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A subdifferential approach to evolution equations in variable exponent Lebesgue spaces

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Abstract

In this résumé, we review recent results reported in the paper [1], where a subdifferential approach to doubly nonlinear parabolic equations involving variable exponents is proposed.

This note is based on a joint work with Giulio Schimperna (Pavia University, Italy).

1 Introduction

Doubly nonlinear parabolic equations have been vigorously studied so far, as they appear in various fields such as phase transition, damage mechanics and fluid dynamics. A typical example is as follows:

$$\beta(\partial_t u) - \Delta u = f \quad \text{in } \Omega \times (0,T) \tag{1}$$

with a maximal monotone graph $\beta : \mathbb{R} \to \mathbb{R}$, a domain Ω of $\mathbb{R}^N$, $T > 0$, and a given function $f = f(x,t) : \Omega \times (0,T) \to \mathbb{R}$. The linear Laplacian Δ is often replaced with a nonlinear variant, e.g., the so-called *m-Laplacian* Δ_m given by

$$\Delta_m u = \operatorname{div}\ \left(|\nabla u|^{m-2}\nabla u\right), \quad 1 < m < \infty,$$

and then, (1) is truly a *doubly nonlinear* parabolic equation. The doubly nonlinear parabolic equation is also classified as a fully nonlinear equation, and due to the severe nonlinearity, it is somewhat delicate which space is chosen as a *base space*, a function space in which the equation is mainly treated throughout analysis, so as to apply an energy method in an effective way.

By setting $u(t) := u(\cdot,t)$, such a nonlinear parabolic equation is interpreted as an abstract evolution equation,

$$A(u'(t)) + B(u(t)) = f(t) \quad \text{in } X, \quad 0 < t < T \tag{2}$$

with unknown function $u : (0,T) \to X$, two (possibly nonlinear) operators A, B in a proper function space X and $f : (0,T) \to X$. Equation (2) is often called a *doubly nonlinear evolution equation.*

In most of studies on nonlinear evolution equations, existence and regularity results are usually established in a proper class of vector-valued functions, e.g., a Lebesgue-Bochner space,

$$L^p(0,T;X) := \left\{ u : (0,T) \to X\text{: "strongly measurable" and } \int_0^T \|u(t)\|_X^p dt < \infty \right\}.$$

Here "strong measurability" of u means that there exists a sequence of simple functions $u_n : (0,T) \to X$ such that $u_n(t) \to u(t)$ strongly in X for a.e. $t \in (0,T)$.

Now, let us recall several results on the existence and regularity of solutions for the doubly nonlinear evolution equation (2) in Bochner space frameworks. Barbu [3], Arai [2] and Senba [12] obtained existence results based on the Hilbert space, $L^2(0,T;H)$. The method of their proofs relies on the time differentiation of the equation, which transforms the equation into another (more tractable) type of doubly nonlinear equations, and a peculiar monotonicity condition called an *A-monotonicity* for B, i.e.,

$$(Bu - Bv, A_\lambda(u-v))_H \geq 0 \quad \text{for all } u, v \in D(B) \text{ and } \lambda > 0,$$

where A_λ denotes the Yosida approximation of A. Colli-Visintin [7] and Colli [6] also treated (2) in the Hilbert space $L^2(0,T;H)$ and in the reflexive Banach space $L^p(0,T;V)$ with $1 < p < \infty$, respectively. However, their approach is totally different from the former one, and instead of differentiating the equation and assuming the A-monotonicity of B, they impose a p-power growth condition,

$$c_0\|u\|_V^p \leq \langle Au, u\rangle_V + C, \quad \|Au\|_{V^*}^{p'} \leq C(\|u\|_V^p + 1) \quad \text{for } u \in V,$$

on the operator $A : V \to V^*$ defined on a Banach space V and its dual space V^* (in the Hilbert space setting, $V = V^* = H$).

These results have been applied to various types of doubly nonlinear parabolic equations, and here, let us take the following example:

$$|\partial_t u|^{p-2}\partial_t u - \Delta_m u = f(x,t) \quad \text{in } Q := \Omega \times (0,T), \tag{3}$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{4}$$
$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{5}$$

where $1 < p, m < \infty$. Equation (3) can be regarded as a sort of *generalized Ginzburg-Landau equations* proposed by Gurtin [9],

$$\beta(u, \nabla u, \partial_t u)\partial_t u = \operatorname{div}\left[\partial_{\nabla u}\psi(u, \nabla u)\right] - \partial_u \psi(u, \nabla u) + \gamma$$

with kinetic coefficient β, free energy density ψ and external microforce γ. In particular, (3)–(5) is well suited to the general theory due to Colli [6] based on the Banach space $L^p(0,T;V)$ by setting $V = L^p(\Omega)$, $Au = |u|^{p-2}u$ and $Bu = -\Delta_m u$. Indeed, one can easily check that the p-power growth condition holds true, more precisely, it holds that

$$\|u\|_V^p = \langle Au, u\rangle_V, \quad \|Au\|_{V^*}^{p'} = \|u\|_V^p.$$

In this note, we shall treat a variant of (3)–(5) involving *variable exponents.* More precisely, let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and consider

$$|\partial_t u|^{p(x)-2}\partial_t u - \Delta_{m(x)} u = f(x,t) \quad \text{in } Q := \Omega \times (0,T), \tag{6}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{7}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{8}$$

where $1 < p(x), m(x) < \infty$ are variable exponents and $\Delta_{m(x)}$ stands for the $m(x)$-Laplacian given by

$$\Delta_{m(x)} u = \mathrm{div}\ \left(|\nabla u|^{m(x)-2}\nabla u\right).$$

It is worth mentioning that Equation (6) can describe mixed settings of several types of (generalized) Ginzburg-Landau models, e.g., the case

$$\underbrace{p(x) \equiv 2 \text{ in } \Omega_1}_{\text{usual GL model}}, \quad \underbrace{p(x) \not\equiv 2 \text{ in } \Omega_2}_{\text{generalized GL model}}, \quad \Omega = \Omega_1 \oplus \Omega_2.$$

Not only does such a generalization extend the scope of the abstract theory developed so far in order to cover Equation (6) but also it would shed new light on the theory of evolution equations by reconsidering whether a vector-valued function space such as a Lebesgue-Bochner space is an optimal choice as a base space.

2 Lebesgue and Sobolev spaces with variable exponents

In this section, we briefly review some material on variable exponent Lebesgue and Sobolev spaces and set up notation. We refer the reader to [8] as a survey of this field.

Define the set of *variable exponents* by

$$\mathcal{P}(\Omega) := \left\{ p \in \mathcal{M}(\Omega) \colon \operatorname*{ess\,inf}_{x\in\Omega} p(x) \geq 1 \right\},$$

where $\mathcal{M}(\Omega)$ denotes the set of Lebesgue measurable functions defined on Ω. For $p(x) \in \mathcal{P}(\Omega)$, denote the (essential) supremum and infimum of $p(x)$ by

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$$

and define the class of *log-Hölder continuous variable exponents* by

$$\mathcal{P}_{\log}(\Omega) := \left\{ p \in \mathcal{P}(\Omega) \colon |p(x) - p(x')| \leq \frac{L}{\log(|x-x'|^{-1} + e)} \right.$$
$$\left. \text{for all } x, x' \in \Omega \text{ and some } L > 0 \right\}.$$

Now, *variable exponent Lebesgue and Sobolev spaces* are defined by

$$L^{p(x)}(\Omega) := \left\{ u \in \mathcal{M}(\Omega) \colon \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\}$$

with norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 \colon \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\},$$

and

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) \colon \partial_{x_i} u \in L^{p(x)}(\Omega) \ \text{ for } \ i = 1, \dots, N \right\},$$

whose norm is given by

$$\|u\|_{W^{1,p(x)}(\Omega)} := \left(\|u\|^2_{L^{p(x)}(\Omega)} + \|\nabla u\|^2_{L^{p(x)}(\Omega)} \right)^{1/2}.$$

3 Difficulties arising from variable exponents

This section is devoted to discussing difficulties of treating Equation (6),

$$|\partial_t u|^{p(x)-2} \partial_t u - \Delta_{m(x)} u = f(x,t),$$

arising from the presence of variable exponents. Following a classical strategy as in constant exponent cases, we set

$$V = L^{p(x)}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) \colon \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\}$$

and note that

$$\left(|u|^{p(x)-2} u \right) u = |u|^{p(x)}, \quad \left| |u|^{p(x)-2} u \right|^{p'(x)} = |u|^{p(x)}.$$

However, it only implies

$$\left\langle |u|^{p(x)-2} u, u \right\rangle_V \ge c \|u\|_V^{p^-}, \quad \left\| |u|^{p(x)-2} u \right\|_{V^*}^{(p^+)'} \le C \|u\|_V^{p^+}$$

with positive constants c, C and $V^* = L^{p'(x)}(\Omega)$. Since $p^+ > p^-$, the equation does not fall within the scope of the general theory of [6].

Let us more precisely discuss a difficulty due to the relation $p^+ > p^-$ peculiar to the variable exponent setting. For simplicity, suppose $f \equiv 0$ and test (6) by $\partial_t u$ to see that

$$\int_\Omega |\partial_t u|^{p(x)} \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{m(x)} |\nabla u|^{m(x)} \mathrm{d}x = 0.$$

The integration of both sides over $(0,t)$ implies

$$\int_0^t \int_\Omega |\partial_\tau u|^{p(x)} \mathrm{d}x \mathrm{d}\tau + \int_\Omega \frac{1}{m(x)} |\nabla u(x,t)|^{m(x)} \mathrm{d}x \le \int_\Omega \frac{1}{m(x)} |\nabla u_0(x)|^{m(x)} \mathrm{d}x. \tag{9}$$

To estimate $|\partial_t u|^{p(x)-2}\partial_t u$ in $V^* = L^{p'(x)}(\Omega)$, we use the relation,

$$\int_\Omega \left||\partial_t u|^{p(x)-2}\partial_t u\right|^{p'(x)} \mathrm{d}x = \int_\Omega |\partial_t u|^{p(x)} \mathrm{d}x. \tag{10}$$

If one works in a framework (called "Frame B" below) based on a Bochner space (e.g., $L^p(0,T;V)$), one needs to derive estimates for $\partial_t u$ and $|\partial_t u|^{p(x)-2}\partial_t u$ in a proper Bochner space and its dual space, respectively. However, in the variable exponent setting, we should pay attention to a gap between the modular and norm of $L^{p(x)}(\Omega)$, that is,

$$\int_\Omega |w(x)|^{p(x)} \mathrm{d}x \neq \|w\|^{p(x)}_{L^{p(x)}(\Omega)} \quad \text{for } w \in L^{p(x)}(\Omega).$$

To overcome this defect, we usually use the following relation between the modular and norm:

$$\sigma^-_{p(x)}(\|w\|_{L^{p(x)}}) \le \int_\Omega |w(x)|^{p(x)} dx \le \sigma^+_{p(x)}(\|w\|_{L^{p(x)}}) \quad \text{for all } w \in L^{p(x)}(\Omega)$$

with $\sigma^-_{p(x)}(s) := \min\{s^{p^-}, s^{p^+}\}$ and $\sigma^+_{p(x)}(s) := \max\{s^{p^-}, s^{p^+}\}$. Then one may obtain estimates in Bochner spaces with some loss of integrability in t (cf. see (9)) such as

$$\int_0^T \|\partial_t u\|_V^{p^-} \, \mathrm{d}t \le \int_0^T \left(\int_\Omega |\partial_t u|^{p(x)} \, \mathrm{d}x\right) \mathrm{d}t,$$
$$\int_0^T \left\||\partial_t u|^{p(x)-2}\partial_t u\right\|_{V^*}^{(p')^-} \mathrm{d}t \le \int_0^T \left(\int_\Omega |\partial_t u|^{p(x)} \, \mathrm{d}x\right) \mathrm{d}t.$$

However, there is no duality between two spaces $L^{p^-}(0,T;V)$ and $L^{(p')^-}(0,T;V^*)$, where $\partial_t u$ and $|\partial_t u|^{p(x)-2}\partial_t u$, respectively, are estimated. Indeed, we find that $L^{(p')^-}(0,T;V^*)$ is not identified with the dual space of $L^{p^-}(0,T;V)$ due to the fact that $(p')^- = (p^+)' < (p^-)'$ by $p^+ > p^-$. Such a lack of a duality prevents us to identify the limit of approximate solutions in a usual manner so as to prove the existence of solutions.

On the other hand, a framework (called "Frame L" below) based on Lebesgue spaces for functions of space-time variables shows us a different picture. Recall the relation (10). Then we immediately observe that

$$\iint_Q \left||\partial_t u|^{p(x)-2}\partial_t u\right|^{p'(x)} \mathrm{d}x\mathrm{d}t = \iint_Q |\partial_t u|^{p(x)} \mathrm{d}x\mathrm{d}t$$

with $Q := \Omega \times (0,T)$, which implies, e.g.,

$$\left\||\partial_t u|^{p(x)-2}\partial_t u\right\|_{L^{p'(x)}(Q)} \le \left(\|\partial_t u\|_{L^{p(x)}(Q)}\right)^{p^+/p^-}$$

(see also (9)). Here it is noteworthy that there is no loss of integrability through these procedures. Moreover, since $L^{p'(x)}(Q)$ is identified with the dual space of $L^{p(x)}(Q)$, there is a duality between two spaces in which the estimate is established. Hence the operator $\mathcal{A} : u \mapsto |u|^{p(x)-2}u$ is well defined from $L^{p(x)}(Q)$ into its dual space $L^{p'(x)}(Q)$. Moreover, we observe that

$$\mathcal{A} : L^{p(x)}(Q) \to L^{p'(x)}(Q) \quad \text{is bounded and coercive.}$$

Therefore it seems better to work in the Lebesgue space, $L^{p(x)}(Q)$, in order to treat Equation (6) without any loss of integrability (in t) (cf. we found loss in view of Frame B). On the other hand, in contrast with constant exponent cases, there is no Bochner space which can be identified with the Lebesgue space $L^{p(x)}(Q)$, as the formal description $L^{p(x)}(0,T;L^{p(x)}(\Omega))$ has no longer sense due to the x-dependence of $p(x)$. Furthermore, in most of studies on evolution equations in view of energy methods, the chain-rule for gradient operators (e.g., subdifferential) is often employed and plays an crucial role. However, chain-rules are always formulated and proved in Frame B (see, e.g., [5] and [10]). So this situation encourages us to develop a combination of two frameworks, Frame B and Frame L, in a suitable way.

4 Main results of [1]

The main results of [1] are concerned with the existence and regularity of solutions for the Cauchy-Dirichlet problem (6)–(8). To prove these results, we shall present a mixed framework of Frame B and Frame L. Moreover, we shall develop some devices of subdifferential calculus, in particular, a chain-rule for subdifferentials in the mixed frame.

To state the main results, let us introduce basic assumptions (H),

$$m \in \mathcal{P}_{\log}(\Omega), \quad p \in \mathcal{P}(\Omega), \quad 1 < p^-, m^-, p^+, m^+ < \infty, \tag{H1}$$

$$\operatorname*{ess\,inf}_{x\in\Omega} (m^*(x) - p(x)) > 0, \quad m^*(x) := \frac{Nm(x)}{(N - m(x))_+}, \tag{H2}$$

$$f \in L^{p'(x)}(Q), \quad u_0 \in W_0^{1,m(x)}(\Omega). \tag{H3}$$

Remark 4.1. (i) By (H1), $L^{p(x)}(\Omega)$ and $W^{1,m(x)}(\Omega)$ are uniformly convex and separable Banach spaces.

(ii) Since $m(\cdot) \in \mathcal{P}_{\log}(\Omega)$, one can define $W_0^{1,p(x)}(\Omega)$ by

$$W_0^{1,m(x)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,m(x)}(\Omega)}, \quad \|u\|_{W_0^{1,m(x)}(\Omega)} := \|\nabla u\|_{L^{m(x)}(\Omega)},$$

and moreover, it has similar properties (e.g., Poincaré and Sobolev inequalities) to the constant exponent case.

(iii) Moreover, (H2) ensures that $W_0^{1,m(x)}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^{p(x)}(\Omega)$.

We are concerned with strong solutions of (6)–(8) defined by

Definition 1 (Strong solutions)

We call $u \in L^{p(x)}(Q)$ a *strong solution* of (6)–(8) in Q whenever the following conditions hold true:

(i) $t \mapsto u(\cdot, t)$ is continuous with values in $L^{p(x)}(\Omega)$ on $[0, T]$, and it is weakly continuous with values in $W_0^{1,m(x)}(\Omega)$ on $[0, T]$,

(ii) $\partial_t u \in L^{p(x)}(Q)$, $\Delta_{m(x)} u \in L^{p'(x)}(Q)$,

(iii) the equation (6) holds for a.e. $(x, t) \in Q$,

(iv) the initial condition (8) is satisfied for a.e. $x \in \Omega$.

In [1], the following theorems are proved.

Theorem 2 (Existence of strong solutions [1])

Assume (H). Then the Cauchy-Dirichlet problem (6)–(8) admits (at least) one strong solution u.

Theorem 3 (Time-regularization of strong solutions [1])

In addition to (H), suppose that

$$t\partial_t f \in L^{p'(x)}(Q).$$

Then, the Cauchy-Dirichlet problem (6)–(8) admits a strong solution u, which additionally satisfies

$$\operatorname*{ess\,sup}_{t \in (\delta, T)} \|\partial_t u(\cdot, t)\|_{L^{p(x)}(\Omega)} < \infty,$$
$$\operatorname*{ess\,sup}_{t \in (\delta, T)} \|\Delta_{m(x)} u(\cdot, t)\|_{L^{p'(x)}(\Omega)} < \infty$$

for any $\delta \in (0, T)$.

5 Two formulations of the equation

We first set up a formulation based on a Bochner space setting, "Frame B," for (6)–(8). Set

$$V = L^{p(x)}(\Omega) \quad \text{and} \quad X = W_0^{1,m(x)}(\Omega)$$

with norms $\|u\|_V := \|u\|_{L^{p(x)}(\Omega)}$, $\|u\|_X := \|\nabla u\|_{L^{m(x)}(\Omega)}$ and duality pairing

$$\langle v, u \rangle_V = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \text{for all } u \in V,\ v \in V^* = L^{p'(x)}(\Omega).$$

By (H2), it follows that $X \overset{\text{compact}}{\hookrightarrow} V$ and $V^* \overset{\text{compact}}{\hookrightarrow} X^*$.

Define functionals ψ and ϕ on V by

$$\psi(u) := \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} \mathrm{d}x \quad \text{for } u \in V$$

$$\phi(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{m(x)} |\nabla u(x)|^{m(x)} \mathrm{d}x & \text{if } u \in X, \\ +\infty & \text{if } u \in V \setminus X. \end{cases}$$

Denote by ∂_Ω the subdifferential in $V = L^{p(x)}(\Omega)$. Then (6)–(8) can be reduced to

$$\partial_\Omega \psi(u'(t)) + \partial_\Omega \phi(u(t)) = Pf(t) \text{ in } V^*, \quad 0 < t < T, \quad u(0) = u_0,$$

where $Pf(t) := f(\cdot, t)$. Here we emphasize that the notion of a subdifferential is essentially needed here. Indeed, we work in $V = L^{p(x)}(\Omega)$ to get rid of any loss of integrability in x; however, the functional ϕ is not smooth in V. So a notion of the derivative for non-smooth functionals is required.

Next, we transform the formulation in "Frame B" into one in "Frame L." To this end, we carefully reconsider the correspondence between functions in two frameworks by taking account of variable exponent Lebesgue spaces. For each $u \in \mathcal{M}(Q)$, write

$$Pu(t) := u(\cdot, t) \quad \text{for } t \in (0, T).$$

Then it follows that

Proposition 4 (Identification between B- and L-spaces [1])

Let $1 \le p < \infty$ and let $p(x)$ be such that $1 \le p^- \le p^+ < \infty$.

(i) P is a linear, bijective, isometric mapping from $L^p(Q)$ to $L^p(0, T; L^p(\Omega))$. Furthermore, if $u \in L^{p(x)}(Q)$, then $Pu \in L^{p^-}(0, T; L^{p(x)}(\Omega))$.

(ii) The inverse $P^{-1} : L^p(0, T; L^p(\Omega)) \to L^p(Q)$ is well defined, and for $u = u(t) \in L^p(0, T; L^p(\Omega))$, $u(t) = P^{-1}u(\cdot, t)$ for a.e. $t \in (0, T)$.

(iii) If $u \in L^{p(x)}(Q)$ with $\partial_t u \in L^{p(x)}(Q)$, then Pu belongs to the space $W^{1,p^-}(0, T; L^{p(x)}(\Omega))$ and $(Pu)' = P(\partial_t u)$.

(iv) If $u \in W^{1,p}(0, T; L^p(\Omega))$, then $\partial_t(P^{-1}u) \in L^p(Q)$ and $\partial_t(P^{-1}u) = P^{-1}(u')$.

Remark 5.1. It is known that $L^\infty(0, T; L^\infty(\Omega))$ is not identified with $L^\infty(Q)$ (see [11]).

Set

$$\mathcal{V} := L^{p(x)}(Q) \quad \text{and} \quad \mathcal{V}^* := L^{p'(x)}(Q) \quad \text{with } Q = \Omega \times (0, T).$$

Let $\varphi : V(= L^{p(x)}(\Omega)) \to (-\infty, \infty]$ be a proper lower semicontinuous convex functional and define $\Phi : \mathcal{V} \to (-\infty, \infty]$ by

$$\Phi(u) := \begin{cases} \displaystyle\int_0^T \varphi(Pu(t))\, \mathrm{d}t & \text{if } \varphi(Pu(\cdot)) \in L^1(0, T), \\ \infty & \text{otherwise.} \end{cases}$$

Here and henceforth, denote by ∂_Q the subdifferential in $\mathcal{V} = L^{p(x)}(Q)$. Then it holds that

Proposition 5 (Identification of subdifferentials [1])

For $u \in \mathcal{V}$, $\xi \in \mathcal{V}^*$ with $1 < p^- \le p^+ < \infty$,

$$\xi \in \partial_Q \Phi(u) \quad \text{iff} \quad P\xi(t) \in \partial_\Omega \varphi(Pu(t)) \text{ for a.e. } t \in (0,T).$$

Now, we are ready to provide a formulation of (6)–(8) based on Frame L. Define functionals Ψ and Φ on $\mathcal{V}$ by

$$\Psi(u) := \iint_Q \frac{1}{p(x)} |u(x,t)|^{p(x)} \mathrm{d}x\, \mathrm{d}t = \int_0^T \psi(Pu(t))\, \mathrm{d}t,$$

$$\Phi(u) := \begin{cases} \displaystyle\int_0^T \phi(Pu(t))\, \mathrm{d}t & \text{if } Pu(t) \in X \text{ for a.e. } t \in (0,T), \\ & \quad t \mapsto \phi(Pu(t)) \in L^1(0,T), \\ \infty & \text{otherwise} \end{cases}$$

for $u \in \mathcal{V}$. Then by Proposition 5, the evolution equation ($\Leftrightarrow$ (6)–(8)),

$$\partial_\Omega \psi(u'(t)) + \partial_\Omega \phi(u(t)) = Pf(t) \text{ in } V^*, \quad 0 < t < T,$$

is equivalently rewritten as the relation,

$$\partial_Q \Psi(\partial_t(P^{-1}u)) + \partial_Q \Phi(P^{-1}u) = f \text{ in } \mathcal{V}^*.$$

6 Construction of a strong solution

In this section, we give an outline of a proof for Theorem 2.

Step 1 (Time-discretization) We consider the following time-discretized equations, for $n = 0, \dots, N-1$,

$$\partial_\Omega \psi\left(\frac{u_{n+1} - u_n}{h}\right) + \partial_\Omega \phi(u_{n+1}) = f_{n+1} \text{ in } V^*$$

with

$$h := T/N, \quad t_n := nh \quad \text{and} \quad f_n := \frac{1}{h} \int_{t_{n-1}}^{t_n} Pf(\theta)\, \mathrm{d}\theta.$$

The existence of $u_{n+1} \in X$ can be proved by using a variational method.

Moreover, define a *piecewise forward constant interpolant* $\overline{u}_N : (0,T) \to X = W_0^{1,m(x)}(\Omega)$ and a *piecewise linear interpolant* $u_N : (0,T) \to X$ by

$$\overline{u}_N(t) := u_{n+1}, \quad u_N(t) := \frac{t - t_n}{h} u_{n+1} + \frac{t_{n+1} - t}{h} u_n \quad \text{for } t \in [t_n, t_{n+1}).$$

Then we have

$$\partial_\Omega \psi(u_N'(t)) + \partial_\Omega \phi(\overline{u}_N(t)) = \overline{f}_N(t) \text{ in } V^*, \quad \text{for a.e. } t \in (0,T)$$

with $u_N(0) = u_0$ in Frame B, and equivalently,

$$\partial_Q \Psi(\partial_t(P^{-1}u_N)) + \partial_Q \Phi(P^{-1}\overline{u}_N) = P^{-1}\overline{f}_N \text{ in } \mathcal{V}^*, \quad u_N(0) = u_0$$

in Frame L.

Step 2 (Energy estimates) Test the discretized equation by $(u_{n+1} - u_n)/h$ to obtain

$$\iint_Q \left|\partial_t(P^{-1}u_N)\right|^{p(x)} \mathrm{d}x\,\mathrm{d}t + \sup_{t\in[0,T]} \phi(\overline{u}_N(t)) \le C,$$

which also gives

$$\left\|\partial_t(P^{-1}u_N)\right\|_{\mathcal{V}} \le C, \quad \sup_{t\in[0,T]} |\overline{u}_N(t)|_X + \sup_{t\in[0,T]} |u_N(t)|_X \le C.$$

Recall that

$$\mathcal{A} = \partial_Q \Psi : v \mapsto |v|^{p(x)-2}v \text{ is bounded from } \mathcal{V} \text{ to } \mathcal{V}^*.$$

Thus we conclude that

$$\left\|\partial_Q \Psi(\partial_t(P^{-1}u_N))\right\|_{\mathcal{V}^*} \le C,$$

which also implies the boundedness of $\partial_Q \Phi(P^{-1}\overline{u}_N)$ in $\mathcal{V}^*$ by comparison.

Step 3 (Convergence) Passing to the limit as $N \to \infty$, up to subsequence, one has the following convergences in both frames:

$$\begin{aligned}
u_N \to u \quad & \text{strongly in } C([0,T];V), \\
& \text{weakly star in } L^\infty(0,T;X), \\
\overline{u}_N \to u \quad & \text{strongly in } L^\infty(0,T;V), \\
& \text{weakly star in } L^\infty(0,T;X), \\
P^{-1}\overline{u}_N \to \hat{u} = P^{-1}u \quad & \text{strongly in } \mathcal{V}, \\
\partial_t(P^{-1}u_N) \to \partial_t \hat{u} \quad & \text{weakly in } \mathcal{V}, \\
\partial_Q \Phi(P^{-1}\overline{u}_N) \to \xi \quad & \text{weakly in } \mathcal{V}^*, \\
\partial_Q \Psi(\partial_t(P^{-1}u_N)) \to \eta \quad & \text{weakly in } \mathcal{V}^*.
\end{aligned}$$

Thus $\eta + \xi = f$ in $\mathcal{V}^*$. From the maximal monotonicity of $\partial_Q \Phi$ in $\mathcal{V} \times \mathcal{V}^*$, one can immediately obtain $\xi \in \partial_Q \Phi(\hat{u})$.

Now, it remains to show $\eta \in \partial_Q \Psi(\partial_t \hat{u})$. To this end, we shall use Minty's trick. One observes that

$$\begin{aligned}
&\iint_Q \partial_Q \Psi\left(\partial_t(P^{-1}u_N)\right) \partial_t(P^{-1}u_N)\,\mathrm{d}x\,\mathrm{d}t \\
&= \iint_Q \left(P^{-1}\overline{f}_N - \partial_Q \Phi(P^{-1}\overline{u}_N)\right) \partial_t(P^{-1}u_N)\,\mathrm{d}x\,\mathrm{d}t \\
&\le \iint_Q \left(P^{-1}\overline{f}_N\right) \partial_t(P^{-1}u_N)\,\mathrm{d}x\,\mathrm{d}t - \phi(u_N(T)) + \phi(u_0).
\end{aligned}$$

Passing to the limit as $N \to \infty$, we have

$$\limsup_{n\to\infty} \iint_Q \partial_Q \Psi\left(\partial_t(P^{-1}u_N)\right) \partial_t(P^{-1}u_N)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \iint_Q f\partial_t\hat{u}\,\mathrm{d}x\,\mathrm{d}t - \phi(u(T)) + \phi(u_0) \overset{?}{=} \iint_Q \eta\partial_t\hat{u}\,\mathrm{d}x\mathrm{d}t.$$

However, the last equality is not obvious at this moment, due to the lack of a chain-rule for the current situation. To justify the equality, we need a new chain-rule for subdifferential operators in a mixed framework.

Proposition 6 (Chain rule in a mixed frame [1])

Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. Let $u \in \mathcal{V}$ be such that $\partial_t u \in \mathcal{V}$. Suppose that there exists $\xi \in \mathcal{V}^*$ such that $\xi \in \partial_Q \Phi(u)$. Then, the function $t \mapsto \varphi(Pu(t))$ is absolutely continuous over $[0,T]$. Moreover, for each $t \in (0,T)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(Pu(t)) = \langle \eta, (Pu)'(t)\rangle_V \quad \text{for all } \eta \in \partial_\Omega\varphi(Pu(t)),$$

whenever Pu and $\varphi(Pu(\cdot))$ are differentiable at t. In particular, for $0 \leq s < t \leq T$, we have

$$\varphi(Pu(t)) - \varphi(Pu(s)) = \iint_{\Omega\times(s,t)} \xi\partial_\tau u\,\mathrm{d}x\,\mathrm{d}\tau.$$

Applying Proposition 6, we deduce that

$$\limsup_{n\to\infty} \iint_Q \partial_Q \Psi\left(\partial_t(P^{-1}u_N)\right) \partial_t(P^{-1}u_N)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \iint_Q f\partial_t\hat{u}\,\mathrm{d}x\,\mathrm{d}t - \phi(u(T)) + \phi(u_0)$$
$$\overset{u=P\hat{u}}{=} \iint_Q f\partial_t\hat{u}\,\mathrm{d}x\,\mathrm{d}t - \phi(P\hat{u}(T)) + \phi(P\hat{u}(0))$$
$$\overset{\text{Prop 6}}{=} \iint_Q f\partial_t\hat{u}\,\mathrm{d}x\mathrm{d}t - \iint_Q \xi\partial_t\hat{u}\,\mathrm{d}x\mathrm{d}t \overset{\xi+\eta=f}{=} \iint_Q \eta\partial_t\hat{u}\,\mathrm{d}x\mathrm{d}t,$$

whence follows

$$\eta \in \partial_Q\Psi(\partial_t\hat{u}).$$

Consequently, $\hat{u}$ solves (6)–(8). □

7 Outline of a proof for Proposition 6

In the section, we exhibit an outline of a proof for the new chain-rule.

Step 1 (Modification of the Moreau-Yosida approximation) We start with approximating the proper lower semicontinuous convex functional φ defined on $V =$

$L^{p(x)}(\Omega)$ by

$$\varphi_\lambda(u) := \min_{v\in V}\left(\int_\Omega \frac{\lambda}{p(x)}\left|\frac{v(x)-u(x)}{\lambda}\right|^{p(x)} dx + \varphi(v)\right) \quad \text{for } \ u \in V,$$

which is called a *modified Moreau-Yosida regularization* of φ. Then φ_λ enjoys similar properties to the usual Moreau-Yosida regularization with the modified resolvent J_λ and modified Yosida approximation A_λ of $A = \partial_\Omega\varphi$ defined below.

Let $A : V \to V^*$ be a maximal monotone operator.

- The *modified resolvent* $J_\lambda : V \to V$ of A is given by, for each $u \in V$, $J_\lambda u := u_\lambda$, which is a unique solution of

$$Z_\Omega\left(\frac{u_\lambda - u}{\lambda}\right) + A(u_\lambda) \ni 0 \quad \text{in } \ V^*,$$

  where $Z_\Omega(u) := |u|^{p(x)-2}u$ for $u \in V$.

- The *modified Yosida approximation* $A_\lambda : V \to V^*$ of A is given by

$$A_\lambda(u) := Z_\Omega\left(\frac{u - J_\lambda u}{\lambda}\right) \in A(J_\lambda u) \quad \text{for each } \ u \in V.$$

One can also define the modified Moreau-Yosida regularization Φ_λ of Φ defined on $\mathcal{V}$.

Step 2 (Correspondence of φ_λ and Φ_λ) Now, we have the following correspondence between φ_λ and Φ_λ:

Lemma 7 (Correspondence of φ_λ and Φ_λ [1])

It follows that

$$\Phi_\lambda(u) = \int_0^T \varphi_\lambda(Pu(t))\,\mathrm{d}t \quad \text{for all } \ u \in \mathcal{V}.$$

In particular, for $u \in \mathcal{V}$ and $\xi \in \mathcal{V}^*$,

$$\xi_\lambda = \partial_Q\Phi_\lambda(u) \quad \text{iff} \quad P\xi_\lambda(t) = \partial_\Omega\varphi_\lambda(Pu(t)) \ \text{ for a.a. } t \in (0,T).$$

A similar relation is known for a setting based on Hilbert spaces H and $\mathcal{H} := L^2(0,T;H)$. However, it cannot be directly extended to Banach spaces V and $\mathcal{V} := L^p(0,T;V)$ for $p \neq 2$.

Step 3 (Chain-rule for φ_λ) Thanks to the notion of the modified Moreau-Yosida regularization, one shall obtain higher integrability for the subdifferentials of regularized functionals φ_λ and be able to apply a standard chain-rule to φ_λ.

Let $u \in \mathcal{V}$ be such that $\partial_t u \in \mathcal{V}$. Then since $u, \partial_t u \in L^{p(x)}(Q)$, we deduce that $Pu \in W^{1,p^-}(0,T;L^{p(x)}(\Omega))$ (see Proposition 4). Moreover, since $\partial\varphi_\lambda$ is bounded, we see that

$$\partial_\Omega\varphi_\lambda(Pu(\cdot)) \in L^\infty(0,T;V).$$

Let $\xi_\lambda := \partial_Q \Phi_\lambda(u)$ and use a standard chain-rule in Frame B to obtain

$$\begin{aligned}\varphi_\lambda(Pu(t)) - \varphi_\lambda(Pu(s)) &\overset{\text{chain}}{=} \int_s^t \langle \partial_\Omega \varphi_\lambda(Pu(\tau)), (Pu)'(\tau)\rangle_V \, \mathrm{d}\tau \\ &\overset{\text{Lem 7}}{=} \int_s^t \langle P\xi_\lambda(\tau), (Pu)'(\tau)\rangle_V \, \mathrm{d}\tau \\ &= \iint_{\Omega\times(s,t)} \xi_\lambda \partial_t u \, \mathrm{d}x\mathrm{d}\tau, \quad 0 \le s \le t \le T.\end{aligned}$$

Step 4 (Convergence) To discuss the convergence of both sides of the relation as $\lambda \to 0$, we first note that

Lemma 8 (Boundedness of modified Yosida approx. [1])

Let $u \in V$, $\eta \in Au$ and let A_λ be the modified Yosida approximation. Then it follows that

$$\int_\Omega \frac{1}{p'(x)} |A_\lambda u(x)|^{p'(x)} \mathrm{d}x \le \int_\Omega \frac{1}{p'(x)} |\eta(x)|^{p'(x)} \mathrm{d}x.$$

An analogous statement also holds for any maximal monotone operator $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$.

Thus since $\xi_\lambda = \partial_Q \Phi_\lambda(Pu)$, we have, for any $\eta \in \partial_Q \Phi(Pu)$,

$$\iint_Q \frac{1}{p'(x)} |\xi_\lambda|^{p'(x)} \, \mathrm{d}x\mathrm{d}t \le \iint_Q \frac{1}{p'(x)} |\eta|^{p'(x)} \, \mathrm{d}x\mathrm{d}t < \infty.$$

Hence

$$\xi_\lambda \to \xi \quad \text{weakly in } \mathcal{V}^* \quad \text{and} \quad \xi \in \partial_Q \Phi(Pu).$$

Thus

$$\varphi_\lambda(Pu(t)) - \varphi_\lambda(Pu(s)) = \iint_{\Omega\times(s,t)} \xi_\lambda \partial_t u \, \mathrm{d}x\mathrm{d}\tau.$$

Using the fact

$$\varphi_\lambda(u) \to \varphi(u) \quad \text{for all } u \in V,$$

we have obtained the formula,

$$\varphi(Pu(t)) - \varphi(Pu(s)) = \iint_{\Omega\times(s,t)} \xi \partial_t u \, \mathrm{d}x\mathrm{d}\tau,$$

which also implies the absolute continuity of $t \mapsto \varphi(Pu(t))$. □

8 Summary

In this note, we reviewed the results obtained in the paper [1]. The main results are concerned with the existence and regularity (in time) of solutions of the Cauchy-Dirichlet problem for the doubly nonlinear parabolic equation involving variable exponents,

$$|\partial_t u|^{p(x)-2} \partial_t u - \Delta_{m(x)} u = f(x,t) \quad \text{in } Q := \Omega \times (0,T).$$

Furthermore, we gave an outline of a proof for the existence result.

- In order to efficiently use energy structures (without loss of integrability in t), we partially worked in "Frame L", a framework based on the Lebesgue space
$$\mathcal{V} := L^{p(x)}(Q) \quad \text{with} \quad Q = \Omega \times (0, T).$$
- To this end, we reformulated the problem both in "Frame L" and "Frame B", a framework based on Bochner spaces, and also investigated the correspondences between these frameworks.
- We presented a new chain-rule for subdifferentials in a mixed framework. In its statement, the assumptions are formulated in "Frame L" and the conclusion is stated in the both frames.

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