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Kyoto University
The behavior of the interfaces in the fast reaction limits of some reaction-diffusion systems with unbalanced interactions

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1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. Hilhorst-Hout-Peletier [2, 3] investigated a simple reaction-diffusion system with a huge positive parameter $k$

\[
\begin{aligned}
&u_t = \Delta u - k uw & \text{in } \Omega, \\
&w_t = -k uw & \text{in } \Omega
\end{aligned}
\] (1)

which describes a "fast reaction" between a diffusive reactant $u$ and a non-diffusive one $w$. Assuming that the initial values of $u$ and $w$ are non-negative and fixing a positive number $T$, they derived the singular limit as $k \to \infty$ of an initial-boundary value problem in $\Omega \times (0, T)$ for a class of reaction-diffusion systems with a parameter $k$ such as (1). Their results are summarized as follows: the solution $(u_k, w_k)$ of their initial-boundary value problem possesses its singular limit $(u_*, w_*)$ as $k \to \infty$ such that $u_* w_* \equiv 0$; therefore, when we use the notation

\[
\begin{aligned}
&\Omega^u(t) = \{x \in \Omega | u_*(x, t) > 0\}, \\
&\Omega^w(t) = \text{Int} \overline{\{x \in \Omega | w_*(x, t) > 0\}}, \\
&\Gamma(t) = \Omega \setminus (\Omega^u(t) \cup \Omega^w(t))
\end{aligned}
\] (2)

the region $\Omega^u(t)$ and the region $\Omega^w(t)$ are divided by an "interface" $\Gamma(t)$; moreover $u_*$ satisfies the one-phase Stefan problem

\[
\begin{aligned}
&u_*,t = \Delta u_* & \text{in } \Omega^u(t), \\
&w_*|_{\Gamma(t) + 0n} = -\frac{\partial u_*}{\partial n}|_{\Gamma(t) - 0n}, \\
&u_*|_{\Gamma(t)} = 0
\end{aligned}
\] (3)

in a weak sense. Here $n$ is the unit normal vector to $\Gamma(t)$ oriented from $\Omega^u(t)$ to $\Omega^w(t)$, and $V_n$ is the velocity of $\Gamma(t)$ in the direction of $n$.

In this article we consider generalized "fast reactions" between $u$ and $w$:

\[
\begin{aligned}
&u_t = \Delta u - k u^{m_1}w^{m_3} & \text{in } \Omega, \\
&w_t = -k u^{m_2}w^{m_4} & \text{in } \Omega
\end{aligned}
\] (4)

where $m_j \geq 1$ ($j = 1, 2, 3, 4$). We are particularly interested in the situations where $(m_1, m_3) \neq (m_2, m_4)$, while Hilhorst-Hout-Peletier [2, 3] investigated situations where $(m_1, m_3) = (m_2, m_4)$. Even in the situations where $(m_1, m_3) \neq (m_2, m_4)$ the corresponding
singular limit \((u_*, w_*)\) of \((u_k, w_k)\) as \(k \to \infty\), if it exists, must formally satisfies \(u_* w_* \equiv 0\). However, the rapid dynamics of \((4)\) in such situations are very different from that in the situations where \((m_1, m_3) = (m_2, m_4)\). The rapid dynamics of \((4)\) is essentially determined by the two-dimensional dynamical system

\[
\begin{align*}
  u_t &= -u^{m_1} w^{m_3}, \\
  w_t &= -u^{m_2} w^{m_4}.
\end{align*}
\]

Note that all the trajectories of \((5)\) are straight and that the trajectories toward the axis \(u = 0\) intersect it slantwise if \((m_1, m_3) = (m_2, m_4)\). If \((m_1, m_3) \neq (m_2, m_4)\), then the trajectories toward the axis \(u = 0\) intersect vertically in some situations; those trajectories touch the axis \(u = 0\) tangentially in other situations; in some situations among the other ones no trajectories possess intersections with the axis \(u = 0\). When \((m_1, m_3) \neq (m_2, m_4)\), these various structures of the trajectories in \((5)\) may cause any different behavior of the interface \(\Gamma(t)\) in the singular limit of \((4)\). Related problems were investigated in [6] from the aspect of numerical simulation (see also [4]).

As the first attempt to solve the behavior of the interface \(\Gamma(t)\) in the situations where \((m_1, m_3) \neq (m_2, m_4)\), we will investigate typical four cases of such "unbalanced interactions" between \(u\) and \(w\): \((m_1, m_2, m_3, m_4) = (1, 1, 1, m),\ (1, 1, m, 1),\ (1, m, 1, 1)\) and \((m, 1, 1, 1)\), where \(m\) is a constant larger than 1. In each case we would like to reveal the interfacial dynamics in the fast reaction limit of \((4)\) as \(k \to \infty\). Hereafter we denote \(\Omega \times (0, T)\) by \(Q_T\) and consider \((4)\) under the initial condition

\[
u |_t = u_0, \quad w|_t = w_0 \quad \text{in } \Omega
\]

and a boundary condition

\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\]

where \(\nu\) denotes the unit outer normal vector of \(\partial \Omega\).

2 Singular limits in Case \((m_1, m_2, m_3, m_4) = (1, 1, 1, m)\) or \((1, 1, m, 1)\): moving interfaces

In these cases we can respectively reduce \((4)\) into a reaction-diffusion system with a "balanced interaction"; namely into a system with \((m_1, m_3) = (m_2, m_4)\) by some transformations of variables. When \((m_1, m_2, m_3, m_4) = (1, 1, 1, m)\) with \(1 \leq m < 2\), we put \(W_k = w_k^{2-m}\) for any solution \((u_k, w_k)\) to \((4)\). Then \((u_k, W_k)\) becomes a solution to

\[
\begin{align*}
  u_t &= \Delta u - ku W^{1/(2-m)} \quad \text{in } \Omega, \\
  W_t &= -(2-m)ku W^{1/(2-m)} \quad \text{in } \Omega.
\end{align*}
\]

The singular limits of \((8)\) with appropriate initial-boundary conditions were studied by Hilhorst, Hout and Peletier [2, 3]. They showed that \(u_*\) of the singular limit \((u_*, W_*) =

---

**Note:** The above text is a simplified representation of the original content. For a more accurate and detailed understanding, please refer to the original academic source.
$$\lim_{k \to \infty} (u_k, W_k)$$ satisfies a one-phase Stefan problem with a finite normal velocity of the interface. In the same manner as the proofs in [2, 3], we can derive the singular limit of (8) with an initial condition

$$u|_{t=0} = u_0, \quad W|_{t=0} = w_0^{2-m} \quad \text{in } \Omega$$

(9) and a boundary condition (7).

Throughout this section, we impose the following assumption on the initial datum $(u_0, w_0)$:

(H1) $(u_0, w_0) \in C(\overline{\Omega}) \times L^\infty(\Omega)$, $w_0$ is continuous in $\text{supp}\ w_0$ and there exist positive constants $M$ and $m_w$ such that

$$u_0w_0 = 0, \quad 0 \leq u_0, \quad w_0 \leq M \quad \text{in } \Omega,$$

$$m_w \leq w_0 \quad \text{in } \text{supp}\ w_0.$$ (10)

Under the assumption (H1), there exists a unique solution $(u_k, W_k)$ of the initial-boundary value problem (8), (9) and (7) satisfying

$$u_k \in C([0, T]; C(\overline{\Omega})) \cap C^1((0, T]; C(\overline{\Omega}) \cap C((0, T]; W^{2,p}(\Omega)) \quad \forall p > 1),$$

$$w_k \in C^1([0, T]; L^\infty(\Omega))$$

(10) (see [1]). We obtain the following theorem in the same manner as the proofs in [2, 3].

**Theorem 2.1 (Hilhorst, Hout and Peletier [2, 3])** Let $(u_k, W_k)$ be the solution of (8) under the initial and boundary conditions (9) and (7), where $1 \leq m < 2$. Then there exist subsequences $\{u_{k_n}\}$, $\{W_{k_n}\}$ and functions $(u_*, W_*) \in L^2(0, T; H^1(\Omega)) \times L^2(Q_T)$ such that

$$u_{k_n} \to u_* \quad \text{strongly in } L^2(\Omega_T) \text{ and weakly in } L^2(0, T; H^1(\Omega)),$$

$$W_{k_n} \to W_* \quad \text{strongly in } L^2(\Omega_T),$$

as $k_n$ tends to infinity, where

$$u_*W_* = 0, \quad u_* \geq 0, \quad W_* \geq 0 \quad \text{a.e. in } Q_T.$$

Moreover, $u_*$ and $W_*$ satisfy

$$\int_{\Omega_T} \left\{ - (u_* - \lambda W_*) \zeta_t + \nabla u_* \cdot \nabla \zeta \right\} dx dt = \int_{\Omega} (u_0 - \lambda w_0^{2-m}) \zeta(\cdot, 0) dx$$

(11)

for all functions $\zeta \in C^\infty(\Omega_T)$ such that $\zeta(x, T) = 0$, where $\lambda = 1/(2 - m)$.

Since $u_*W_* \equiv 0$, we can rewrite (11) as a classical one-phase Stefan problem with a finite propagation speed. Here we use $\Omega^u(t), \Omega^w(t)$ and $\Gamma(t)$ defined by (2) where $w_* = W_*^{1/(2-m)}$ with $1 \leq m < 2$. Also we use the following notation:

$$Q_T^u = \bigcup_{0 < t < T} \Omega^u(t) \times \{t\}, \quad Q_T^w = \bigcup_{0 < t < T} \Omega^w(t) \times \{t\}, \quad \Gamma = \bigcup_{0 < t < T} \Gamma(t) \times \{t\}.$$ (12)
Theorem 2.2 Set \((m_1, m_2, m_3, m_4) = (1, 1, 1, m)\) where \(1 \leq m < 2\). Let \((u_k, w_k)\) be the solution of (4) under the initial-boundary conditions (6)-(7) and set \(W_k = w_k^{2-m}\). Namely \((u_k, W_k)\) is the solution of (8) satisfying (9) and (7). Let \((u_*, W_*)\) be the limit given in Theorem 2.1 and set \(w_* = W_*^{1/(2-m)}\). Suppose that \(\Gamma(t)\) is a smooth, closed and orientable hypersurface satisfying \(\Gamma(t) \cap \partial \Omega = \emptyset\) for all \(t \in [0, T]\). Also assume that \(\Gamma(t)\) smoothly moves with a normal velocity \(V_n\) from \(\Omega^u(t)\) to \(\Omega^w(t)\), and \(u_*\) is continuous in QT and smooth on \(\overline{Q_T^w}\), and \(w_*\) is smooth on \(\overline{Q_T^w}\). Then the following relations hold.

\[
w_*(t) = w_0, \quad \text{in } Q_T^w.
\]

\[
\left\{ \begin{array}{ll}
u_* = \Delta u_* & \quad \text{in } Q_T^u, \\
u_* = 0, & \quad \text{on } \Gamma, \\
_{\partial u_*} = 0 & \quad \text{on } \partial \Omega \times (0, T), \\
u_* = u_0 & \quad \text{on } \Omega^u(0) \times \{0\}.
\end{array} \right.
\]

Taking the fast reaction limit of (13) under the boundary condition (7) and an initial condition

\[
u|_{t=0} = u_0, \quad W|_{t=0} = w_0^m \quad \text{in } \Omega,
\]
we can similarly derive the same conclusions as those of Theorem 2.1 where \(\lambda = 1/m\). Thus we obtain the following theorem. Here we use the notation \(\Omega^u(t), \Omega^w(t), \Gamma(t), Q_T^u, Q_T^w\) and \(\Gamma\) defined by (2) and (12) where \(w_* = W_*^{1/m}\) with \(m \geq 1\).

Theorem 2.3 Set \((m_1, m_2, m_3, m_4) = (1, 1, m, 1)\) where \(m \geq 1\). Let \((u_k, w_k)\) be the solution of (4) under the initial-boundary conditions (6)-(7) and set \(W_k = w_k^m\). Namely \((u_k, W_k)\) is the solution of (13) satisfying (14) and (7). Set \(w_* = W_*^{1/m}\) for the limit \((u_*, W_*)\) given in Theorem 2.1 where (8), (9) and (11) are replaced by (13), (14) and

\[
\int_{Q_T} \{-(w_* - \lambda W_* )\zeta_t + \nabla u_* \cdot \nabla \zeta \} dx dt = \int_{\Omega} (u_0 - \lambda w_0^m) \zeta(\cdot, 0) dx
\]

with \(\lambda = 1/m\), respectively. Suppose that \(\Gamma(t)\) is a smooth, closed and orientable hypersurface satisfying \(\Gamma(t) \cap \partial \Omega = \emptyset\) for all \(t \in [0, T]\). Also assume that \(\Gamma(t)\) smoothly moves with a normal velocity \(V_n\) from \(\Omega^u(t)\) to \(\Omega^w(t)\), and \(u_*\) is continuous in QT and smooth
on $\overline{Q_T^u}$, and $w_*$ is smooth on $\overline{Q_T^w}$. Then the following relations hold.

$$w_*(t) = w_0, \quad \text{in } Q_T^w;$$

$$\begin{cases}
    u_{*,t} = \Delta u_* & \text{in } Q_T^u, \\
    u_0 = u_0 & \text{on } \Gamma, \\
    \frac{\partial u_*}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T), \\
    u_* = u_0 & \text{on } \Omega^u(0) \times \{0\}.
\end{cases}$$

3 Singular limits in Case $(m_1, m_2, m_3, m_4) = (1, m, 1, 1)$: immovable interfaces

A free boundary appears in the fast reaction limit also in this case; however, this free boundary does not move.

Throughout this section, we impose (H1) on the initial datum $(u_0, w_0)$ again, and assume $m > 1$. Under the assumption (H1), there exists a unique solution $(u_k, w_k)$ of the initial-boundary value problem (4), (6) and (7) satisfying (10).

We give a result on the convergence of $(u_k, w_k)$.

**Theorem 3.1** Set $(m_1, m_2, m_3, m_4) = (1, m, 1, 1)$ where $m > 1$. Let $(u_k, w_k)$ be the solution of (4) under the initial and boundary conditions (6) and (7). Then there exist subsequences $\{u_{k_n}\}$ and $\{w_{k_n}\}$ of $\{u_k\}$ and $\{w_k\}$, respectively, and functions $u_*, w_*$ and a distribution $U_*$ such that

$$\begin{align*}
    u_*, u_*^\frac{m}{2} &\in L^\infty(Q_T) \cap L^2(0, T; H^1(\Omega)), \ w_* \in L^\infty(Q_T), \ U_* \in H^{-1}(Q_T), \\
    0 \leq u_*, w_* &\leq M, \ u_* w_* = 0 \ \text{a.e. in } Q_T, \ U_* \geq 0 \ \text{in } H^{-1}(Q_T), \\
    u_{k_n} &\to u_* \ \text{strongly in } L^p(Q_T)(\forall p \geq 1), \ \text{a.e. in } Q, \\
    w_{k_n} &\to w_* \ \text{weakly in } L^p(0, T; H^1(\Omega)) \text{ and weakly } \ast \text{ in } L^\infty(Q_T), \\
    \left|\nabla u_{k_n}^\frac{m}{2}\right|^2 &\to U_* \ \text{weakly in } H^{-1}(Q_T)
\end{align*}$$

as $k_n$ tends to infinity. Moreover $u_*, w_*$ and $U_*$ satisfy

$$\int_Q \left\{ -\left(\frac{1}{m_1} u_*^m - w_*\right) \zeta_t + \frac{2}{m} u_*^\frac{m}{2} \nabla u_* \cdot \nabla \zeta \right\} dx \, dt + \frac{4(m-1)}{m^2} \left\langle U_*, \zeta \right\rangle_{H^1_0(Q_T)} = 0$$

for all $\zeta \in H^1_0(Q_T)$.

We can prove $U_* = \left|\nabla u_*^\frac{m}{2}\right|^2 \in L^1(Q_T)$ under additional conditions. Here we use the notation $\Omega^u(t), \Omega^w(t), \Gamma(t), Q_T^u, Q_T^w$ and $\Gamma$ defined by (2) and (12). Then we can give an explicit equation of motion for the free boundary.
Theorem 3.2 Let \( u_*, w_*, U_* \) be the functions satisfying (16)-(20). Suppose that \( \Gamma(t) \) is a smooth, closed and orientable hypersurface satisfying \( \Gamma(t) \cap \partial \Omega = \emptyset \) for all \( t \in [0, T] \). Also assume that \( \Gamma(t) \) smoothly moves with a normal velocity \( V_n \) from \( \Omega^u(t) \) to \( \Omega^w(t) \), and \( u_* \) is continuous in \( Q_T \) and smooth on \( \overline{Q_T^u} \), and \( w_* \) is smooth on \( \overline{Q_T^w} \). Then the following relations hold.

\[
V_n = 0 \text{ on } \Gamma, \quad \text{that is, } \quad \Omega^u(t) \equiv \Omega^u(0), \quad \Omega^w(t) \equiv \Omega^w(0), \quad \Gamma(t) \equiv \Gamma(0);
\]

\[
w_*(t) = w_0, \quad U_* = |\nabla u^{\frac{m}{2}}|^2 \quad \text{in } Q_T;
\]

\[
\begin{cases}
  u_{*,t} = \Delta u_* & \text{in } Q_T^u = \Omega^u(0) \times (0, T), \\
  u_* = 0 & \text{on } \Gamma = \Gamma(0) \times (0, T), \\
  \frac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T), \\
  u_* = u_0 & \text{on } \Omega^u(0) \times \{0\}.
\end{cases}
\]

See [5] for the proofs of Theorems 3.1 and 3.2.

4 Singular limits in Case \( (m_1, m_2, m_3, m_4) = (m, 1, 1, 1) \): vanishing interfaces

In this case the non-diffusive reactant \( w \) consumes much faster than diffusive one \( u \) in the limit as \( k \to \infty \). This fact makes the propagation speed of \( \Gamma(t) \) too rapid. At least if \( m > 2 \), then \( \Omega^u(t) \) spread too rapidly for us to follow its boundary \( \Gamma(t) \): actually we cannot observe any free boundary.

Throughout this section, we impose the following assumptions on the initial data:

(H2) \((u_0, w_0) \in C^2(\overline{\Omega}) \times C^\alpha(\overline{\Omega})\) satisfy

\[
u_0(x)w_0(x) = 0, \quad 0 \leq u_0(x) \leq M_u, \quad 0 \leq w_0(x) \leq M_w
\]

for any \( x \in \Omega \), where \( \alpha \in (0, 1) \) represents a Hölder exponent and

\[
M_u := \max_{x \in \Omega} |u_0|, \quad M_w := \max_{x \in \Omega} |w_0|.
\]

(H3) \( u_0 \) holds the homogeneous Neumann boundary condition:

\[
\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial \Omega.
\]

We can derive the following result on the singular limit of (4) (see [5]).

Theorem 4.1 Set \( (m_1, m_2, m_3, m_4) = (m, 1, 1, 1) \) where \( m \geq 1 \). Let \((u_k, w_k)\) be the solution of (4) under the initial and boundary conditions (6) and (7). Then

\[
\begin{align*}
  u_k & \to u_* \quad \text{in } C^0(\overline{Q_T}) \quad \text{as } k \to \infty, \\
  w_k & \to 0 \quad \text{in } C^0(\Omega \times [\epsilon, T]) \quad \text{as } k \to \infty \quad \text{for any } \epsilon \in (0, T),
\end{align*}
\]
where $u_*(x, t)$ belongs to $C^{2,1}(Q_T)$ and satisfies the heat equation in the whole domain as follows:

$$\begin{align*}
    u_{*,t} &= \Delta u_* \quad \text{in } Q_T, \\
    \frac{\partial u_*}{\partial \nu} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
    u_* &= u_0 \quad \text{on } \Omega \times \{0\}.
\end{align*}$$

References


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