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On a semilinear evolution equation

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1 Problem

We shall consider a semilinear parbolic evolution equation arising from fluid mechanics. More specifically, we deal with the following system of equations of the motion of incompressible micropolar fluids.

$$\frac{\partial u}{\partial t}-\mu\Delta u+(u\cdot\nabla)u+\nabla p=f+2\chi\nabla\times\left(\omega-\frac{1}{2}\nabla\times u\right),$$
$$\frac{\partial\omega}{\partial t}-\alpha\Delta\omega-\beta\nabla(\nabla\cdot\omega)+(u\cdot\nabla)\omega=g-4\chi\left(\omega-\frac{1}{2}\nabla\times u\right),$$
$$\nabla\cdot u=0,$$

where the unknown functions are the velocity vector field u, the microrotation vector ω and the pressure p, and f and g are given vector fields, α, β, μ, χ are constants satisfying $\alpha>0$, $\alpha+\beta>0$, $\mu>0$ and $\chi>0$.

Micropolar fluid model is one of generalizations of the classical Navier-Stokes model. We shall discuss the solvability of the initial-boundary value problem of this system of equations in a bounded domain Ω with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$.

As for the boundary conditions, we assume that

$$u|_{\partial\Omega}=0,\quad \omega|_{\partial\Omega}=\left.\frac{\theta}{2}\nabla\times u\right|_{\partial\Omega},$$

where θ is a constant belonging to the interval $[0,1]$ (see [1], [3]).

For the case where $\theta=0$, i.e., both u and ω yield the homogeneous Dirichlet boundary conditions, there have been many results concerning this system of equations [4]. On the other hand, the case where $\theta\neq 0$ is not fully pursued yet. Furthermore, this boundary condition is not based on any physical principles, the well-posedness of the initial-boundary value problem is not clear. That is why we consider this problem.

Let us introduce a new variable $v:=\omega-\frac{\theta}{2}\nabla\times u$ and change the variables (u,ω) to (u,v). Then the original system of equations for u and ω will be rewritten in the following form.

$$\frac{\partial u}{\partial t} - (\mu+\chi)\Delta u + (u\cdot\nabla)u + \nabla p = f + 2\chi\nabla\times v - \theta\chi\Delta u,$$

$$\begin{aligned}\frac{\partial v}{\partial t} &- \alpha\Delta v - \beta\nabla(\nabla\cdot v) + (u\cdot\nabla)v + 4\chi v + \theta\chi\nabla\times(\nabla\times v)\\ &= g - \frac{\theta}{2}\nabla\times f - \frac{\theta}{2}\{(\mu+(1-\theta)\chi-\alpha)\nabla\times\Delta u\\ &\qquad + ((\nabla\times u)\cdot\nabla)u\} + 2(1-\theta)\chi\nabla\times u,\end{aligned}$$

$$\nabla\cdot u = 0,$$

$$u|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega} = 0,$$

$$u(\cdot,0) = u_0(\cdot), \qquad v(\cdot,0) = v_0(\cdot).$$

Notice that the boundary condition for the new unknown function v is reduced to the homogeneous boundary condition.

Our aim is to show the existence of a solution (u,v) local in time to this system of equations in the L^2-framework.

2 Function spaces and operators

We refer the details of mathematical facts to be mentioned below to the books [2], [4] and [5].

Let $\mathbb{C}_\sigma^\infty(\Omega)$ be the set of all smooth solenoidal vector functions in Ω with compact support, and $\mathbb{L}_\sigma^2(\Omega)$ and $\mathbb{H}_\sigma^1(\Omega)$ the closure of $\mathbb{C}_\sigma^\infty(\Omega)$ in $\mathbb{L}^2(\Omega)$ and in $\mathbb{H}^1(\Omega)$ respectively. We denote by $|\cdot|$ the norm of $\mathbb{L}^2(\Omega)$, $(\cdot,\cdot)$ the inner product of $\mathbb{L}^2(\Omega)$; $|\cdot|_\sigma$ the norm of $\mathbb{L}_\sigma^2(\Omega)$, $(\cdot,\cdot)_\sigma$ the inner product of $\mathbb{L}_\sigma^2(\Omega)$; $\|\cdot\|$ the norm of $\mathbb{H}_0^1(\Omega)$, $((\cdot,\cdot))$ the inner product of $\mathbb{H}_0^1(\Omega)$; $\|\cdot\|_\sigma$ the norm of $\mathbb{H}_\sigma^1(\Omega)$, $((\cdot,\cdot))_\sigma$ the inner product of $\mathbb{H}_\sigma^1(\Omega)$. Due to the Poincaré inequality, we equip $\mathbb{H}_\sigma^1(\Omega)$ and $\mathbb{H}_0^1(\Omega)$ with the norm

$$\|u\|_\sigma := \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2},$$

$$\|v\| := \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}$$

for $u \in \mathbb{H}_\sigma^1(\Omega)$ and $v \in \mathbb{H}_0^1(\Omega)$, respectively.

The orthogonal projection from $\mathbb{L}^2(\Omega)$ to $\mathbb{L}_\sigma^2(\Omega)$ is denoted by P. The Stokes operator A is defined by $A := -P\Delta$ with $D(A) = \mathbb{H}^2(\Omega)\cap\mathbb{H}_\sigma^1(\Omega)$.

Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $\mathbb{H}^{-1}(\Omega)$ and $\mathbb{H}_0^1(\Omega)$. The differential operator L, which maps $\mathbb{H}_0^1(\Omega)$ to $\mathbb{H}^{-1}(\Omega)$, is defined by

$$\langle Lv, w\rangle := \alpha\int_\Omega \nabla v\cdot\nabla w dx + \beta\int_\Omega (\nabla\cdot v)(\nabla\cdot w)dx + 4\chi\int_\Omega vw dx \qquad (1)$$

for all $v, w \in \mathbb{H}_0^1(\Omega)$. By integration by parts, we obtain

$$\int_\Omega |\nabla w|^2 dx = \int_\Omega |\nabla\cdot w|^2 dx + \int_\Omega |\nabla\times w|^2 dx \qquad (2)$$

for $w \in \mathbb{H}_0^1(\Omega)$. Hence there are positive constants $\alpha_\star$ and $\alpha^\star$ such that

$$\alpha_\star|\nabla w|^2 + 4\chi|w|^2 \le \langle Lw, w\rangle \le \alpha^\star|\nabla w|^2 + 4\chi|w|^2 \tag{3}$$

for $w \in \mathbb{H}_0^1(\Omega)$.

For $v \in \mathbb{L}^2(\Omega)$, we also define $\nabla \times v \in \mathbb{H}^{-1}(\Omega)$ by

$$\langle \nabla \times v, w\rangle := \int_\Omega v \cdot \nabla \times w dx \tag{4}$$

with $w \in \mathbb{H}_0^1(\Omega)$.

We identify the space $\mathbb{L}^2(\Omega)$ with its dual. Then $\|\nabla \times v\|_\star \le |v|$ holds for $v \in \mathbb{L}^2(\Omega)$, where $\|\cdot\|_\star$ stands for the norm of the space $\mathbb{H}^{-1}(\Omega)$.

3 Main result

In the above settings, our system can be regarded as a system of abstract evolution equations of the following form.

$$\begin{aligned}
&\frac{du}{dt} + (\mu+\chi)Au = Pf + b_1(u,v) \quad \text{in } \mathbb{L}_\sigma^2(\Omega),\\
&\frac{dv}{dt} + Lv = g - \frac{\theta}{2}\nabla \times f + b_2(u,v) \quad \text{in } \mathbb{H}^{-1}(\Omega),\\
&u(0) = u_0, \quad v(0) = v_0,
\end{aligned}$$

where $b_1(u,v)$ and $b_2(u,v)$ are defined by

$$\begin{aligned}
b_1(u,v) &:= 2\chi\nabla \times v + \theta\chi Au - P(u\cdot\nabla)u,\\
b_2(u,v) &:= -\frac{\theta}{2}\{(\mu + (1-\theta)\chi - \alpha)\nabla \times \Delta u + ((\nabla \times u)\cdot\nabla)u\}\\
&\qquad - (u\cdot\nabla)v + 2(1-\theta)\chi\nabla \times u.
\end{aligned}$$

Notice that $\nabla \times w \in \mathbb{L}_\sigma^2(\Omega)$ whenever $w \in \mathbb{H}_0^1(\Omega)$ since $\mathbb{C}_c^\infty(\Omega)$ is dense in $\mathbb{H}_0^1(\Omega)$ and if $w_0 \in \mathbb{C}_c^\infty(\Omega)$ and $q \in \mathbb{H}^1(\Omega)$, the integration by parts gives

$$(\nabla \times w_0, \nabla q) = (-\nabla\cdot(\nabla \times w_0), q) = 0.$$

Now our main result reads as follows.

Theorem *There exists a constant $\theta_0 \in (0,1]$ satisfying the following property: Given $\theta \in [0,\theta_0]$, $T > 0$, $(u_0, v_0) \in \mathbb{H}_\sigma^1(\Omega) \times \mathbb{L}^2(\Omega)$, $f \in L^2(0,T;\mathbb{L}^2(\Omega))$ and $g \in L^2(0,T;\mathbb{H}^{-1}(\Omega))$, there exist a $T_\star \in (0,T]$ and a unique solution (u,v) to our system on the time interval $(0,T_\star)$ with*

$$\begin{aligned}
&u \in C([0,T_\star];\mathbb{H}_\sigma^1(\Omega)) \cap W^{1,2}(0,T_\star;\mathbb{L}_\sigma^2(\Omega)),\\
&v \in C([0,T_\star];\mathbb{L}^2(\Omega)) \cap W^{1,2}(0,T_\star;\mathbb{H}^{-1}(\Omega)).
\end{aligned}$$

4 Sketch of a proof of Theorem

Our idea for proof is the following: First, solve a linear problem

$$\frac{du}{dt} + (\mu + \chi)Au = Pf + h \quad \text{in } \mathbb{L}^2_\sigma(\Omega),$$
$$\frac{dv}{dt} + Lv = g - \frac{\theta}{2}\nabla \times f + k \quad \text{in } \mathbb{H}^{-1}(\Omega),$$
$$u(0) = u_0, \quad v(0) = v_0,$$

for given $h \in L^2(0,T;\mathbb{L}^2_\sigma(\Omega))$ and $k \in L^2(0,T;\mathbb{H}^{-1}(\Omega))$. It is well-known that there is a unique solution (u,v) such that

$$u \in C([0,T);\mathbb{H}^1_\sigma(\Omega)) \cap W^{1,2}(0,T;\mathbb{L}^2_\sigma(\Omega)),$$
$$v \in C([0,T);\mathbb{L}^2(\Omega)) \cap W^{1,2}(0,T;\mathbb{H}^{-1}(\Omega)).$$

Then one can define a mapping S from $L^2(0,T;\mathbb{L}^2_\sigma(\Omega)) \times L^2(0,T;\mathbb{H}^{-1}(\Omega))$ into itself as $S(h,k) := (b_1(u,v), b_2(u,v))$. If S has a fixed point, then (u,v) is a solution of our problem. Since b_1 and b_2 are sums of terms which are linear or quadratic in u and v, it is natural to expect that S would be a contraction mapping. This conjecture turns out to be true with the following trick. Let $\eta \in (0,1]$ be a constant to be fixed later and $U := \eta u$.Then our system becomes as follows.

$$\frac{dU}{dt} + (\mu + \chi)AU = \eta Pf + B_1(U,v) \quad \text{in } \mathbb{L}^2_\sigma(\Omega),$$
$$\frac{dv}{dt} + Lv = g - \frac{\theta}{2}\nabla \times f + B_2(U,v) \quad \text{in } \mathbb{H}^{-1}(\Omega),$$
$$U(0) = \eta u_0, \quad v(0) = v_0,$$

where $B_1(U,v)$ and $B_2(U,v)$ are defined by

$$B_1(U,v) := 2\eta\chi\nabla \times v + \theta\chi AU - \frac{1}{\eta}P(U\cdot\nabla)U,$$
$$B_2(U,v) := -\frac{\theta}{2\eta}(\mu + (1-\theta)\chi - \alpha)\nabla \times \Delta U - \frac{\theta}{2\eta^2}((\nabla \times U)\cdot\nabla)U$$
$$- \frac{1}{\eta}(U\cdot\nabla)v + \frac{2(1-\theta)\chi}{\eta}\nabla \times U.$$

We are going to show this modified system of equations has a solution (U,v).

Let data $T > 0$, u_0, v_0, f and g be given and take a positive number R satisfying

$$R \geq \max\{\|u_0\|_\sigma, |v_0|, \|f\|_{L^2(0,T;\mathbb{L}^2(\Omega))}, \|g\|_{L^2(0,T;\mathbb{H}^{-1}(\Omega))}\}.$$

Let τ be a positive number in $(0,T]$ and is also to be fixed later. Denote by $\mathcal{B}_R$ the set of functions (h,k) such that $h \in L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))$ and $k \in L^2(0,\tau;\mathbb{H}^{-1}(\Omega))$ with $\|h\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))} \leq R$ and $\|k\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))} \leq R$.

It is well-known that there is a unique solution (U,v) to the problem

$$\frac{dU}{dt} + (\mu + \chi)AU = \eta Pf + h \quad \text{in } \mathbb{L}^2_\sigma(\Omega), \tag{5}$$
$$\frac{dv}{dt} + Lv = g - \frac{\theta}{2}\nabla \times f + k \quad \text{in } \mathbb{H}^{-1}(\Omega), \tag{6}$$
$$U(0) = \eta u_0, \quad v(0) = v_0, \tag{7}$$

which satisfies

$$U \in C([0,\tau);\mathbb{H}^1_\sigma(\Omega)) \cap L^2(0,\tau;\mathbb{H}^2(\Omega)\cap\mathbb{H}^1_\sigma(\Omega)) \cap W^{1,2}(0,\tau;\mathbb{L}^2_\sigma(\Omega)),$$
$$v \in C([0,\tau);\mathbb{L}^2(\Omega)) \cap L^2(0,\tau;\mathbb{H}^1_0(\Omega)) \cap W^{1,2}(0,\tau;\mathbb{H}^{-1}(\Omega)).$$

Multiplying (5) by U, we get

$$\frac{1}{2}\frac{d}{dt}|U|^2_\sigma + (\mu+\chi)\|U\|^2_\sigma \le C(|f| + |h|_\sigma)\|U\|_\sigma.$$

Here and henceforth C or C_i (i is a positive number) denotes a constant which may depend only on $\mu, \chi, \alpha, \beta, \Omega$ and may take different values line by line. Then we have

$$\|U\|_{L^\infty(0,\tau;\mathbb{L}^2_\sigma(\Omega))} \le C_1 R,$$
$$\|U\|_{L^2(0,\tau;\mathbb{H}^1_\sigma(\Omega))} \le C_2 R.$$

Multiplying (5) by AU, we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2_\sigma + (\mu+\chi)|AU|^2_\sigma \le C(|f| + |h|_\sigma)|AU|_\sigma,$$

whence follows

$$\|U\|_{L^\infty(0,\tau;\mathbb{H}^1_\sigma(\Omega))} \le C_3 R,$$
$$\|U\|_{L^2(0,\tau;\mathbb{H}^1_\sigma(\Omega)\cap\mathbb{H}^2(\Omega))} \le C_4 R.$$

We here use the estimate from the elliptic regularity theory:

$$\|w\|_{\mathbb{H}^2(\Omega)} \le C_0\|Aw\|_\sigma,$$

which holds for $w \in D(A)$.

Taking the duality pairing between (6) and v, we obtain

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \alpha_\star\|v\|^2 + 4\chi|v|^2 \le C(\|g\|_\star + |f| + |k|)\|v\|.$$

From this it follows that

$$\|v\|_{L^\infty(0,\tau;\mathbb{L}^2(\Omega))} \le C_5 R,$$
$$\|v\|_{L^2(0,\tau;\mathbb{H}^1_0(\Omega))} \le C_6 R.$$

Now we shall show η, θ and τ can be chosen so that $(B_1(U,v), B_2(U,v))$ also belongs to the set $\mathcal{B}_R$.

Let ϕ and ψ be scalar functions. $D\phi$ denotes any one of the partial derivative of ϕ. We need the following well-known inequalities in order to estimate the nonlinear termes.

If $\phi \in H^1_0(\Omega)$, we have

$$\|\phi\|_{L^3(\Omega)} \le C\|\phi\|^{1/2}_{L^2(\Omega)}\|\phi\|^{1/2}_{H^1(\Omega)},$$
$$\|\phi\|_{L^6(\Omega)} \le C\|\phi\|_{H^1(\Omega)}.$$

If we assume further that $\phi \in H^2(\Omega)$, then $\phi \in L^\infty(\Omega)$ and

$$\|\phi\|_{L^\infty(\Omega)} \le C\|\phi\|_{H^1(\Omega)}^{1/2}\|\phi\|_{H^2(\Omega)}^{1/2}.$$

If $\phi \in H_0^1(\Omega)$ and $\psi \in H^2(\Omega)$ or $\phi \in H^2(\Omega)$ and $\psi \in H_0^1(\Omega)$, the product $\phi D\psi$ belongs to $L^2(\Omega)$ and

$$\|\phi D\psi\|_{L^2} \le \begin{cases} C\|\phi\|_{H^1}^{1/2}\|\phi\|_{H^2}^{1/2}\|\psi\|_{H^1} & \text{for } \phi \in H^1(\Omega),\ \psi \in H^2(\Omega), \\ C\|\phi\|_{H^1}\|\psi\|_{H^2}^{1/2}\|\psi\|_{H^1}^{1/2} & \text{for } \phi \in H^2(\Omega),\ \psi \in H^1(\Omega). \end{cases} \tag{8}$$

From these estimate, we obtain

$$\begin{aligned} \int_0^\tau |P(U\cdot\nabla)U(s)|_\sigma^2 ds &\le \int_0^\tau |(U\cdot\nabla)U(s)|^2 ds \\ &\le C_7\int_0^\tau \|U(s)\|^3\|U(s)\|_{\mathbb{H}^2} ds \\ &\le C_3^3C_7R^3\int_0^\tau \|U(s)\|_{\mathbb{H}^2} ds \\ &\le C_3^3C_7R^3\tau^{1/2}\|U\|_{L^2(0,\tau;\mathbb{H}^2(\Omega)\cap\mathbb{H}_\sigma^1(\Omega))} \\ &\le C_3^3C_4C_7R^4\tau^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} &\|B_1(U,v)\|_{L^2(0,\tau;\mathbb{L}_\sigma^2(\Omega))} \\ &\le 2\eta\chi\|\nabla\times v\|_{L^2(0,\tau;\mathbb{L}_\sigma^2(\Omega))} + \theta\chi\|AU\|_{L^2(0,\tau;\mathbb{L}_\sigma^2(\Omega))} + \frac{1}{\eta}\|(U\cdot\nabla)U\|_{L^2(0,\tau;\mathbb{L}^2(\Omega))} \\ &\le \left(2C_6\eta\chi + C_4\theta\chi + \frac{C_3^{3/2}C_4^{1/2}C_7^{1/2}R\tau^{1/4}}{\eta}\right)R. \end{aligned}$$

Suppose that $w_1, w_2, w_3 \in \mathbb{H}^1(\Omega)$, $\nabla\cdot w_1 = 0$ and at least one of these functions vanishes on the boundary $\partial\Omega$. Then $((w_1\cdot\nabla)w_2, w_3)$ is well-defined and it holds that $((w_1\cdot\nabla)w_2, w_3) = -((w_1\cdot\nabla)w_3, w_2)$.

For $w \in \mathbb{H}_0^1(\Omega)$ we have

$$\begin{aligned} |\langle\nabla\times\Delta U, w\rangle| &= |(\Delta U, \nabla\times w)| \\ &\le C_8\|U\|_{\mathbb{H}^2(\Omega)}\|w\|, \\ |\langle((\nabla\times U)\cdot\nabla)U, w\rangle| &= |(((\nabla\times U)\cdot\nabla)U, w)| \\ &= |-(((\nabla\times U)\cdot\nabla)w, U)| \\ &\le C\|\nabla U\|_{\mathbb{L}^3(\Omega)}\|\nabla w\|_{\mathbb{L}^2(\Omega)}\|U\|_{\mathbb{L}^6(\Omega)} \\ &\le C_9\|U\|_\sigma^{3/2}\|U\|_{\mathbb{H}^2}^{1/2}\|w\|, \\ |\langle(U\cdot\nabla)v, w\rangle| &= |((U\cdot\nabla)v, w)| \\ &= |-((U\cdot\nabla)w, v)| \\ &\le C\|U\|_{\mathbb{L}^6(\Omega)}\|\nabla w\|_{\mathbb{L}^2(\Omega)}\|v\|_{\mathbb{L}^3(\Omega)} \\ &\le C_{10}\|U\|_\sigma|v|^{1/2}\|v\|^{1/2}\|w\| \\ |\langle\nabla\times U, w\rangle| &\le |U|_\sigma\|w\|, \end{aligned}$$

and further

$$\|B_2(U,v)\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))}$$
$$\leq \left[\frac{\theta}{2\eta}C_4C_8(\mu+\chi+\alpha)+\frac{\theta}{2\eta^2}C_3^{3/2}C_7^{1/2}C_9R\tau^{1/4} + \frac{C_3C_5^{1/2}C_6^{1/2}C_{10}R\tau^{1/4}}{\eta}+\frac{2\chi C_1\tau^{1/2}}{\eta}\right]R.$$

Next, let (h_i,k_i) $(i=1,2)$ be taken from $\mathcal{B}_R$ and (U_i,v_i) $(i=1,2)$ be the solution of

$$\frac{dU_i}{dt}+(\mu+\chi)AU_i=\eta Pf+h_i,$$
$$\frac{dv_i}{dt}+Lv_i=g-\frac{\theta}{2}\nabla\times f+k_i,$$
$$U_i(0)=\eta u_0,\quad v_i(0)=v_0.$$

Then it is easy to see that the differences $\tilde{U}:=U_1-U_2$ and $\tilde{v}:=v_1-v_2$ can be estimated as

$$\|\tilde{U}\|_{L^\infty(0,\tau;\mathbb{L}^2_\sigma(\Omega))}\leq C_{11}\|\tilde{h}\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))},$$
$$\|\tilde{U}\|_{L^2(0,\tau;\mathbb{H}^1_\sigma(\Omega))}\leq C_{12}\|\tilde{h}\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))},$$
$$\|\tilde{U}\|_{L^\infty(0,\tau;\mathbb{H}^1_\sigma(\Omega))}\leq C_{13}\|\tilde{h}\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))},$$
$$\|\tilde{U}\|_{L^2(0,\tau;\mathbb{H}^1_\sigma(\Omega)\cap\mathbb{H}^2(\Omega))}\leq C_{14}\|\tilde{h}\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))},$$
$$\|\tilde{v}\|_{L^\infty(0,\tau;\mathbb{L}^2(\Omega))}\leq C_{15}\|\tilde{k}\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))},$$
$$\|\tilde{v}\|_{L^2(0,\tau;\mathbb{H}^1_0(\Omega))}\leq C_{16}\|\tilde{k}\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))}.$$

Since

$$B_1(U_1,v_1)-B_1(U_2,v_2)=2\eta\chi\nabla\times\tilde{v}+\theta\chi A\tilde{U}-\frac{1}{\eta}[P(U_1\cdot\nabla)\tilde{U}+P(\tilde{U}\cdot\nabla)U_2],$$

then

$$\|B_1(U_1,v_1)-B_1(U_2,v_2)\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))}$$
$$\leq 2\eta\chi C_{16}\|\tilde{k}\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))}+\left[\theta\chi C_{14}+\frac{2C_3C_{13}^{1/2}C_{14}^{1/2}R\tau^{1/4}}{\eta}\right]\|\tilde{h}\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))}.$$

Similarly we obtain

$$\|B_2(U_1,v_1)-B_2(U_2,v_2)\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))}$$
$$\leq\left[\frac{\theta}{2\eta}C_{14}(\mu+\chi+\alpha)+\frac{C_2^{1/2}C_6^{1/2}C_{13}R\tau^{1/4}}{\eta}+\frac{\theta C_3^{1/2}C_4^{1/2}C_{13}R\tau^{1/4}}{\eta^2} +\frac{2\chi C_{11}\tau^{1/2}}{\eta}\right]\|\tilde{h}\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))}+\frac{C_3C_{15}^{1/2}C_{16}^{1/2}R\tau^{1/4}}{\eta}\|\tilde{k}\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))}.$$

Now, set the number $\eta\in(0,1]$ so that the following inequalities hold:

$$2C_6\chi\eta\leq\frac{1}{2},\quad 2C_{16}\chi\eta\leq\frac{1}{4}.$$

After that, chose $\theta \in (0,1]$ and $\tau \in (0,T]$ so small that

$$\|B_1(U,v)\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))} \le R,$$
$$\|B_2(U,v)\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))} \le R,$$
$$\|B_1(U,v) - B_1(U,v)\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))} \le \frac{1}{2}\|\tilde{h}\|_{L^2(0,\tau;\mathbb{L}^2_\sigma(\Omega))},$$
$$\|B_2(U,v) - B_2(U,v)\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))} \le \frac{1}{2}\|\tilde{k}\|_{L^2(0,\tau;\mathbb{H}^{-1}(\Omega))}.$$

Thus the mapping $(h,k) \mapsto (B_1(U,v), B_2(U,v))$ turns out to be a contraction, and the existence of a solution to our problem follows. The uniqueness of solution (U,v) logically follows from the above argument.

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