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# A simple way to derive a priori estimates for solutions to chemotaxis systems

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## 1. Introduction

This paper summarizes some recent results on several kinds of Keller-Segel systems and presents how to derive a priori estimates for solutions which play a key role in the analysis of the systems. These are mainly based on joint works (with Ishida) [**14, 15, 16, 17**], (with Ishida and Maeda) [**11**], (with Ishida and Ono) [**12**], (with Ishida and Seki) [**13**], (with Fujie and Winkler) [**4**], (with Fujie) [**3**].

The Keller-Segel system is proposed by Keller and Segel [**18**] in 1970. This system describes a part of the life cycle of cellular slime molds with chemotaxis. In more detail, slime molds move towards higher concentration of the chemical substance when they plunge into hunger. We denote by $u(x,t)$ the density of the cell population and by $v(x,t)$ the concentration of the signal substance at place $x$ and time $t$. A number of variations of the original Keller-Segel system are proposed and studied (see Hillen-Painter [**6**]).

In this paper we consider some versions of the following Keller-Segel system:

$$\text{(KS)}\qquad \begin{cases} u_t = \nabla\cdot(D(u)\nabla u - A(u,v)\nabla v), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain or $\Omega=\mathbb{R}^N$, $\tau=1$ (parabolic-parabolic system) or $\tau=0$ (parabolic-elliptic system), and typical examples of $D$ and $A$ are given by

$$D(u)\equiv 1,\quad D(u)=mu^{m-1}\ (m>1),$$
$$A(u,v)=u^{q-1}\ (q\ge 2),\quad A(u,v)=u\frac{\chi_0}{v}\ (\chi_0>0).$$

This paper deals with the following three topics:

- $L^p$-estimates in (KS) with $D(u)=mu^{m-1}$, $A(u,v)=u^{q-1}$, $\tau=1$, $\Omega=\mathbb{R}^N$ (Section 2).
- Energy estimates in (KS) with $D(u)=mu^{m-1}$, $A(u,v)=u^{q-1}$, $\tau=1$ (Section 3).
- Uniform $L^p$-estimates in (KS) with $D(u)\equiv 1$, $A(u,v)=u\frac{\chi_0}{v}$ and $\tau=0$ (Section 4).

These estimates yield some new results on the global existence, blow-up and boundedness of solutions. Our way to derive a priori estimates for solutions is much simple, because we effectively use the structures of the equations in (KS). Indeed, concerning the first equation in (KS), we will do only *multiplication by* $u^{p-1}$ and *integration by parts.* Thus the key to our derivation of a priori estimates is how we combine the effect by the second equation with the first one.

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## 2. Global existence of weak solutions to quasilinear degenerate parabolic-parabolic Keller-Segel systems on $\mathbb{R}^N$

In this section we discuss the global existence of solutions to the following quasilinear degenerate parabolic-parabolic Keller-Segel system on $\mathbb{R}^N$:

$$(\mathrm{KS})_{\mathbb{R}^N} \qquad \begin{cases} u_t = \nabla\cdot(\nabla u^m - u^{q-1}\nabla v), & x\in\mathbb{R}^N,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\mathbb{R}^N,\ t>0,\end{cases}$$

with initial condition $u(x,0)=u_0(x)$ and $v(x,0)=v_0(x)$, where $N\in\mathbb{N}$, $m\geq 1$, $q\geq 2$, $\tau=1$ or $\tau=0$. We study the case where $\tau=1$; however, we use $\tau$ for the comparison with the case where $\tau=0$. We assume that the initial data $(u_0,v_0)$ satisfies

$$(2.1) \qquad u_0\geq 0,\ u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),$$

$$(2.2) \qquad v_0\geq 0,\ v_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\ \Delta v_0\in L^{p_0}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N) \text{ for some } p_0>1.$$

Problem $(\mathrm{KS})_{\mathbb{R}^N}$ was first studied by Sugiyama [**22**] when $q=2$ and by Sugiyama-Kunii [**23**] when $q\geq 2$. Their result can be summarized as follows:

(i) $\tau=1,\ m\geq q\ \Rightarrow$ (KS) possesses a global weak solution with (large) initial data.
(ii) $\tau=0,\ m>q-\frac{2}{N}\ \Rightarrow$ (KS) has a global weak solution with (large) initial data.
(iii) $\tau=0,\ m\leq q-\frac{2}{N}\ \Rightarrow$ (KS) admits a global weak solution with small initial data.

In view of the above result there is a difference between $\tau=1$ and $\tau=0$. More precisely, there is a gap $\frac{2}{N}$ between $\tau=1$ and $\tau=0$ in the global solvability without any restriction on the size of initial data (compare (i) with (ii)). Moreover, the case $\tau=1$ and $m\leq q-\frac{2}{N}$ was not discussed. This would be caused by the following difficulty in the case $\tau=1$. Roughly speaking, one can *directly* substitute the second equation into the first one in the case $\tau=0$. Indeed, the first equation in (KS) is rewritten as

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla u^{q-1}\cdot\nabla v - u^{q-1}\Delta v.$$

In the case $\tau=0$ one can replace $\Delta v$ with $v-u$ in the third term on the right-hand side, so that we have the nonlinear effect as $u^q$. Then by comparing the diffusion term $\Delta u^m$ with $u^q$, a priori estimate for $u$ can be obtained when $\tau=0$ and $m>q-\frac{2}{N}$ or $m\leq q-\frac{2}{N}$. On the other hand, when $\tau=1$, it is *impossible* to use such direct substitution, because the second equation has $v_t$. This is the most difficult point in the case $\tau=1$.

To overcome the difficulty we employ the following inequality which is a particular consequence of well-known results on maximal Sobolev regularity in parabolic evolution equations (see e.g., Hieber-Prüss [**5**, Theorem 3.1]):

$$(2.3) \qquad \|\Delta v\|_{L^p(0,T;L^p(\mathbb{R}^N))} \leq \|\Delta v_0\|_{L^p(\mathbb{R}^N)} + C_{\langle p\rangle}\|u\|_{L^p(0,T;L^p(\mathbb{R}^N))},$$

where $C_{\langle p\rangle}>0$ is a constant. This inequality produces the same situation as in the case $\tau=0$. Consequently, we can adjust the difference between $\tau=1$ and $\tau=0$ in [**23**].

Before stating our results we define global weak solutions to $(\mathrm{KS})_{\mathbb{R}^N}$.

**Definition 2.1.** Let $T > 0$. A pair $(u, v)$ of nonnegative functions defined on $\mathbb{R}^N \times (0, T)$ is called a *weak solution* to (KS) *on* $[0, T)$ if

(a) $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ $(\forall p \in [1, \infty])$, $u^m \in L^2(0, T; H^1(\mathbb{R}^N))$,

(b) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$,

(c) $(u, v)$ satisfies $(\mathrm{KS})_{\mathbb{R}^N}$ in the sense of distributions, i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^N \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}^N} (\nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \varphi_t)\, dx\, dt = \int_{\mathbb{R}^N} u_0(x) \varphi(x, 0)\, dx,$$

$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \varphi + v \varphi - u \varphi - \tau v \varphi_t)\, dx\, dt = \tau \int_{\mathbb{R}^N} v_0(x) \varphi(x, 0)\, dx.$$

In particular, if $T > 0$ can be taken arbitrary, then $(u, v)$ is called a *global weak solution* to $(\mathrm{KS})_{\mathbb{R}^N}$.

We now state our main results in this section.

**Theorem 2.1** (Ishida-Y. [**14**]). *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\tau = 1$, $T > 0$. Let $(u_0, v_0)$ satisfy* (2.1) *and* (2.2). *Assume that*

$$q < m + \frac{2}{N}.$$

*Then there exists a nonnegative (global) weak solution $(u, v)$ to $(\mathrm{KS})_{\mathbb{R}^N}$ on $[0, T)$. Moreover, $u^m \in C((0, T); L^p_{\mathrm{loc}}(\mathbb{R}^N))$ $(\forall p \in [1, \infty))$ and the following estimates hold:*

$$\|u\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} + \|v\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} \leq K_1 \quad (\forall r \in [1, \infty]),$$
$$\|v_t\|_{L^{p_0}(0,T;L^{p_0}(\mathbb{R}^N))} + \|v\|_{L^{p_0}(0,T;H^{p_0}(\mathbb{R}^N))} \leq K_2,$$

*where* $K_1 = K_1(\|u_0\|_{L^1}, \|u_0\|_{L^\infty}, \|v_0\|_{L^1}, \|v_0\|_{L^\infty}, \|\Delta v_0\|_{L^{p_0}}, \|\Delta v_0\|_{L^\infty}, m, q, N, T) > 0$ *and* $K_2 = K_2(K_1, T) > 0$ *are constants.*

**Theorem 2.2** (Ishida-Y. [**15**]). *Let $N \geq 2$, $m \geq 1$, $q \geq 2$, $\tau = 1$, $T > 0$. Let $(u_0, v_0)$ satisfy* (2.1) *and* (2.2). *Suppose that*

$$q \geq m + \frac{2}{N}.$$

*Then there exist $\delta_u = \delta_u(m, q, N)$, $\delta_v = \delta_v(m, q, N)$ such that if*

$$\|u_0\|_{L^r} < \delta_u, \quad \|\Delta v_0\|_{L^{r+q-1}},\ \|\Delta v_0\|_{L^{r+1}} < \delta_v \ (r = \tfrac{N(q-m)}{2},\ \tfrac{N}{2}),$$

*then $(\mathrm{KS})_{\mathbb{R}^N}$ admits a nonnegative (global) weak solution $(u, v)$ to* (KS) *on $[0, T)$. Moreover, $u^m \in C((0, T); L^p_{\mathrm{loc}}(\mathbb{R}^N))$ $(\forall p \in [1, \infty))$ and the following estimates hold:*

$$\|u\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} + \|v\|_{L^\infty(0,T;L^r(\mathbb{R}^N))} \leq K_1 \quad \Big(\forall r \in \Big[\frac{N}{2} + 1, \infty\Big]\Big), \tag{2.4}$$

$$\|v_t\|_{L^r(0,T;L^r(\mathbb{R}^N))} + \|v\|_{L^r(0,T;W^{2,r}(\mathbb{R}^N))} \leq K_2, \quad \Big(\forall r \in \Big[\frac{N}{2} + 1, \infty\Big)\Big), \tag{2.5}$$

*where* $K_1 = K_1(r, \|u_0\|_{L^1}, \|u_0\|_{L^\infty}, \|v_0\|_{L^1}, \|v_0\|_{L^\infty}, \|\Delta v_0\|_{L^{\frac{N}{2}+1}}, \|\Delta v_0\|_{L^\infty}, m, q, N, T) > 0$ *and* $K_2 = K_2(K_1, T) > 0$ *are constants.*

**Remark 2.1.** Theorems 2.1 and 2.2 improve the pioneer work by Sugiyama-Kunii [**23**] in which $q \le m$ was assumed and the case $q > m$ was left as an open problem. We solved this open problem completely (without boundedness).

**Remark 2.2.** In Theorem 2.2, using the Besov space, we can lessen a kind of differentiability for $v_0$ and can construct a global solution under only two kinds of smallness which is independent of $\|u_0\|_{L^1}$; moreover, we can obtain the same result as Theorem 2.2 aslo in the one dimensional case (for more details, refer to Ishida-Y. [**15**]).

**Remark 2.3.** In Theorems 2.1 and 2.2 we can see that the mass conservation law holds:

$$\|u(t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)} \quad (t \ge 0),$$

which was rigorously proved by Ishida-Maeda-Y. [**11**].

**Remark 2.4.** Theorems 2.1 and 2.2 say only the existence of global weak solutions to $(\mathrm{KS})_{\mathbb{R}^N}$ and it is open whether the solution is uniformly-in-time bounded or not. Recently, when $q \ge m + \frac{2}{N}$ and the initial data $(u_0, v_0)$ is small in some sense, Ishida [**10**] succeeded in showing uniform-in-time boundedness of weak solutions to $(\mathrm{KS})_{\mathbb{R}^N}$. As to the case $q < m + \frac{2}{N}$, boundedness in the Neumann boundary problem on bounded domains was proved by Ishida-Seki-Y. [**13**].

**Remark 2.5.** The constant $q_c := m + \frac{2}{N}$ coincides with the critical exponent which divides the global solvability of the quasilinear parabolic equation

$$u_t = \Delta u^m + u^q.$$

As discussed in the next section, as to the Neumann boundary problem for $(\mathrm{KS})_{\mathbb{R}^N}$ in a ball, if $q > m + \frac{2}{N}$, then the solution with large negative energy blows up. Therefore the condition in Theorem 2.1 might be best possible one in a sense.

We can prove Theorems 2.1 and 2.2 as follows. We first consider an approximate problem of $(\mathrm{KS})_{\mathbb{R}^N}$. Indeed, we replace the diffusion term $\Delta u^m$ with

$$\Delta(u+\varepsilon)^m \quad (\varepsilon > 0).$$

Next we derive some estimates for approximate solutions Finally we discuss convergence of approximate solutions as $\varepsilon \downarrow 0$. The key to the proof lies in $L^r$-estimates for the first component of approximate solutions. In the rest of this section we explain how to derive a priori estimates for solutions by a formal computation. For the rigorous proof see [**14, 15, 16**] and Ishida [**9**].

**Proofs of Theorems 2.1 and 2.2 ($L^r$-estimates).** As stated above, we derive only $L^r$-estimates for solutions to $(\mathrm{KS})_{\mathbb{R}^N}$. Let $r \in (1, \infty)$. Multiplying the first equation in $(\mathrm{KS})_{\mathbb{R}^N}$ by $u^{r-1}$ and integrating it over $\mathbb{R}^N$, we obtain

$$\begin{aligned} \text{(2.6)} \qquad \frac{1}{r}\frac{d}{dt}\|u(t)\|^r_{L^r(\mathbb{R}^N)} &= -\int_{\mathbb{R}^N} \nabla u^m \cdot \nabla u^{r-1}\,dx + \int_{\mathbb{R}^N} u^{q-1}\nabla v \cdot \nabla u^{r-1}\,dx \\ &=: -\boldsymbol{I}_1 + \boldsymbol{I}_2. \end{aligned}$$

First it follows that

$$(2.7)\qquad -\int_0^t \boldsymbol{I}_1\,ds = -m(r-1)\int_0^t \left(\int_{\mathbb{R}^N} u^{m-1}\nabla u\cdot u^{r-2}\nabla u\,dx\right)ds$$

$$= -m(r-1)\int_0^t \left(\int_{\mathbb{R}^N} |u^{\frac{r+m-3}{2}}\nabla u|^2\,dx\right)ds$$

$$= -\frac{4m(r-1)}{(r+m-1)^2}\int_0^t \|\nabla u^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\,ds.$$

Next we consider the estimate for $\boldsymbol{I}_2$. Integration by parts and Hölder's inequality give

$$\boldsymbol{I}_2 = (r-1)\int_{\mathbb{R}^N} u^{q-1}\nabla u\cdot u^{r-2}\nabla v\,dx$$

$$= \frac{r-1}{r+q-2}\int_{\mathbb{R}^N}\nabla[u^{r+q-2}]\cdot\nabla v\,dx$$

$$= \frac{r-1}{r+q-2}\int_{\mathbb{R}^N} u^{r+q-2}(-\Delta v)\,dx$$

$$\le \frac{r-1}{p-1}\|u(t)\|^{p-1}_{L^p(\mathbb{R}^N)}\|\Delta v(t)\|_{L^p(\mathbb{R}^N)},$$

Integrating this inequality over $(0,t)$ and using Hölder's inequality again, we obtain

$$(2.8)\qquad \int_0^t \boldsymbol{I}_2\,ds \le \frac{r-1}{p-1}\left(\int_0^t \|u(s)\|^p_{L^p(\mathbb{R}^N)}\,ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|\Delta v(s)\|^p_{L^p(\mathbb{R}^N)}\,ds\right)^{\frac{1}{p}}.$$

We now recall the maximal Sobolev regularity (2.3):

$$\|\Delta v\|_{L^p(0,t;L^p(\mathbb{R}^N))} \le \|\Delta v_0\|_{L^p(\mathbb{R}^N)} + C_{\langle p\rangle}\|u\|_{L^p(0,t;L^p(\mathbb{R}^N))}.$$

Applying this inequality to the right-hand side of (2.8), we see from Young's inequality that

$$(2.9)\qquad \int_0^t \boldsymbol{I}_2\,ds \le \frac{r-1}{p-1}\left(\int_0^t \|u(s)\|^p_{L^p(\mathbb{R}^N)}\,ds\right)^{\frac{p-1}{p}}\|\Delta v_0\|_{L^p(\mathbb{R}^N)}$$

$$+\frac{r-1}{p-1}\left(\int_0^t \|u(s)\|^p_{L^p(\mathbb{R}^N)}\,ds\right)^{\frac{p-1}{p}} C_{\langle p\rangle}\left(\int_0^t \|u(s)\|^p_{L^p(\mathbb{R}^N)}\,ds\right)$$

$$\le \frac{r-1}{p-1}\left[\|\Delta v_{0\varepsilon}\|^p_{L^p(\mathbb{R}^N)} + (C_{\langle p\rangle}+1)\int_0^t \|u_\varepsilon(s)\|^p_{L^p(\mathbb{R}^N)}\,ds\right].$$

Integrating (2.6) over $(0,t)$, we deduce from (2.8) and (2.9) that

$$\|u(t)\|^r_{L^r(\mathbb{R}^N)}$$

$$\le \|u_0\|^r_{L^r(\mathbb{R}^N)} + \frac{r(r-1)}{p-1}\|\Delta v_0\|^p_{L^p(\mathbb{R}^N)}$$

$$+\frac{r(r-1)}{p-1}\int_0^t\left[(C_{\langle p\rangle}+1)\|u_\varepsilon(s)\|^p_{L^p(\mathbb{R}^N)} - \frac{4m(p-1)}{(r+m-1)^2}\|\nabla u^{\frac{r+m-1}{2}}(t)\|^2_{L^2(\mathbb{R}^N)}\right]ds.$$

This makes the same situation as in the quasilinear parabolic equation $u_t = \Delta u^m + u^q$. Therefore the standard argument using the Gagliardo-Nirenberg type inequality yields the desired estimate. □

## 3. Blow-up in quasilinear degenerate parabolic-parabolic Keller-Segel systems on $\Omega$

We discuss the existence of blow-up solutions to the following quasilinear degenerate parabolic-parabolic Keller-Segel system:

$$
(\mathrm{KS})_\Omega \qquad \begin{cases} u_t = \nabla\cdot(\nabla u^m - u^{q-1}\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\end{cases}
$$

with $u(x,0)=u_0(x)$, $v(x,0)=v_0(x)$ and $\frac{\partial u^m}{\partial\nu}=\frac{\partial v}{\partial\nu}=0$ $(x\in\partial\Omega,\ t>0)$, where $m\ge 1$, $q\ge 2$ and

$$\Omega = B := \{x\in\mathbb{R}^N; |x|<1\} \quad \text{with } N\ge 2.$$

We assume that the initial data $(u_0,v_0)$ satisfies

$$
\begin{aligned}
&u_0\ge 0,\ u_0\in L^\infty(B) \text{ with } \nabla u_0^m\in L^2(B),\\
&v_0\ge 0,\ v_0\in W^{1,\infty}(B).
\end{aligned}
$$

In the case of nondegenerate diffusion, Winkler [**24**] showed that there exist initial data such that the solution blows up in either finite or infinite time under the condition corresponding to $q>m+\frac{2}{N}$. Recently, Winkler [**25**] and Cieślak-Stinner [**2**] succeeded in constructing a finite time blow-up solution when $N\ge 3$. Thus we can expect that the same assertion holds in the case of degenerate diffusion. Ishida-Ono-Y. [**12**] found initial data such that every radially symmetric strong solution blows up in either finite or infinite time by assuming the existence of radially symmetric "strong solutions". However, in general, one can not expect that the system with degenerate diffusion has a strong solution with nonnegative initial data, so it still remains an open question.

To give an answer to the question, we define "energy solutions" to $(\mathrm{KS})_\Omega$ as follows.

**Definition 3.1.** Let $T\in(0,\infty]$. Then a pair $(u,v)$ of nonnegative functions defined on $B\times(0,T)$ is called an *energy solution* to $(\mathrm{KS})_\Omega$ on $[0,T)$ if

- $u\in L^\infty(0,T;L^\infty(B))$, $\nabla u^m\in L^\infty(0,T;L^2(B))$, $(u^{\frac{m+1}{2}})_t\in L^2(0,t;L^2(B))$ $(\forall t<T)$,
- $v\in L^\infty(0,T;H^1(B))$, $v_t\in L^2(0,T;L^2(B))$,
- $(u,v)$ satisfies $(\mathrm{KS})_\Omega$ in the sense of distributions, i.e., for all $\varphi\in L^1(0,T;H^1(B))\cap W^{1,1}(0,T;L^2(B))$ with compact support $\operatorname{supp}\varphi(x)\subset[0,T)$ (a.a. $x\in B$),

$$
\begin{aligned}
\int_0^T\!\!\int_B(\nabla u^m\cdot\nabla\varphi - u^{q-1}\nabla v\cdot\nabla\varphi - u\varphi_t)\,dxdt &= \int_B u_0(x)\varphi(x,0)\,dx,\\
\int_0^T\!\!\int_B(\nabla v\cdot\nabla\varphi + v\varphi - u\varphi - v\varphi_t)\,dxdt &= \int_B v_0(x)\varphi(x,0)\,dx,
\end{aligned}
$$

- $(u,v)$ satisfies the following energy estimate for a.a. $t\in(0,T)$,

$$
\frac{2e^{-2t}}{(m+1)^2}\int_0^t\!\!\int_B\left|(u^{\frac{m+1}{2}})_t\right|^2dxds + \frac{1}{2m}\int_B|\nabla(u^m(t))|^2\,dx \le K, \tag{3.1}
$$

where $K$ is a positive constant depending on $\|u_0\|_{L^1\cap L^2}$, $\|\nabla u_0^m\|_{L^2}$, $\|v_0\|_{H^1\cap W^{1,\infty}}$, $\|u\|_{L^\infty(0,T;L^\infty(B))}$, $m$, $q$, $N$, $|B|$.

We next define a maximal existence time and a blow-up for $(\mathrm{KS})_\Omega$.

**Definition 3.2.** A *maximal existence time* $T_{\max}$ for (KS) is defined as

$$T_{\max} := \sup\left\{T > 0;\ \text{there exists an energy solution to (KS) on } [0,T)\right\}.$$

**Definition 3.3.** For $T \in (0,\infty]$ let $(u,v)$ be an energy solution to (KS) on $[0,T)$. If

$$\begin{aligned} &\text{ess-}\limsup_{t\to T}\|u(t)\|_{L^\infty(B)} = \infty,\\ &\text{i.e., } \forall M > 0\ \exists T_0 < T\ \forall t \geq T_0;\ \|u(s)\|_{L^\infty(B)} \geq M \text{ for a.a. } s \in (t,T), \end{aligned}$$

then we say that $(u,v)$ *blows up* at $T$.

Now, we state the main theorem.

**Theorem 3.1** (Ishida-Y. [**17**]). *Let $N \geq 2$, $m \geq 1$ and $q \geq 2$. Then the following hold:*
**(I)** (Local existence) *Assume that $m$ and $q$ satisfy*

$$q \geq \frac{m+1}{2}.$$

*Then for every nonnegative initial data $(u_0, v_0) \in L^\infty(B) \times W^{1,\infty}(B)$ with $\nabla u_0^m \in L^2(B)$, there exists $T > 0$ such that $(\mathrm{KS})_\Omega$ admits an energy solution $(u,v)$ on $[0,T)$. Moreover, if $(u_0,v_0)$ is radially symmetric, then so is $(u,v)$.*
**(II)** (Blow-up) *Assume that $m$ and $q$ satisfy*

$$q > m + \frac{2}{N}.$$

*Let $T_{\max}$ be a maximal existence time for $(\mathrm{KS})_\Omega$. Then there exists a positive constant $C := C(\|u_0\|_{L^1}, N)$ such that every radially symmetric energy solution to $(\mathrm{KS})_\Omega$ with nonnegative initial data $(u_0,v_0) \in L^\infty(B) \times W^{1,\infty}(B)$ with $\nabla u_0^m \in L^2(B)$ fulfilling*

$$\exists\, r_0 > 0;\ G(u_0) := \int_{r_0}^{u_0}\int_{r_0}^{\sigma} \xi^{m-q}\, d\xi d\sigma \in L^1(B)$$

*as well as*

$$L(u_0,v_0) := \int_B \left(G(u_0) - u_0 v_0 + \frac{1}{2}|\nabla v_0|^2 + \frac{1}{2}v_0^2\right) dx < -C,$$

*blows up in either finite or infinite time.*

The strategy for the proof of this theorem follows the well-known strategy introduced to chemotaxis problems independently in Horstmann [8], Senba-Suzuki [**21**]. They consist of finding the lower bound $c_0$ of the Lyapunov function on the radially symmetric steady states and showing that one can find initial data admitting the value of the Lyapunov function smaller than $c_0$. However, the proof of Theorem 3.1 has two difficulties. One is to construct a local-in-time "energy solution" to (KS), the other is to show that any energy solution satisfies an important estimate for the Lyapunov function for (KS). In particular, the energy estimate (3.1) plays a central role in our argument that we derive a contradiction by assuming uniform-in-time boundedness of $u(t)$ on $(0,\infty)$, because (3.1) assures a limit of $u(t)$ as $t \to \infty$ under boundedness of $u$ via compactness methods.

**Proof of Theorem 3.1 (energy estimates).** We derive only the energy estimate (3.1) for solutions to $(\mathrm{KS})_\Omega$ which is key to the proof as stated above. Let $n \in \mathbb{N}$ and $T \in (0,\infty]$. Let $(u,v)$ be a solution to $(\mathrm{KS})_\Omega$ on $[0,T)$. By a suitable approximation procedure we may assume that $u$ is smooth and the following mass conservation low holds:

$$\|u(t)\|_{L^1(B)} = \|u_0\|_{L^1}, \quad t \in [0,T). \tag{3.2}$$

Assume that $u$ is bounded on $B \times [0,T)$, that is,

$$\|u\|_{L^\infty(0,T;L^\infty(B))} < \infty.$$

Then the standard technique for inhomogeneus linear heat equations entails that the following estimates hold:

$$\|v(t)\|_{W^{1,\infty}(B)} \le K_1 \quad (\forall t \in [0,T)), \tag{3.3}$$

$$\|v(t)\|^2_{L^2(B)} + 2\int_0^t e^{2(s-t)} \int_B |\nabla v(s)|^2\, dxds \le K_2 \quad (\forall t \in [0,T)), \tag{3.4}$$

$$\|\nabla v(t)\|^2_{L^2(B)} + \int_0^t e^{2(s-t)} \int_B |\Delta v(s)|^2\, dxds \le K_3 \quad (\forall t \in [0,T)), \tag{3.5}$$

where

$$\begin{aligned}
K_1 &:= \|v_0\|_{W^{1,\infty}} + (1 + C(N)\sqrt{\pi})\|u\|_{L^\infty(0,T;L^\infty(B))},\\
K_2 &:= \|v_0\|^2_{L^2} + 2K_1\|u_0\|_{L^1},\\
K_3 &:= \|\nabla v_0\|^2_{L^2} + |B|\|u\|^2_{L^\infty(0,T;L^\infty(B))},
\end{aligned}$$

where $C(N)$ is a positive constant. We now multiply the first equation in $(\mathrm{KS})_\Omega$ by $u$ and integrate it over $B$. Then using the Young inequality and noting

$$2q - m - 1 \ge 0,$$

we obtain the following estimate:

$$\begin{aligned}
\frac{d}{dt}\int_B u^2\, dx &\le -m\int_B u^{m-1}|\nabla u|^2\, dx + \frac{1}{m}\int_B u^{2q-m-1}|\nabla v|^2\, dx\\
&\le -\frac{4m}{(m+1)^2}\int_B \left|\nabla (u^{\frac{m+1}{2}})\right|^2 dx + \frac{1}{m}\|u\|^{2q-m-1}_{L^\infty(B\times(0,t))}\int_B |\nabla v|^2\, dx.
\end{aligned} \tag{3.6}$$

Multiplying (3.6) by $e^{2s}$ and integrating it over $(0,t)$, we see from (3.4) that

$$\begin{aligned}
&\|u(t)\|^2_{L^2(B)} + \frac{4m}{(m+1)^2}\int_0^t e^{2(s-t)}\int_B \left|(\nabla u)^{\frac{m+1}{2}}\right|^2 dxds\\
&\le e^{-2t}\|u_0\|^2_{L^2(B)} + (1-e^{-2t})\|u\|^2_{L^\infty(0,t;L^2(B))} + \frac{K_2}{2m}\|u\|^{2q-m-1}_{L^\infty(0,t;L^\infty(B))}\\
&\le e^{-2t}\|u_0\|^2_{L^2(B)} + |B|\|u\|^2_{L^\infty(0,t;L^\infty(B))} + \frac{K_2}{2m}\|u\|^{2q-m-1}_{L^\infty(0,t;L^\infty(B))}\\
&=: K_4'.
\end{aligned} \tag{3.7}$$

Next, multiplying the first equation in $(\mathrm{KS})_\Omega$ by $u^{m-1}u_t = \frac{1}{m}(u^m)_t$ and integrating it over $B$, we have

$$\int_B u^{m-1}|u_t|^2\,dx = -\frac{1}{2m}\frac{d}{dt}\int_B |\nabla(u^m)|^2\,dx - \int_B \nabla\cdot\left(u^{q-1}\nabla v\right)\,u^{m-1}u_t\,dx.$$

It follows from the inequality $ab \le \frac{1}{2}(a^2+b^2)$ $(a, b \ge 0)$ that

$$\frac{1}{2}\int_B u^{m-1}|u_t|^2\,dx \le -\frac{1}{2m}\frac{d}{dt}\int_B |\nabla(u^m)|^2\,dx + \frac{1}{2}\int_B \left|\nabla\cdot\left(u^{q-1}\nabla v\right)\right|^2 u^{m-1}\,dx. \tag{3.8}$$

We consider the estimate for the last term on the right-hand side of (3.8). Noting that $|\nabla\cdot(A\nabla B)|^2 = |\nabla A\cdot\nabla B + A\Delta B|^2 \le 2\left(|\nabla A\cdot\nabla B|^2 + |A\Delta B|^2\right)$, we see from (3.3) that

$$\begin{aligned}\frac{1}{2}\int_B \left|\nabla\cdot\left((u^{q-1}\nabla v\right)\right|^2 u^{m-1}\,dx &\le \int_B \left\{\left|u^{q-1}\Delta v\right|^2 + \left|\nabla\left(u^{q-1}\right)\cdot\nabla v\right|^2\right\}u^{m-1}\,dx \\ &\le \|u\|^{2q+m-3}_{L^\infty(B\times(0,t))}\int_B |\Delta v|^2\,dx \\ &\quad + \frac{4(q-1)^2K_1^2}{(m+1)^2}\|u\|^{2(q-2)}_{L^\infty(B\times(0,t))}\int_B \left|\nabla\left(u^{\frac{m+1}{2}}\right)\right|^2 dx.\end{aligned}$$

Combining this inequality with (3.8) and noting that $u^{m-1}|u_t|^2 = \frac{4}{(m+1)^2}\left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2$, we deduce that

$$\begin{aligned}\frac{2}{(m+1)^2}\int_B \left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 dx \le -\frac{1}{2m}\frac{d}{dt}\int_B |\nabla(u^m)|^2 + \|u\|^{2q+m-3}_{L^\infty(B\times(0,t))}\int_B |\Delta v|^2\,dx \\ + \frac{4(q-1)^2K_1^2}{(m+1)^2}\|u\|^{2(q-2)}_{L^\infty(B\times(0,t))}\int_B \left|\nabla\left(u^{\frac{m+1}{2}}\right)\right|^2 dx.\end{aligned}$$

Multiplying this inequality by $e^{2t}$ and integrating it over $(0, t)$ yield that

$$\begin{aligned}&\frac{2}{(m+1)^2}\int_0^t e^{2s}\int_B \left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 dxds + \frac{1}{2m}e^{2t}\int_B |\nabla(u(t)^m)|^2\,dx \\ &\le \frac{1}{2m}\int_B |\nabla(u_0^m)|^2\,dx + \frac{1}{m}\int_0^t e^{2s}\int_B |\nabla(u^m)|^2\,dxds + \|u\|^{2q+m-3}_{L^\infty(B\times(0,t))}\int_0^t e^{2s}\int_B |\Delta v|^2\,dxds \\ &\quad + \frac{4(q-1)^2K_1^2}{(m+1)^2}\|u\|^{2(q-2)}_{L^\infty(B\times(0,t))}\int_0^t e^{2s}\int_B \left|\nabla\left(u^{\frac{m+1}{2}}\right)\right|^2 dxds.\end{aligned}$$

Then, noting that

$$|\nabla(u^m)|^2 \le \frac{4m^2}{(m+1)^2}\|u\|^{m-1}_{L^\infty(B\times(0,t))}\left|\nabla\left(u^{\frac{m+1}{2}}\right)\right|^2,$$

we obtain

$$\begin{aligned}&\frac{2}{(m+1)^2}\int_0^t e^{2(s-t)}\int_B \left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 dxds + \frac{1}{2m}\int_B |\nabla(u(t)^m)|^2\,dx \\ &\le \frac{1}{2m}e^{-2t}\int_B |\nabla(u_0^m)|^2\,dx + \|u\|^{2q+m-3}_{L^\infty(B\times(0,t))}\int_0^t e^{2(s-t)}\int_B |\Delta v|^2\,dxds \\ &\quad + \frac{4}{(m+1)^2}\left(\|u\|^{m-1}_{L^\infty(B\times(0,t))} + (q-1)^2K_1^2\|u\|^{2(q-2)}_{L^\infty(B\times(0,t))}\right)\int_0^t e^{2(s-t)}\int_B \left|\nabla\left(u^{\frac{m+1}{2}}\right)\right|^2 dxds.\end{aligned} \tag{3.9}$$

Applying (3.5) and (3.7) to the second and last terms on the right-hand side of (3.9), respectively, we see that

$$(3.10)\qquad \begin{aligned} &\frac{2}{(m+1)^2}\int_0^t e^{2(s-t)}\int_B \left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 dxds + \frac{1}{2m}\int_B |\nabla(u(t)^m)|^2\,dx \\ &\le \frac{1}{2m}e^{-2t}\int_B |\nabla(u_0^m)|^2\,dx + K_3\|u\|_{L^\infty(B\times(0,t))}^{2q+m-3} \\ &\quad + \frac{K_4'}{m}\left(\|u\|_{L^\infty(B\times(0,t))}^{m-1} + (q-1)^2K_1^2\|u\|_{L^\infty(B\times(0,t))}^{2(q-2)}\right), \end{aligned}$$

where $K_3$ and $K_4'$ are the same constants as in (3.5) and (3.7). Since $e^{2(s-t)} \ge e^{-2t}$ for $s \in (0,t)$, the energy estimate (3.1) follows from (3.10). □

## 4. Boundedness in parabolic-elliptic Keller-Segel systems with signal-dependent sensitivity on $\Omega$

In this section we especially focus on a model of chemotaxis processes where movement towards higher signal concentrations is inhibited at points where these concentrations are high. Such saturation effects are usually accounted for by introducing a signal dependent sensitivity function $\chi(v)$, i.e., by setting

$$A(u,v) = u\chi(v)$$

in (KS). Here of particular importance seems to be the prototypical choice

$$\chi(v) = \frac{\chi_0}{v}, \quad v > 0,$$

with some constant $\chi_0 > 0$, thus meaning that stimulus perception is governed by the Weber-Fechner law. This model was first proposed by Keller-Segel [**19**].

Thus we consider the questions of global existence and boundedness in the following parabolic-elliptic Keller-Segel system with signal-dependent sensitivity:

$$(\mathrm{KS})_{\chi(v)}\qquad \begin{cases} u_t = \Delta u - \nabla\cdot(u\chi(v)\nabla v), & x\in\Omega,\ t>0, \\ 0 = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases}$$

with $u(x,0) = u_0(x)$ and $\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0$ on $\partial\Omega$, where $\Omega\subset\mathbb{R}^N$ ($N\ge 2$) is a bounded domain with smooth boundary $\partial\Omega$. We assume that

$$(4.1)\qquad u_0 \ge 0, \quad u_0 \in C(\overline{\Omega}), \quad \int_\Omega u_0 > 0,$$

$$(4.2)\qquad \chi \in C^1((0,\infty)), \quad \chi > 0 \text{ on } (0,\infty).$$

When $\chi(v) = \frac{\chi_0}{v}$ ($\chi_0 > 0$), Biler [**1**] proved the global existence of weak solutions under the condition $\chi_0 < \frac{2}{N}$; however, the *boundedness* is left as an open problem. Independently, Nagai and Senba [**20**] studied radially symmetric solutions to the same system $(\mathrm{KS})_{\frac{\chi_0}{v}}$, and they showed that solutions are global and remain *bounded* when either $N\ge 3$ and $0<\chi_0<\frac{N}{N-2}$, or $N=2$ and $\chi_0>0$ is arbitrary. Concerning nonradial solutions, the boundedness question is still open even for the particular system $(\mathrm{KS})_{\frac{\chi_0}{v}}$.

The purpose of this section is to report a recent result by Fujie-Winkler-Y. [**4**] which gives an answer to the open question not only for $\chi(v) = \frac{\chi_0}{v}$ ($\chi_0 < \frac{2}{N}$) but also for a rather general $\chi(v)$. In order to formulate our main results in this direction, given a nonnegative $0 \not\equiv u_0 \in C^0(\bar{\Omega})$, let us introduce a positive constant $\gamma$ by defining

$$\gamma := \|u_0\|_{L^1(\Omega)} \int_0^\infty \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-t-\frac{(\operatorname{diam}\Omega)^2}{4t}}\, dt < \infty, \tag{4.3}$$

where $\operatorname{diam}\Omega := \max_{x,y\in\bar{\Omega}} |x-y|$. The particular role of $\gamma$ stems from the fact that it marks an a priori pointwise lower bound on the solution component $v$, as we shall see below.

**Theorem 4.1** (Fujie-Winkler-Y. [**4**]). *Let $N \geq 2$, and suppose that $u_0$ and $\chi$ satisfy* (4.1) *and* (4.2), *respectively. Moreover, assume that $\chi$ satisfies*

$$\chi(s) \leq \frac{\chi_0}{s^k} \quad \textit{for all } s \in [\gamma, \infty),$$

*with some $k \geq 1$ and some $\chi_0 > 0$ fulfilling*

$$\chi_0 < \begin{cases} \dfrac{2}{N} & \textit{if } k = 1, \\ \dfrac{2}{N} \cdot \dfrac{k^k}{(k-1)^{k-1}} \gamma^{k-1} & \textit{if } k > 1. \end{cases}$$

*Then* $(\mathrm{KS})_{\chi(v)}$ *possesses a unique global classical solution*

$$u \in C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\bar{\Omega})),$$
$$v \in C^{2,0}(\bar{\Omega} \times (0,\infty)) \cap C^0((0,\infty); C^0(\bar{\Omega})).$$

*Moreover, the solution component $u$ is uniformly bounded*:

$$\|u(\cdot,t)\|_{L^\infty} \leq M_\infty \quad \textit{for all } t \in [0,\infty)$$

*for some constant $M_\infty > 0$.*

**Remark 4.1.** We firstly remark that our result for $k = 1$ goes somewhat beyond that given in [**1**] in that it provides classical solutions, rather than weak solutions, and moreover it asserts their boundedness, thus ruling out any blow-up phenomenon in infinite time.

**Remark 4.2.** Secondly, unlike in [**1**] our proof does not depend on any particular structure of the system $(\mathrm{KS})_{\chi(v)}$ with $\chi(v) = \frac{\chi_0}{v}$.

**Remark 4.3.** We thirdly note that $\gamma$ depends on $\operatorname{diam}\Omega$ in such a way that $\gamma \to \infty$ as $\operatorname{diam}\Omega \to 0$; in particular, in the case $k > 1$ for each $\chi_0 > 0$ and any choice of the mass $m > 0$, our above condition will be satisfied for any $\Omega$ with sufficiently small diameter and all nonnegative $u_0 \in C^0(\bar{\Omega})$ having mass $\int_\Omega u_0 = m$.

**Remark 4.4.** Finally we observe that the assertion of Theorem 4.1 can be generalized to the case of the system $(\mathrm{KS})_{\chi(v)}$ with the growth (death) term $f(u)$, provided that $\lambda_1 - \mu_1 u \leq f(u) \leq \lambda_2 - \mu_2 u$ ($\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$). For details see Fujie-Y. [**3**].

We conclude this paper by giving the main part of the proof of Theorem 4.1.

**Proof of Theorem 4.1 ($L^p$-estimates).** We first give an a priori pointwise lower bound on the solution component $v$. In the same way as in the proof of Hillen-Painter-Winkler [**7**, Lemma 3.1], we can obtain the pointwise estimate from below

$$e^{t\Delta}\varphi(x) \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}\Omega)^2}{4t}} \cdot \int_\Omega w > 0 \quad (x \in \Omega, t > 0) \quad \text{for all} \quad (0 \leq)\ w \in C^0(\bar{\Omega}),$$

for the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ in $\Omega$. In light of the formula $(I-\Delta)^{-1}w = \int_0^\infty e^{-t}e^{t\Delta}w\,dt$, we have

$$(I-\Delta)^{-1}w = \int_0^\infty e^{-t}e^{t\Delta}w\,dt \geq \Big(\int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-t-\frac{(\operatorname{diam}\Omega)^2}{4t}}\,dt\Big) \cdot \int_\Omega w.$$

This explains the role of the constant $\gamma$ defined in (4.3). Namely, since $(\mathrm{KS})_{\chi(v)}$ evidently preserves the norm of the first solution component $u$ in $L^1(\Omega)$ and the second solution component $v$ is represented by $v = (I-\Delta)^{-1}u$, we can thereby estimate $v$ from below according to

$$\begin{aligned} v(x,t) &\geq \Big(\int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-t-\frac{(\operatorname{diam}\Omega)^2}{4t})}\,dt\Big) \cdot \int_\Omega u(x,t)\,dx \\ &= \|u_0\|_{L^1(\Omega)} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-(t+\frac{(\operatorname{diam}\Omega)^2}{4t})}\,dt \\ &= \gamma \quad \text{for all } x \in \Omega \text{ and } t \in (0,T), \end{aligned} \tag{4.4}$$

whenever $(u,v)$ solves $(\mathrm{KS})_{\chi(v)}$ in $\Omega \times (0,T)$ for some $T > 0$. We next derive the $L^p$-estimate for $u$. By virtue of the first equation in $(\mathrm{KS})_{\chi(v)}$, we have

$$\frac{d}{dt}\int_\Omega u^p \;=\; -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + p(p-1)\int_\Omega u^{p-1}\chi(v)\nabla u \cdot \nabla v.$$

In light of Young's inequality we deduce that

$$\frac{d}{dt}\int_\Omega u^p \leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p\chi^2(v)|\nabla v|^2. \tag{4.5}$$

Now let $\varphi \in C^1([\gamma,\infty))$ be nonnegative and such that there exists a constant $M > 0$ satisfying

$$s\varphi(s) \leq M \quad \text{for all } s \in [\gamma,\infty).$$

Using the second equation in $(\mathrm{KS})_{\chi(v)}$, we see that $\int_\Omega u^p\varphi(v)(\Delta v - v + u) = 0$. Here from the Neumann boundary condition it follows that

$$-p\int_\Omega u^{p-1}\varphi(v)\nabla u \cdot \nabla v - \int_\Omega u^p\varphi'(v)|\nabla v|^2 - \int_\Omega u^p\varphi(v)v + \int_\Omega u^{p+1}\varphi(v) = 0.$$

Noting that $u \geq 0$ and $\varphi(v) \geq 0$ imply that $\int_\Omega u^{p+1}\varphi(v) \geq 0$, we thus find that

$$\begin{aligned} -\int_\Omega u^p\varphi'(v)|\nabla v|^2 &\leq p\int_\Omega u^{p-1}\varphi(v)\nabla u\cdot\nabla v + \int_\Omega u^p\varphi(v)v \\ &\leq \frac{A^2}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{B^2}{2}\int_\Omega u^p\varphi^2(v)|\nabla v|^2 + M\int_\Omega u^p, \end{aligned}$$

where $A := \sqrt{p(p-1)-\varepsilon}$ and $B := \frac{p}{\sqrt{p(p-1)-\varepsilon}}$ $(\varepsilon < p(p-1))$. This implies that

$$\int_\Omega u^p\Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v)\Big)|\nabla v|^2 \leq \frac{A^2}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p. \tag{4.6}$$

By assumption we can find a function $\varphi$ such that the Riccati inequality

$$\frac{p(p-1)}{2}\chi^2(v) \leq -\varphi'(v) - \frac{B^2}{2}\varphi^2(v)$$

holds for $p \in \Big[1, \frac{1}{\chi_0}\cdot\frac{k^k}{(k-1)^{k-1}}\gamma^{k-1}\Big)$ (for details see [**4**]). By virtue of this inequality, we can now combine (4.6) with (4.5) to achieve the inequality

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p &\leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega u^p\chi^2(v)|\nabla v|^2 \\ &\leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^p\Big(-\varphi'(v) - \frac{B^2}{2}\varphi^2(v)\Big)|\nabla v|^2 \\ &\leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p(p-1)-\varepsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p \\ &= -\frac{\varepsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + M\int_\Omega u^p. \end{aligned} \tag{4.7}$$

Now invoking the Gagliardo-Nirenberg inequality, we see that

$$\int_\Omega u^p = \|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} \leq C_{GN}\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\Big)^{2a}\|u^{\frac{p}{2}}\|^{2(1-a)}_{L^{\frac{2}{p}}(\Omega)}, \tag{4.8}$$

where $C_{GN}$ is a positive constant and

$$a := \frac{\frac{p}{2}-\frac{1}{2}}{\frac{p}{2}+\frac{1}{N}-\frac{1}{2}} \in (0,1). \tag{4.9}$$

Since according to the mass conservation property we have

$$\|u^{\frac{p}{2}}(\cdot,t)\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} = \int_\Omega u(x,t)\,dx = \int_\Omega u_0(x), \tag{4.10}$$

we infer from (4.8) and (4.10) that $\int_\Omega u^p \leq K\big(\|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} + 1\big)^a$ for some $K > 0$, so that we have

$$\int_\Omega u^{p-2}|\nabla u|^2 = \frac{4}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \geq \frac{4}{K^{\frac{1}{a}}p^2}\Big(\int_\Omega u^p\Big)^{\frac{1}{a}} - \frac{4}{p^2}. \tag{4.11}$$

Inserting (4.11) into (4.7), we obtain

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{2\varepsilon}{K^{\frac{1}{a}}p^2}\Big(\int_\Omega u^p\Big)^{\frac{1}{a}} + M\int_\Omega u^p + \frac{2\varepsilon}{p^2}.$$

Consequently, $y(t) := \int_\Omega u^p(x,t)\,dx$ satisfies $y'(t) \le -C_1 y^{\frac{1}{a}}(t) + C_2 y(t) + C_3$ with certain positive constants $C_1$, $C_2$ and $C_3$. In view of (4.9), we have $\frac{1}{a} > 1$ and thus a standard ODE comparison argument implies the boundedness of $y$ on $(0, T_{\max})$. Thus we conclude that $\|u(\cdot,t)\|_{L^p(\Omega)} \le M_p < \infty$ holds for all $t \in (0,T)$ and some $M_p > 0$. From this estimate we can obtain the assertion of Theorem 4.1 (see [**4**]). □

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