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Multiple points blow-up for the Keller–Segel system

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1 Introduction

This is an announcement of the authors' recent study on aggregation phenomena for the two-dimensional Keller–Segel system:

$$u_t = \Delta u - \nabla \cdot (u \nabla v), \qquad x \in \mathbb{R}^2,\ t > 0, \tag{1.1a}$$

$$0 = \Delta v + u, \qquad x \in \mathbb{R}^2,\ t > 0, \tag{1.1b}$$

supplemented with initial data $u(\cdot, 0) = u_0 \in (C^2 \cap L^1)(\mathbb{R}^2)$, $u_0 \geq 0 (\not\equiv 0)$ satisfying:

$$\int_{\mathbb{R}^2} |x|^2 u_0(x) dx < \infty. \tag{1.2}$$

Under these assumptions the Cauchy problem (1.1) admits a loca-in-time classical solution satisfying

$$u > 0, \qquad \int_{\mathbb{R}^2} u(x,t) dx = \int_{\mathbb{R}^2} u_0(x) dx, \qquad 0 < t < T, \tag{1.3}$$

where $T \leq +\infty$ stands for the maximal existence time of the solution. Here the solution is unique up to addition of constant to v. It is well known that if

$$M := \int_{\mathbb{R}^2} u_0(x) dx > 8\pi$$

and condition (1.2) is satisfied, then the solution (u, v) of (1.1) blows up in a finite time (cf. [7]). Namely, the maximal existence time T is finite and there holds

$$\limsup_{t \nearrow T} \|u(\cdot, t)\|_\infty = +\infty,$$

where $\|\cdot\|_\infty$ stands for the L^∞ norm. The time T is called blow-up time. A point $x_0 \in \mathbb{R}^2$ is called a blow-up point if there exist $\{(x_n, t_n)\}$ with $x_n \to x_0$ and $t_n \nearrow T$ such that $u(x_n, t_n) \to \infty$ as $n \to \infty$. Blow-up set S is defined, as usual, as the set of all blow-up points.

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System (1.1) was introduced in [6] as a simplified system of the original fully parabolic system (1.1) with equation (1.1b) replaced by a paraboic equation:

$$v_t = \Delta v + u, \qquad x \in \mathbb{R}^2,\, t > 0.$$

It is a classical model to describe aggregation phenomenon and has been intensively studied by many researchers. In this model u and v denote the density of a biological organism and the concentration of a chemical substance produced by the organism having chemoattractant properties, respectively.

A remarkable features of system (1.1) is the existence of critical mass m_0 such that for solutions with initial total mass of organism of u_0 larger than m_0, a finite-time blow-up takes place, whereas smaller values of u_0 yield global-in-time solutions. See [1, 3, 6, 8, 9].

An important aspect of the Keller–Segel system consists in the onset of chemotactic aggregation (referred also as chemotactic collapse) cf. [2, 4]. This term refers to the fact that concentration of biological organism happens in finite time. Mathematically, this amounts to the question that whether or not Dirac mass would be produced in u at the blow-up time. The papers mentioned above don't conclude that chemotactic aggregation occurs as a consequence of blow-up. In fact, the description of such blow-up mechanisms is far from obvious. The first example of a blow-up describing chemotactic aggregation was obtained by Herrero and Velázquez in [4] by matched asymptotic expansions and its actual existence was rigorously proven in [5]. More precisely, they constructed a radial blow-up solution satisfying:

$$\begin{aligned} u(x,t) &\rightharpoonup 8\pi\delta + \psi \quad \text{in the sense of measures as } t \nearrow T, \\ \psi(x) &= \frac{C}{|x|^2} \exp\left(-2\sqrt{|\log|x||}\right) \quad \text{as } |x| \to 0, \end{aligned}$$

where T is the blow-up time and C is a positive constant. We note that much more detailed asymptotic behavior of the solution is obtained therein. As for the general case, Senba and Suzuki [14] studied the Cauchy–Neumann problem on a bounded domain $\Omega \subset \mathbb{R}^2$:

$$u_t = \Delta u - \nabla \cdot (u \nabla v), \quad x \in \Omega,\, t > 0, \tag{1.4a}$$

$$0 = \Delta v - v + u, \quad x \in \Omega,\, t > 0, \tag{1.4b}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad t > 0, \tag{1.4c}$$

and obtained the following result: if a solution of (1.1) blows up in a finite time T, then its blow-up set S consists of a finite number of points and there exist

$$f \in L^1(\Omega) \cap C\left(\overline{\Omega} \setminus S\right), \; m(q) \geq m^*(q) := \begin{cases} 8\pi, & q \in \Omega, \\ 4\pi, & q \in \partial\Omega \end{cases} \tag{1.5}$$

such that

$$u(\cdot, t) \rightharpoonup \sum_{q \in S} m(q)\delta + f \quad \text{as } t \nearrow T \tag{1.6}$$

in the sense of measures. It was also shown there that

$$8\pi\sharp(S \cap \Omega) + 4\pi\sharp(S \cap \partial\Omega) \leq \|u_0\|_{L^1(\Omega)}. \tag{1.7}$$

This estimate yields an upper bound for the number of blow-up points. Similar results may be obtained for our system posed in the whole $\mathbb{R}^2$. This is because most of the argument in [14] is local in space and the term $-v$ in (1.4b) plays no important role on the blow-up behavior.

Questions that naturally arise from the above results are:

1. Can the constant $m(q)$ in (1.6) be strictly larger than 8π?

2. Can the number blow-up points be greater than one?

It is worth pointing out that the answers are negative for the both questions if only radial blow-up solutions are concerned [13]. Therefore we have to study blow-up of non-radial solutions to get affirmative answers to those questions. As for the first question, numerical simulation by [11] shows that there exits a blow-up mechanism with $m(q)$ being a multiple of 8π. Moreover, the authors have formally constructed such a blow-up solution by means of matched asymptotics [12].

This announcement reports a positive answer to the second question. The result may be roughly stated as follows:

Theorem 1.1. *Given any positive integer n, one may construct an initial datum such that the corresponding solution of* (1.1) *blows up in a finite time at exactly n-points in* $\mathbb{R}^2$.

More precise statement of this theorem is to be given in §3. See Theorem 3.1.

The rest of this report consists of two sections. In §2 we consider a sufficient condition that ensures a local bound near a given point in $\mathbb{R}^2$. This is a key tool to prove Theorem 1.1. We introduce a sketch of the proof of Theorem 1.1 in §3.

2 ε-regularity

The following ε-regularity result was implicitly but essentially obtained by Senba and Suzuki [14] for problem (1.4) on bounded domains. This result may be regarded as a localized version of L^p bounds obtained by Nagai, Senba, and Yoshida [10] for a fully parabolic system. The idea of estimating L^p norms goes back to Jäger and Luckhaus [6]. The formulation of Proposition 2.1 below is due to Sugiyama [15], where the same type of ε-regularity result was established for a system of equations with degenerate diffusion in the whole $\mathbb{R}^2$ that includes system (1.4) as a particular case.

Proposition 2.1. *Assume that condition* (1.2) *holds. Let T denote the blow-up time of the solution (u,v) with initial data u_0. Then there exists a positive constant ε such that if*

$$\sup_{0<t<T} \int_{B_R(a)} u(x,t)dx < \varepsilon \tag{2.1}$$

for some $R>0$ and $a \in \mathbb{R}^2$, then for every $R' < R$ there holds

$$\sup_{0<t<T} \|u(\cdot,t)\|_{L^\infty(B_{R'}(a))} \leq C^* \tag{2.2}$$

for some positive constant C^* depending only on ε, T, M, R', and R. In particular, the point a cannot be a blow-up point.

Remark 2.2. The constant ε is given, for example, by $1/16K^2$ with K being a positive constant appearing in Gagliard–Nirenberg's inequality:

$$\|u\|^q_{L(\Omega)} \leq K\|u\|^{1-\lambda}_{W^{1,2}(\Omega)}\|u\|^{\lambda}_{L^1(\Omega)},$$
$$\lambda = 1 - \frac{1}{q}, \quad 1 \leq q < +\infty,$$

where Ω us a domain of $\mathbb{R}^2$ with C^1 boundary.

It is possible, however, to replace ε by an optimal constant 8π in (2.1), although the constant C^* in (2.2) may depend also on the center of the ball, i.e., a in that case. Indeed, Senba and Suzuki [14, Lemma 9] proves that if a solution of (1.4) blows up in a finite time T, then for every blow-up point x_0 and $0 < R$ sufficiently small, we have

$$\liminf_{t \nearrow T} \int_{B_R(x_0)\cap\Omega} u(x,t)dx \geq m^*, \tag{2.3}$$

where m^* is the constant in (1.5). Hence condition (2.1) with $\varepsilon = 8\pi$ implies that the solution is locally bounded in a neighborhood of x_0 provided that x_0 is not on the boundary $\partial\Omega$.

Remark 2.3. The following proof is nothing but a modification of the argument in [14] so as to work for system (1.1) in unbounded domains. If equation (1.1b) is replaced by $0 = \Delta v - v + u$, function v may be represented by the Bessel kernel that decays exponentially as $|x| \to \infty$. On the other hand, the Newtonian kernel grows logarithmically as $|x| \to \infty$. This is a main difference from the previous articles mentioned above.

Proof. A key point of [14] is to prove the uniform bound of $\int_\Omega u \log u dx$ in $(0,T)$ implies a local L^∞ bound of u, where Ω is a bounded domain under consideration. To this aim uniform L^p bounds of v play essential roles. Since our problem for (1.1) is posed on the whole $\mathbb{R}^2$, we need to take into account of the logarithmic growth of the Newtonian kernel at spatial infinity.

We only have to prove :

$$\sup_{0<t<T} \|v(\cdot,t)\|_{L^p(\Omega)} < +\infty \tag{2.4}$$

for any bounded domain Ω of $\mathbb{R}^2$, since there is no change from [14] in remaining arguments. Let $L > 0$ be a constant such that the ball $B = B_L := \{|x| < L\}$ contains Ω. Since we know $v = G * u$ with $G = -(1/2\pi)\log|x|$ being the Newtonian kernel, we may write

$$\begin{aligned} v(x,t) &= \int_{\mathbb{R}^2} G(x-y)u(y,t)dy \\ &= \left(\int_{|y|\leq L} + \int_{|y|\geq L}\right) G(x-y)u(y,t)dy =: v_1(x,t) + v_2(x,t). \end{aligned}$$

Application of Young's inequality for convolution to v_1 readily yields

$$\|v_1(\cdot,t)\| \leq \|G\|_{L^p(B)}\|u(\cdot,t)\|_{L^1(\Omega)} \leq C(p,L,M)$$

for $0 < t < T$. To estimate v_2, let us write

$$v_2(x,t) \le \int_{|y|\ge L} \frac{|G(x-y)|}{1+|x-y|^2}\left(1+|x-y|^2\right)u(y,t)dy$$
$$\le C(L)\left(\tilde{G} * f_t\right)(x) \tag{2.5}$$

with $C(L) = 1 + 2L^2$, $\tilde{G}(x) = |G(x)|/(1+|x|^2)$, and $f_t(x) = (1+2|x|^2)u(x,t)\chi_{\{|x|>L\}}$ is the indicator function of the set $\{x \in \mathbb{R}^2;\ |x| > L\}$. An elementary computation shows

$$\sup_{0<t<T} \|\tilde{G}\|_{L^p(B)} < +\infty \qquad \text{for every } p > 1.$$

On the other hand, the uniform estimate

$$\sup_{0<t<T} \|f_t\|_{L^1(B)} < +\infty$$

follows from a standard symmetrization argument if the second moment of u_0 is finite, that is, condition (1.2) holds. The desired bound for v_2 thus follows from Young's inequality for convolution. We therefore obtain (2.4). Once estimate (2.4) is established, we may obtain a uniform local L^1 estimate for $u \log u$ by using an argument in [14] and thus get a uniform local L^∞ estimate as in (2.2) by Moser's iteration scheme. □

3 Multiple points blow-up

In this section we state our result on the construction of a blow-up solution having multiple blow-up points and introduce some ideas of the proof. Given an integer $n \ge 2$, we set an initial datum as follows:

$$u_0(x) = u_{0,n}(x) := \sum_{k=0}^{n-1} \phi_\rho(x - a_k), \tag{3.1a}$$

$$\phi_\rho(x) = \frac{1}{\rho^2}\phi_1\left(\frac{x}{\rho}\right), \quad \rho > 0, \tag{3.1b}$$

$$a_k = S(k\theta_n)a, \quad a \in \mathbb{R}^2 \setminus \{0\},\ k = 1, 2, ..., n-1 \tag{3.1c}$$

with

$$S(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2), \quad \theta_n = \frac{2\pi}{n}, \tag{3.1d}$$

where $\phi \in C_0^2(\mathbb{R}^2)$ is a radially symmetric nonnegative function satisfying

$$8\pi < \int_{\mathbb{R}^2} \phi_1(x)dx < \left(8 + \frac{1}{n}\right)\pi, \quad \operatorname{supp}\phi_1 = \overline{B_1(0)} \tag{3.2}$$

and a is a point on the horizontal axis. Under the particular choice of initial datum as in (3.1c), the solution of (1.1) with $u_0 = u_{0,n}$ is $2\pi/n$-rotation invariant, i.e.,

$(u(x,t), v(x,t)) = (u(S(\theta_n)x,t), v(S(\theta_n)x,t))$ for any $x \in \mathbb{R}^2$ and $t \in (0,T)$ due to uniqueness of classical solution. As a matter of fact, the solution blows up in a finite time T since

$$8n\pi < M = \|u_{0,n}\|_{L^1(\mathbb{R}^2)} < (8n+1)\,\pi. \tag{3.3}$$

Notice that the last inequality of (3.3) implies that the number of blow-up point cannot be greater than n (cf. (1.7)). We are going to choose ρ sufficiently small and $|a|$ large enough in order that the solution may blow up at exactly n points. Indeed, the blow-up set of the solution consists of either

a singleton at the origin

or

n distinct points at the vertexes of a regular n-sided polygon if $n \geq 3$;
two points at the ends of a line segment if $n = 2$.

We want to select the free constants ρ and $|a|$ so appropriately that the former possibility is ruled out. To state our theorem, we set

$$\varepsilon_0 := 8n\pi - M > 0.$$

We may now state the claim of Theorem 1.1 in a quantitative manner.

Theorem 3.1. *Let n be a positive integer and let $u_{0,n}$ be the function as in* (3.1c). *Consider the solution of* (1.1) *with initial datum $u_0 = u_{0,n}$. Then there exist positive constants ρ^* and α^* depending only on ε, ε_0, and M such that if $|a| \geq \alpha^*$ and $0 < \rho < \rho^*$, then the blow-up set of the solution consists only of n distinct points.*

Let us introduce a way of constructing the blow-up solution. The essence of Theorem 3.1 may be explained as follows. We shall consider the case $n = 2$ for simplicity, for which the blow-up set S is of the form: $S = \{\pm a'\}$ for some $a' \in \mathbb{R}^2$. To prove the desired result it suffices to see $a' \neq 0$, i.e., $0 \notin S$. Proposition 2.1 works well for this aim. In order to estimate a local mass around the origin, we pick a constant $R > 1$ such that

$$2R < |a|\sqrt{2(1-\cos\theta_n)} - \rho \tag{3.4}$$

and consider a cut-off function $\zeta_R \in C_0^2(\mathbb{R}^2)$ such that

$$0 \leq \zeta_R \leq 1, \tag{3.5a}$$

$$\zeta_R \equiv 1 \quad \text{for } |x| \leq L := \sqrt{R}, \tag{3.5b}$$

$$\zeta_R \equiv 0 \quad \text{for } |x| \geq R, \tag{3.5c}$$

$$|\nabla\zeta_R| \leq \frac{A}{R}, \quad |\nabla^2\zeta_R| \leq \frac{A}{R^2}, \tag{3.5d}$$

where $A > 0$ is a universal constant. We then multiply the function ζ_R to equation (1.1a) and integrate in x over $\mathbb{R}^2$. Having $\int_{\mathbb{R}^2} u_{0,n}(x)\zeta_R(x)dx = 0$ from the definition of ζ_R, we easily obtain

$$\sup_{0<t<T}\int_{\mathbb{R}^2} u(x,t)\zeta_R(x)dx \leq \frac{A(M+M^2)}{R^2}T \tag{3.6}$$

by using a symmetrization argument. Proposition 2.1 now becomes relevant. Notice that as $|a| \to +\infty$, the constant R may be chosen as large as we want to. We are then led to the question that if the blow-up time T would be much smaller than R^2 as $|a| \to +\infty$.

In other word, the graph of function $x \mapsto u(x,t)$ has two peaks initially located at $x = \pm a$ and they would approach each other as $t \to T$. One may expect that if $|a|$ is large enough, that is, the distance between these peaks are very far, then the blow-up would occur before these peaks collapse, whence the blow-up takes place at two points. This heuristic argument would sound canonical, but ignores how the blow-up time T can be large when $|a|$ is made large. Indeed, a standard argument using the second moment of function u proves that a finite-time blow-up does occur whenever $M > 8\pi$ and gives simultaneously an upper bound of the blow-up time by virtue of initial data. This rather standard estimate is, however, insufficient for our aim. Indeed, we have

$$\int_{\mathbb{R}^2} |x-b|^2 u(x,t)dx - \int_{\mathbb{R}^2} |x-b|^2 u_0(x)dx = -\frac{M}{2\pi}(M-8\pi)\,t \tag{3.7}$$

for every $0 < t < T$ and any $b \in \mathbb{R}^2$, whence

$$T \le D^{-1} \inf_{b\in\mathbb{R}^2} \int_{\mathbb{R}^2} |x-b|^2 u_0(x)dx, \tag{3.8}$$

where $D = M(2\pi)^{-1}(M-8\pi) > 0$. Substituting $u_0(x) = u_{0,n}(x)$ in (3.8), we obtain

$$T \le D^{-1} \inf_{b\in\mathbb{R}^2} \sum_{k=0}^{n-1} \int_{|x-a_k|<\rho} |x-b|^2 \phi_\rho(x-a_k)dx = O\left(|a|^2\right) \tag{3.9}$$

as $|a| \to \infty$. Estimate (3.9) is too rough and doesn't imply $T/R^2 \ll 1$ as $|a| \to \infty$ so that Proposition 2.1 may not apply.

A better estimate for the blow-up time may be obtained by introducing a local second moment:

$$F_k(t) := \int_{\mathbb{R}^2} |x-a_k|^2 u(x,t)\zeta(x-a_k)dx, \qquad k = 0,1,...,n-1, \tag{3.10}$$

where $\zeta = \zeta_R$ is as before. Thanks to the cut-off function ζ, we have

$$F_k(0) = \int_{|x-a_k|<\rho} \phi_\rho(x-a_k)|x-a_k|^2 dx \le \left(8+\frac{1}{n}\right)\pi\rho^2, \qquad k = 0,1,...,n-1. \tag{3.11}$$

A bit long computation then reveals that

$$\frac{dF_k(t)}{dt} = 4\int_{\mathbb{R}^2} u(x,t)\zeta(x-a_k)dx - \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} u(x,t)\zeta(x-a_k)dx\right)^2 + \mathcal{E}_{k,R}(t) \tag{3.12a}$$

with

$$|\mathcal{E}_{k,R}(t)| \le \frac{C_M}{R^{1/4}}(1+t^2), \qquad 0 < t < T, \tag{3.12b}$$

where C_M is a positive constant depending only on M.

By a similar argument to that leading to (3.6), we may obtain

$$\int_{\mathbb{R}^2} u(x,t)\zeta(x-a_k)dx \geq 8\pi + \frac{\varepsilon_0}{n} - \frac{A(M+M^2)}{R^2}t, \qquad 0<t<T. \tag{3.13}$$

We are now ready for proving $T = O(1)$ as $|a| \to \infty$ by contradiction. Suppose that we have $T > 1$, which implies that the solution is defined at least for $0 \leq t \leq 1$. It then follows from (3.13) that $\int_{\mathbb{R}^2} u(x,t)\zeta(x-a_k)dx \geq 8\pi + \varepsilon_0/(2n)$, $0 < t \leq 1$, whence, by (3.11), (3.12a), and (3.12b),

$$F_k(t) \leq \left(8+\frac{1}{n}\right)\pi\rho^2 - \frac{\varepsilon_0}{8n\pi}\left(8\pi + \frac{\varepsilon_0}{2n}\right) + \frac{13C_M}{24R^{1/4}} < 0 \quad \text{at } t = \frac{1}{2}, \tag{3.14}$$

if R is chosen appropriately large and ρ is taken sufficiently small. We have thus arrived at a contradiction.

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