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On a semilinear evolution equation

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1 Problem

We shall consider a semilinear parabolic evolution equation arising from fluid mechanics. More specifically, we deal with the following system of equations of the motion of incompressible micropolar fluids.

\[
\frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f + 2\chi \nabla \times \left( \omega - \frac{1}{2} \nabla \times u \right),
\]

\[
\frac{\partial \omega}{\partial t} - \alpha \Delta \omega - \beta \nabla (\nabla \cdot \omega) + (u \cdot \nabla) \omega = g - 4\chi \left( \omega - \frac{1}{2} \nabla \times u \right),
\]

\[\nabla \cdot u = 0,\]

where the unknown functions are the velocity vector field \( u \), the microrotation vector \( \omega \) and the pressure \( p \), and \( f \) and \( g \) are given vector fields, \( \alpha, \beta, \mu, \chi \) are constants satisfying \( \alpha > 0, \alpha + \beta > 0, \mu > 0 \) and \( \chi > 0 \).

Micropolar fluid model is one of generalizations of the classical Navier-Stokes model. We shall discuss the solvability of the initial-boundary value problem of this system of equations in a bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \) in \( \mathbb{R}^3 \).

As for the boundary conditions, we assume that

\[u|_{\partial \Omega} = 0, \quad \omega|_{\partial \Omega} = \frac{\theta}{2} \nabla \times u\bigg|_{\partial \Omega},\]

where \( \theta \) is a constant belonging to the interval \( [0, 1] \) (see \([1], [3])\).

For the case where \( \theta = 0 \), i.e., both \( u \) and \( \omega \) yield the homogeneous Dirichlet boundary conditions, there have been many results concerning this system of equations \([4]\). On the other hand, the case where \( \theta \neq 0 \) is not fully pursued yet. Furthermore, this boundary condition is not based on any physical principles, the well-posedness of the initial-boundary value problem is not clear. That is why we consider this problem.

Let us introduce a new variable \( v := \omega - \frac{\theta}{2} \nabla \times u \) and change the variables \( (u, \omega) \) to \( (u, v) \). Then the original system of equations for \( u \) and \( \omega \) will be
rewritten in the following form.

\[
\begin{align*}
\frac{\partial u}{\partial t} - (\mu + \chi) \Delta u + (u \cdot \nabla) u + \nabla p &= f + 2\chi \nabla \times v - \theta \chi \Delta u, \\
\frac{\partial v}{\partial t} - \alpha \Delta v - \beta (\nabla \cdot v) + (u \cdot \nabla) v + 4\chi v + \theta \chi \nabla \times (\nabla \times v) &= g - \frac{\theta}{2} \nabla \times f - \frac{\theta}{2} \{((\mu + (1 - \theta)\chi - \alpha) \nabla \times \Delta u \} + 2(1 - \theta)\chi \nabla \times u, \\
\nabla \cdot u &= 0, \\
u|_{\partial \Omega} &= 0, \quad v|_{\partial \Omega} = 0, \\
u(\cdot, 0) &= u_0(\cdot), \quad v(\cdot, 0) = v_0(\cdot).
\end{align*}
\]

Notice that the boundary condition for the new unknown function $v$ is reduced to the homogeneous boundary condition.

Our aim is to show the existence of a solution $(u, v)$ local in time to this system of equations in the $L^2$-framework.

## 2 Function spaces and operators

We refer the details of mathematical facts to be mentioned below to the books [2], [4] and [5].

Let $C_0^\infty(\Omega)$ be the set of all smooth solenoidal vector functions in $\Omega$ with compact support, and $L_2^2(\Omega)$ and $H^1_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ and in $H^1(\Omega)$ respectively. We denote by $|\cdot|$ the norm of $L^2(\Omega)$, $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$; $|\cdot|_\sigma$ the norm of $L_2^2(\Omega)$, $(\cdot, \cdot)_\sigma$ the inner product of $L_2^2(\Omega)$; $\|\cdot\|$ the norm of $H^1_0(\Omega)$, $(\cdot, \cdot)$ the inner product of $H^1_0(\Omega)$; $\|\cdot\|_\sigma$ the norm of $H^1_\sigma(\Omega)$, $(\cdot, \cdot)_\sigma$ the inner product of $H^1_\sigma(\Omega)$. Due to the Poincaré inequality, we equip $H^1_\sigma(\Omega)$ and $H^1_0(\Omega)$ with the norm

\[
\|u\|_\sigma := \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2},
\]

\[
\|v\| := \left(\int_\Omega |\nabla v|^2 dx\right)^{1/2}
\]

for $u \in H^1_\sigma(\Omega)$ and $v \in H^1_0(\Omega)$, respectively.

The orthogonal projection from $L^2(\Omega)$ to $L^2_\sigma(\Omega)$ is denoted by $P$. The Stokes operator $A$ is defined by $A := -P\Delta$ with $D(A) = H^2(\Omega) \cap H^1_\sigma(\Omega)$.

Let $(\cdot, \cdot)$ denote the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. The differential operator $L$, which maps $H^1_0(\Omega)$ to $H^{-1}(\Omega)$, is defined by

\[
(Lv, w) := \alpha \int_\Omega \nabla v \cdot \nabla w dx + \beta \int_\Omega (\nabla \cdot v)(\nabla \cdot w) dx + 4\chi \int_\Omega vw dx \tag{1}
\]

for all $v, w \in H^1_0(\Omega)$. By integration by parts, we obtain

\[
\int_\Omega |\nabla w|^2 dx = \int_\Omega |\nabla \cdot w|^2 dx + \int_\Omega |\nabla \times w|^2 dx \tag{2}
\]
for \( w \in \mathbb{H}_0^1(\Omega) \). Hence there are positive constants \( \alpha_* \) and \( \alpha^* \) such that

\[
\alpha_*|\nabla w|^2 + 4\chi|w|^2 \leq \langle Lw, w \rangle \leq \alpha^*|\nabla w|^2 + 4\chi|w|^2
\] (3)

for \( w \in \mathbb{H}_0^1(\Omega) \).

For \( v \in L^2(\Omega) \), we also define \( \nabla \times v \in \mathbb{H}^{-1}(\Omega) \) by

\[
(\nabla \times v, w) := \int_\Omega v \cdot \nabla \times w dx
\] (4)

with \( w \in \mathbb{H}_0^1(\Omega) \).

We identify the space \( L^2(\Omega) \) with its dual. Then \( \|\nabla \times v\|_* \leq |v| \) holds for \( v \in L^2(\Omega) \), where \( \| \cdot \|_* \) stands for the norm of the space \( \mathbb{H}^{-1}(\Omega) \).

### 3 Main result

In the above settings, our system can be regarded as a system of abstract evolution equations of the following form.

\[
\frac{du}{dt} + (\mu + \chi)Au = Pf + b_1(u, v) \quad \text{in} \quad L^2_\sigma(\Omega),
\]

\[
\frac{dv}{dt} + Lv = g - \frac{\theta}{2}\nabla \times f + b_2(u, v) \quad \text{in} \quad \mathbb{H}^{-1}(\Omega),
\]

\[
u(0) = u_0, \quad v(0) = v_0,
\]

where \( b_1(u, v) \) and \( b_2(u, v) \) are defined by

\[
b_1(u, v) := 2\chi \nabla \times v + \theta \chi Au - P(u \cdot \nabla)u,
\]

\[
b_2(u, v) := -\frac{\theta}{2}\left((\mu + (1-\theta)\chi - \alpha)\nabla \Delta u + ((\nabla \times u) \cdot \nabla)u\right) - (u \cdot \nabla)v + 2(1-\theta)\chi \nabla \times u.
\]

Notice that \( \nabla \times w \in L^2_\sigma(\Omega) \) whenever \( w \in \mathbb{H}_0^1(\Omega) \) since \( C_c^\infty(\Omega) \) is dense in \( \mathbb{H}_0^1(\Omega) \) and if \( w_0 \in C_c^\infty(\Omega) \) and \( q \in H^1(\Omega) \), the integration by parts gives

\[
(\nabla \times w_0, \nabla q) = (-\nabla \cdot (\nabla \times w_0), q) = 0.
\]

Now our main result reads as follows.

**Theorem** There exists a constant \( \theta_0 \in (0, 1] \) satisfying the following property: Given \( \theta \in [0, \theta_0] \), \( T > 0 \), \((u_0, v_0) \in \mathbb{H}_\sigma^1(\Omega) \times L^2(\Omega) \), \( f \in L^2(0, T; L^2(\Omega)) \) and \( g \in L^2(0, T; H^{-1}(\Omega)) \), there exist a \( T_* \in (0, T] \) and a unique solution \((u, v)\) to our system on the time interval \((0, T_*]\) with

\[
u \in C([0, T_*]; \mathbb{H}_\sigma^1(\Omega)) \cap W^{1,2}(0, T_*; L^2(\Omega)),
\]

\[
u \in C([0, T_*]; L^2(\Omega)) \cap W^{1,2}(0, T_*; H^{-1}(\Omega)).
\]
4 Sketch of a proof of Theorem

Our idea for proof is the following: First, solve a linear problem

\[
\begin{align*}
\frac{du}{dt} + (\mu + \chi)Au &= Pf + h & \text{in } L^2_2(\Omega), \\
\frac{dv}{dt} + Lv &= g - \frac{\theta}{2} \nabla \times f + k & \text{in } H^{-1}(\Omega), \\
u(0) &= u_0, & v(0) &= v_0,
\end{align*}
\]

for given \( h \in L^2(0,T;L^2_2(\Omega)) \) and \( k \in L^2(0,T;H^{-1}(\Omega)) \). It is well-known that there is a unique solution \((u, v)\) such that

\[
\begin{align*}
u &\in C([0, T); H^1_\sigma(\Omega)) \cap W^{1,2}(0, T; L^2_2(\Omega)), \\
v &\in C([0, T); L^2(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega)).
\end{align*}
\]

Then one can define a mapping \( S \) from \( L^2(0,T;L^2_2(\Omega)) \times L^2(0,T;H^{-1}(\Omega)) \) into itself as \( S(h, k) := (b_1(u, v), b_2(u, v)) \). If \( S \) has a fixed point, then \((u, v)\) is a solution of our problem. Since \( b_1 \) and \( b_2 \) are sums of terms which are linear or quadratic in \( u \) and \( v \), it is natural to expect that \( S \) would be a contraction mapping. This conjecture turns out to be true with the following trick. Let \( \eta \in (0, 1] \) be a constant to be fixed later and \( U := \eta u \). Then our system becomes as follows.

\[
\begin{align*}
\frac{dU}{dt} + (\mu + \chi)AU &= \eta Pf + B_1(U, v) & \text{in } L^2_2(\Omega), \\
\frac{dv}{dt} + Lv &= g - \frac{\theta}{2} \nabla \times f + B_2(U, v) & \text{in } H^{-1}(\Omega), \\
U(0) &= \eta u_0, & v(0) &= v_0,
\end{align*}
\]

where \( B_1(U, v) \) and \( B_2(U, v) \) are defined by

\[
\begin{align*}
B_1(U, v) &= 2\eta \chi \nabla \times v + \theta \chi AU - \frac{1}{\eta} P(U \cdot \nabla)U, \\
B_2(U, v) &= \frac{\theta}{2\eta} (\mu + (1 - \theta) \chi - \alpha) \nabla \times \Delta U - \frac{\theta}{2\eta^2} ((\nabla \times U) \cdot \nabla)U \\
&\quad - \frac{1}{\eta} (U \cdot \nabla)\nabla \times U.
\end{align*}
\]

We are going to show this modified system of equations has a solution \((U, v)\).

Let \( T > 0, u_0, v_0, f \) and \( g \) be given and take a positive number \( R \) satisfying

\[
R \geq \max\{\|u_0\|_\sigma, |v_0|, \|f\|_{L^2(0,T;L^2_2(\Omega))}, \|g\|_{L^2(0,T;H^{-1}(\Omega))}\}.
\]

Let \( \tau \) be a positive number in \((0, T]\) and is also to be fixed later. Denote by \( B_R \) the set of functions \((h, k)\) such that \( h \in L^2(0, \tau; L^2_2(\Omega)) \) and \( k \in L^2(0, \tau; H^{-1}(\Omega)) \) with \( \|h\|_{L^2(0, \tau; L^2_2(\Omega))} \leq R \) and \( \|k\|_{L^2(0, \tau; H^{-1}(\Omega))} \leq R \).

It is well-known that there is a unique solution \((U, v)\) to the problem

\[
\begin{align*}
\frac{dU}{dt} + (\mu + \chi)AU &= \eta Pf + h & \text{in } L^2_2(\Omega), \\
\frac{dv}{dt} + Lv &= g - \frac{\theta}{2} \nabla \times f + k & \text{in } H^{-1}(\Omega), \\
U(0) &= \eta u_0, & v(0) &= v_0.
\end{align*}
\]

\(\square\)
which satisfies
\[
U \in C([0, \tau]; \mathbb{H}_{\sigma}^{1}(\Omega)) \cap L^{2}(0, \tau; \mathbb{H}^{2}(\Omega) \cap \mathbb{H}_{\sigma}^{1}(\Omega)) \cap W^{1,2}(0, \tau; \mathbb{L}_{\sigma}^{2}(\Omega)),
\]
\[
v \in C([0, \tau]; \mathbb{L}^{2}(\Omega)) \cap L^{2}(0, \tau; \mathbb{H}_{0}^{1}(\Omega)) \cap W^{1,2}(0, \tau; \mathbb{H}^{-1}(\Omega)).
\]

Multiplying (5) by \( U \), we get
\[
\frac{1}{2} \frac{d}{dt}|U|_{\sigma}^{2} + (\mu + \chi)\|U\|_{\sigma}^{2} \leq C(|f| + |h|_{\sigma})\|U\|_{\sigma}.
\]

Here and henceforth \( C \) or \( C_i \) (\( i \) is a positive number) denotes a constant which may depend only on \( \mu, \chi, \alpha, \beta, \Omega \) and may take different values line by line. Then we have
\[
\|U\|_{L^\infty(0,\tau;\mathbb{L}_{\sigma}^{2}(\Omega))} \leq C_{1}R,
\]
\[
\|U\|_{L^{2}(0,\tau;\mathbb{H}_{\sigma}^{1}(\Omega))} \leq C_{2}R.
\]

Multiplying (5) by \( AU \), we have
\[
\frac{1}{2} \frac{d}{dt}\|U\|_{\sigma}^{2} + (\mu + \chi)|AU|_{\sigma}^{2} \leq C(|f| + |h|_{\sigma})|AU|_{\sigma},
\]
whence follows
\[
\|U\|_{L^\infty(0,\tau;\mathbb{H}_{\sigma}^{1}(\Omega))} \leq C_{3}R,
\]
\[
\|U\|_{L^{2}(0,\tau;\mathbb{H}_{\sigma}^{1}(\Omega) \cap \mathbb{H}^{2}(\Omega))} \leq C_{4}R.
\]

We here use the estimate from the elliptic regularity theory:
\[
\|w\|_{\mathbb{H}^{2}(\Omega)} \leq C_{0}\|Aw\|_{\sigma},
\]
which holds for \( w \in D(A) \).

Taking the duality pairing between (6) and \( v \), we obtain
\[
\frac{1}{2} \frac{d}{dt}|v|^{2} + \alpha_{\star}\|v\|^{2} + 4\chi|v|^{2} \leq C(\|g\|_{\ast} + |f| + |k|)\|v\|.
\]
From this it follows that
\[
\|v\|_{L^\infty(0,\tau;\mathbb{L}^{2}(\Omega))} \leq C_{5}R,
\]
\[
\|v\|_{L^{2}(0,\tau;\mathbb{H}_{0}^{1}(\Omega))} \leq C_{6}R.
\]

Now we shall show \( \eta, \theta \) and \( \tau \) can be chosen so that \((B_{1}(U, v), B_{2}(U, v))\) also belongs to the set \( B_{R} \).

Let \( \phi \) and \( \psi \) be scalar functions. \( D\phi \) denotes any one of the partial derivative of \( \phi \). We need the following well-known inequalities in order to estimate the nonlinear terms.

If \( \phi \in H_{0}^{1}(\Omega) \), we have
\[
\|\phi\|_{L^{2}(\Omega)} \leq C\|\phi\|_{L^{2}(\Omega)}^{1/2}\|\phi\|_{H^{1}(\Omega)}^{1/2},
\]
\[
\|\phi\|_{L^{6}(\Omega)} \leq C\|\phi\|_{H^{1}(\Omega)}.
\]
If we assume further that $\phi \in H^2(\Omega)$, then $\phi \in L^\infty(\Omega)$ and
$$
\|\phi\|_{L^\infty(\Omega)} \leq C \|\phi\|_{H^1(\Omega)}^{1/2} \|\phi\|_{H^2(\Omega)}^{1/2}.
$$
If $\phi \in H^1_0(\Omega)$ and $\psi \in H^2(\Omega)$ or $\phi \in H^2(\Omega)$ and $\psi \in H^1_0(\Omega)$, the product $\phi D\psi$ belongs to $L^2(\Omega)$ and
$$
\|\phi D\psi\|_{L^2} \leq \begin{cases}
C \|\phi\|_{H^1}^{1/2} \|\phi\|_{H^2}^{1/2} \|\psi\|_{H^1} & \text{for } \phi \in H^1(\Omega), \psi \in H^2(\Omega), \\
C \|\phi\|_{H^2} \|\psi\|_{H^1}^{1/2} \|\phi\|_{H^1}^{1/2} & \text{for } \phi \in H^2(\Omega), \psi \in H^1(\Omega).
\end{cases}
\tag{8}
$$
From this estimate, we obtain
\[
\int_0^\tau |P(U \cdot \nabla)U(s)|_\sigma^2 ds \leq \int_0^\tau |(U \cdot \nabla)U(s)|^2 ds \leq C_7 \int_0^\tau \|U(s)\|^3 \|U(s)\|_{E^2} ds \leq C_3^3 C_7 R^3 \tau^{1/2} \|U\|_{L^2(0,\tau; H^2(\Omega) \cap H^1_\sigma(\Omega))}.
\]
Therefore
\[
\|B_1(U, v)\|_{L^2(0,\tau; L^2_{\sigma}(\Omega))} \leq 2\eta \chi \|\nabla \times v\|_{L^2(0,\tau; L^2_{\sigma}(\Omega))} + \theta \chi \|AU\|_{L^2(0,\tau; L^2_{\sigma}(\Omega))} + \frac{1}{\eta} \|(U \cdot \nabla)U\|_{L^2(0,\tau; L^2(\Omega))} \leq 2C_6 \eta \chi + C_4 \theta \chi + \frac{C_3^{3/2} C_4^{1/2} C_7^{1/2} R \tau^{1/4}}{\eta} R.
\]
Suppose that $w_1, w_2, w_3 \in H^1(\Omega)$, $\nabla \cdot w_1 = 0$ and at least one of these functions vanishes on the boundary $\partial \Omega$. Then $((w_1 \cdot \nabla)w_2, w_3)$ is well-defined and it holds that $((w_1 \cdot \nabla)w_2, w_3) = -((w_1 \cdot \nabla)w_3, w_2)$.
For $w \in H^1_0(\Omega)$ we have
\[
|\langle \nabla \times \Delta U, w \rangle| = |(\Delta U, \nabla \times w)| \leq C_8 \|U\|_{H^1(\Omega)} \|w\|,
\]
\[
|\langle (\nabla \times U) \cdot \nabla \rangle U, w \rangle| = |((\nabla \times U) \cdot \nabla) U, w) = |(U \cdot \nabla)U, w)| \leq C \|\nabla U\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \|U\|_{L^6(\Omega)} \leq C_9 \|U\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \|U\|_{L^6(\Omega)} \leq C_9 \|U\|_{H^1(\Omega)}^{1/2} \|U\|_{H^2(\Omega)}^{1/2} \|w\|,
\]
\[
|\langle U \cdot \nabla v, w \rangle| = |((U \cdot \nabla) v, w)| = |-(U \cdot \nabla)w, v)| \leq C \|U\|_{L^6(\Omega)} \|\nabla w\|_{L^2(\Omega)} \|v\|_{L^3(\Omega)} \leq C_{10} \|U\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \|v\|_{L^3(\Omega)} \leq C_{10} \|U\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}^{1/2} \|v\|_{L^3(\Omega)}^{1/2} \|w\| \leq |U|_\sigma \|w\|.
\]
and further
\[
\|B_2(U, v)\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))} \\
\leq \left[ \frac{\theta}{2\eta} C_4 C_8 (\mu + \chi + \alpha) + \frac{\theta}{2\eta^2} C_3^{1/2} C_9 R \tau^{1/4} \\
+ \frac{C_3 C_5^{1/2} C_6^{1/2} C_9 R \tau^{1/4}}{\eta} + \frac{2\chi C_1 \tau^{1/2}}{\eta} \right] R.
\]

Next, let \((h_i, k_i) (i = 1, 2)\) be taken from \(B_R\) and \((U_i, v_i) (i = 1, 2)\) be the solution of
\[
\frac{dU_i}{dt} + (\mu + \chi) A U_i = \eta P f + h_i, \\
\frac{dv_i}{dt} + L v_i = g - \frac{\theta}{2} \nabla \times f + k_i, \\
U_i(0) = \eta u_0, \quad v_i(0) = v_0.
\]
Then it is easy to see that the differences \(\tilde{U} := U_1 - U_2\) and \(\tilde{v} := v_1 - v_2\) can be estimated as
\[
\|\tilde{U}\|_{L^\infty(0, \tau; \mathbb{L}_\sigma^2(\Omega))} \leq C_{11} \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}, \\
\|\tilde{U}\|_{L^2(0, \tau; \mathbb{H}_\sigma^1(\Omega))} \leq C_{12} \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}, \\
\|\tilde{U}\|_{L^\infty(0, \tau; \mathbb{H}_\sigma^1(\Omega))} \leq C_{13} \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}, \\
\|\tilde{v}\|_{L^\infty(0, \tau; \mathbb{L}^2(\Omega))} \leq C_{15} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))}, \\
\|\tilde{v}\|_{L^2(0, \tau; \mathbb{H}_0^1(\Omega))} \leq C_{16} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))}.
\]
Since
\[
B_1(U_1, v_1) - B_1(U_2, v_2) = 2\eta \chi \nabla \times \tilde{v} + \theta \chi A \tilde{U} - \frac{1}{\eta} [P(U_1 \cdot \nabla) \tilde{U} + P(\tilde{U} \cdot \nabla) U_2],
\]
then
\[
\|B_1(U_1, v_1) - B_1(U_2, v_2)\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))} \\
\leq 2\eta \chi C_{16} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))} + \left[ \theta \chi C_{14} + \frac{2C_3 C_{13}^{1/2} C_{14}^{1/2} R \tau^{1/4}}{\eta} \right] \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))}.
\]
Similarly we obtain
\[
\|B_2(U_1, v_1) - B_2(U_2, v_2)\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))} \\
\leq \left[ \frac{\theta \chi}{2\eta} C_{14} (\mu + \chi + \alpha) + \frac{C_2^{1/2} C_6^{1/2} C_{13} R \tau^{1/4}}{\eta} + \frac{\theta C_3^{1/2} C_4^{1/2} C_{13} R \tau^{1/4}}{\eta^2} \\
+ \frac{2\chi \gamma \tau^{1/2}}{\eta} \right] \|\tilde{h}\|_{L^2(0, \tau; \mathbb{L}_\sigma^2(\Omega))} + \frac{C_3 C_{15}^{1/2} C_{16}^{1/2} R \tau^{1/4}}{\eta} \|\tilde{k}\|_{L^2(0, \tau; \mathbb{H}^{-1}(\Omega))}.
\]
Now, set the number \(\eta \in (0, 1]\) so that the following inequalities hold:
\[
2C_6 \chi \eta \leq \frac{1}{2}, \quad 2C_{16} \chi \eta \leq \frac{1}{4}.
\]
After that, chose $\theta \in (0, 1]$ and $\tau \in (0, T]$ so small that

$\|B_1(U, v)\|_{L^2(0, \tau; L^2(\Omega))} \leq R,$
$\|B_2(U, v)\|_{L^2(0, \tau; H^{-1}(\Omega))} \leq R,$
$\|B_1(U, v) - B_1(U, v)\|_{L^2(0, \tau; L^2(\Omega))} \leq \frac{1}{2}\|\tilde{h}\|_{L^2(0, \tau; L^2(\Omega))},$
$\|B_2(U, v) - B_2(U, v)\|_{L^2(0, \tau; H^{-1}(\Omega))} \leq \frac{1}{2}\|\tilde{k}\|_{L^2(0, \tau; H^{-1}(\Omega))}.$

Thus the mapping $(h, k) \mapsto (B_1(U, v), B_2(U, v))$ turns out to be a contraction, and the existence of a solution to our problem follows. The uniqueness of solution $(U, v)$ logically follows from the above argument.

References


