}_{t\to\infty} F_\epsilon(q^2B^2\alpha \cdot x/t \le c_1)f(H)e^{-itH} = 0 \tag{4.7}$$

holds.

Now we will state the outline of the proof of Theorem 1.3: We put $\varepsilon = |\alpha|/10 = |E|/(10B)$ and $\hat{\alpha} = \alpha/|\alpha|$. Since $|c(t) - t\alpha| \le 2|E|/(|\omega|B)$ (see §1), we see that $\hat{\alpha} \cdot t\alpha/t = |\alpha| = 10\varepsilon$ and

$$\hat{\alpha} \cdot c(t)/t \ge |\alpha| - 2|E|/(|\omega|Bt) \ge 9\varepsilon \tag{4.8}$$

for $t \ge 20/|\omega|$, which is important for understanding the behavior of the charged particle.

Here we note that besides $(V2)$, the short-range condition $V^1 = 0$ is assumed in Theorem 1.3. As is well known, one has only to prove the existence of

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_0} e^{-itH} P_c(H), \tag{4.9}$$

where $P_c(H)$ is the spectral projection onto the continuous spectral subspace $L^2_c(H)$ of the Hamiltonian H. To this end, we will show the existence of

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_0} f(H) e^{-itH} \tag{4.10}$$

for any real-valued $f \in C_0^\infty(\boldsymbol{R} \setminus \sigma_{pp}(H))$. By virtue of (4.7), we have

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_0} F_\varepsilon(\hat{\alpha} \cdot x/t \le 8\varepsilon) f(H) e^{-itH} = 0. \tag{4.11}$$

Taking account of that $1 - F_\varepsilon(\hat{\alpha} \cdot x/t \le 8\varepsilon)$ may be written as $F_\varepsilon(\hat{\alpha} \cdot x/t \ge 7\varepsilon)$ by definition, we have only to prove the existence of

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_0} F_\varepsilon(\hat{\alpha} \cdot x/t \ge 7\varepsilon) f(H) e^{-itH}. \tag{4.12}$$

By taking $f_1 \in C_0^\infty(\boldsymbol{R})$ such that $f_1(s)f(s) = f(s)$, one has only to show the existence of

$$\operatorname*{s-lim}_{t\to\infty} e^{itH_0} f_1(H_0) F_\varepsilon(\hat{\alpha} \cdot x/t \ge 7\varepsilon) f(H) e^{-itH}, \tag{4.13}$$

which can be proved by Proposition 4.4 and

$$V^s(x) F_\varepsilon(\hat{\alpha} \cdot x/t \ge 7\varepsilon) = O(t^{-\rho_{s,0}}) \tag{4.14}$$

with $\rho_{s,0} > 1$. This yields the asymptotic completeness of W^+.

In dealing with the long-range case, one needs the propagation estimates for e^{-itH} analogous to Proposition 3.1, which is much sharper than Proposition 4.4. One of the keys in the proof of Theorem 1.2 is that σ in Proposition 3.1 can be taken as $0 < \sigma < \rho_l \le 1$. Unfortunately such sharp estimates have not been obtained for e^{-itH} yet.

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