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Global existence results for some double-diffusive convection system based on the Brinkman-Forchheimer equation with homogeneous Neumann boundary conditions †

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1 Introduction

We consider the following system which describes double-diffusive convection phenomena of an incompressible viscous fluid in some porous medium:

$$
\text{(BF)}\begin{cases}
\partial_t \boldsymbol{u} = \nu\Delta \boldsymbol{u} - a\boldsymbol{u} - \nabla p + \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1 & (x,t) \in \Omega\times[0,S],\\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t) \in \Omega\times[0,S],\\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t) \in \Omega\times[0,S],\\
\nabla\cdot\boldsymbol{u} = 0 & (x,t) \in \Omega\times[0,S],\\
\boldsymbol{u} = 0,\ \dfrac{\partial T}{\partial n} = 0,\ \dfrac{\partial C}{\partial n} = 0, & (x,t) \in \partial\Omega\times[0,S],\\
\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot),\ T(\cdot,0) = T_0(\cdot),\ C(\cdot,0) = C_0(\cdot), & \text{(Initial boundary value problem)},\\
\boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S),\ T(\cdot,0) = T(\cdot,S),\ C(\cdot,0) = C(\cdot,S), & \text{(Time periodic problem)},
\end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, n denotes the unit outward normal vector on $\partial\Omega$ and $\frac{\partial T}{\partial n} := \nabla T \cdot n$. Unknown functions $\boldsymbol{u} = (u_1, u_2, \cdots, u_N)^t, T, C$ and p represent the fluid velocity, the temperature of the fluid, the concentration of a solute and the pressure of the fluid respectively. Positive constants ν, ρ, a are called the viscosity coefficient, Soret's coefficient and Darcy's coefficient respectively. Constant vectors $\boldsymbol{g}, \boldsymbol{h}$ are derived from gravity and $\boldsymbol{f}_1 = (f_1^1, f_1^2, \cdots, f_1^N)^t, f_2, f_3$ are given external forces. Furthermore, we impose the solenoidal condition on the fluid velocity $\boldsymbol{u}$.

Double-diffusive convection is a model of convection in the fluid reflecting some interactions between the temperature and the concentration of solute. The double-diffusive convection phenomena can be represented by the second and the third equation of (BF) which originate from a result of the irreversible thermodynamics. The term $\rho\Delta T$, which is called Soret's effect term, describes a certain interaction between the temperature of the fluid and the concentration of a solute. This interaction makes the behavior of the fluid become more complicated than the simplified diffusion model and this Soret's effect mainly characterizes the double-diffusive convection. Originally, the second equation also contains a interaction term $\rho'\Delta C$, which is called Dufour's effect term. However, Dufour's effect is generally much smaller than Soret's effect, especially for the case where we deal with the liquid fluid. Therefore we here consider only Soret's effect term (for further details of physical background, see [1] and [8]).

The first equation of (BF) comes from Brinkman-Forchheimer equation, which describes the behavior of the fluid velocity in some porous medium with a relatively large porosity (the rate of void space in

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a porous medium). Originally, Brinkman-Forchheimer equation has some nonlinear terms and a space-dependent function which stands for the porosity. However, under some physical assumptions, such as the uniformity of the porosity, we can derive the linearized Brinkman-Forchheimer equation given in the first equation of (BF). Here $\boldsymbol{g}T, \boldsymbol{h}C$ are effects from gravity.

There are many studies for (BF), for example, about the continuous dependence of the solutions on Soret's coefficient ρ and so on. However, to the best of our knowledge, it seems that there are very few studies for the solvability of (BF). The first attempt in this direction is made in [13], where the initial boundary value problem for (BF) with homogeneous Dirichlet boundary conditions is considered. In [13], they showed that this problem admits a unique global solution when $N \leqslant 3$.

In [11], the global solvability of the time periodic problem is shown for (BF) with homogeneous Dirichlet boundary conditions both for 2 and 3-dimensional cases.

Due to the convection terms $\boldsymbol{u} \cdot \nabla T, \boldsymbol{u} \cdot \nabla C$, which are quite similar to that appearing in the Navier-Stokes equations, it apparently seems that it would be very difficult to obtain " the global solvability " of (BF) in 3-dimensional case, i.e., the existence of the unique global solution of the initial boundary-value problem for arbitrarily large initial data or the existence of time-periodic solutions for arbitrarily large external forces. However, it is revealed that the global solvability holds true for these problems even for the 3-dimensional case in [13] and [11].

The main purpose of this paper is to show that the global solvability results similar to [13] and [11] still hold true for (BF) with homogeneous Neumann boundary conditions for T and C. In order to carry out this purpose, we follow the basic strategy adopted in [13] and [11], i.e., we reduce our problem to some abstract equation in an appropriate Hilbert space and we rely on the abstract theory developed in [9] and [10]. However, the lack of the coercivity of the Laplacian with homogeneous Neumann boundary conditions causes some difficulties in this procedure. Especially for the periodic problem, we need to introduce some approximate system involving some dissipation terms and cut-off functions as in [11]. Unfortunately this hinders establishing desirable a priori estimates under the Neumann boundary condition. In order to cope with this difficulty, we introduce another step of approximations for the original system.

In section 2, we fix some notations for later use and we introduce abstract results. In section 3, our main results are stated. In section 4 and 5, we give proofs of main results for the initial boundary value problem and the periodic problem respectively.

2 Preliminaries

2.1 Notation

In order to formulate our results, we fix the following notations.

$$\mathbb{C}_\sigma^\infty(\Omega) = \{\, \boldsymbol{u} = (u^1, u^2, \cdots, u^N)^t \,;\, u^j \in C_0^\infty(\Omega) \ \forall j = 1, 2, \cdots, N,\ \nabla\cdot\boldsymbol{u} = 0\,\},$$

$\mathbb{L}^2(\Omega) = (L^2(\Omega))^N, \quad \mathbb{H}^1(\Omega) = (H^1(\Omega))^N = (W^{1,2}(\Omega))^N,$

$\mathbb{L}_\sigma^2(\Omega)$: The closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{L}^2(\Omega)$-norm,

$\mathbb{H}_\sigma^1(\Omega)$: The closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{H}^1(\Omega)$-norm,

$H = \mathbb{L}_\sigma^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$: Hilbert space,

$C_\pi([0,S]; H) = \{\, U \in C([0,S]\,;\, H)\,;\, U(0) = U(S)\,\},$

$\mathcal{P}_\Omega$: The orthogonal projection from $\mathbb{L}^2(\Omega)$ onto $\mathbb{L}_\sigma^2(\Omega)$,

$\mathcal{A} = -\mathcal{P}_\Omega\Delta$: The Stokes operator with domain $D(\mathcal{A}) = \mathbb{H}^2(\Omega) \cap \mathbb{H}_\sigma^1(\Omega)$,

$$A_N = -\Delta \quad \text{with domain} \quad D(A_N) = \{\, u \in H^2(\Omega)\,;\, \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,\},$$

$\mathcal{A}^\alpha$, A_N^α denote the fractional powers of $\mathcal{A}$, A_N of order α.

2.2 Subdifferential Operator and Nonlinear Interpolation Class

Let φ be a proper lower semi-continuous convex function from H into $(-\infty, +\infty]$. Define the effective domain of φ by $D(\varphi) = \{U \in H; \varphi(U) < +\infty\}$ and the subdifferential of φ by

$$\partial\varphi(U) = \{f \in H; \varphi(V) - \varphi(U) \leqslant (f, V - U)_H \text{ for all } V \in H\}$$

with domain $D(\partial\varphi) = \{U \in H; \partial\varphi(U) \neq \varnothing\}$.

Generally, subdifferential operators are multivalued maximal monotone operators. However, since subdifferential operators used in this paper are always single-valued, we restrict ourselves to the single-valued subdifferential operators. It will be shown that the leading terms of the system (BF) can be given as the subdifferential of some lower semi-continuous convex function in the next subsection.

It is well known that for any maximal monotone operator A in H, the resolvent of A; $J_\lambda = (I + \lambda A)^{-1}$ $(\lambda > 0)$, is well defined on H and $J_\lambda U \to U$ as $\lambda \to 0$ for all $U \in \overline{D(A)}$. Then for $\alpha \in (0,1)$, $p \in [1, \infty]$, by measuring how fast $J_\lambda U$ converges to U, we can define a nonlinear interpolation class $\mathcal{B}_{\alpha,p}(A)$ associated with A by

$$\mathcal{B}_{\alpha,p}(A) = \{\, U \in \overline{D(A)}\,;\, t^{-\alpha}\,|U - J_t U|_H \in L_*^p(0,1)\,\},$$

where $L_*^p = L^p(dt/t)$, i.e., $|f|_{L_*^p(0,S)} = \left(\int_0^S |f(t)|^p t^{-1}\,dt\right)^{1/p}$ for $1 \leqslant p < \infty$ and $L_*^\infty = L^\infty$. We often use the notation

$$|U|_{\mathcal{B}_{\alpha,p}(A)} = \left|\, t^{-\alpha}|U - J_t U|_H \,\right|_{L_*^p(0,1)}.$$

This nonlinear interpolation class $\mathcal{B}_{\alpha,p}(A)$ covers a very wide class of interpolation spaces already known such as Besov spaces. In particular, if A is non-negative self-adjoint operator, then the domain of the fractional power of A of order α is given by $D(A^\alpha) = \mathcal{B}_{\alpha,2}(A)$ (see [2], [3] and [4]). In what follows, we use this nonlinear interpolation theory for the special case where $A = \partial\varphi$.

2.3 Reduction to an Abstract Problem

In this subsection, we reduce our problem to an abstract problem in some Hilbert space. Operating the projection $\mathcal{P}_\Omega$ to the first equation of (BF) to erase the pressure term ∇p. Then we obtain the following equations:

$$\begin{cases} \partial_t \boldsymbol{u} + \nu\, \mathcal{A}\, \boldsymbol{u} = -a\, \boldsymbol{u} + \mathcal{P}_\Omega\, \boldsymbol{g}\, T + \mathcal{P}_\Omega\, \boldsymbol{h}\, C + \mathcal{P}_\Omega\, \boldsymbol{f}_1, \\ \partial_t T + A_N\, T + \boldsymbol{u}\cdot\nabla T = f_2, \\ \partial_t C + A_N\, C + \boldsymbol{u}\cdot\nabla C = -\rho\, A_N\, T + f_3. \end{cases} \tag{2.1}$$

We introduce the Hilbert space H_η for each parameter $\eta \in (0,1]$, which designates the Hilbert space H endowed with the following inner product:

$$(U_1,\ U_2)_{H_\eta} = (\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathbb{L}^2_\sigma} + (T_1, T_2)_{L^2} + \frac{\eta^2}{9\rho^2}\,(C_1, C_2)_{L^2} \quad \text{for } U_i = (\boldsymbol{u}_i, T_i, C_i)^t,\ (i = 1, 2). \tag{2.2}$$

Here, in order to deal with the perturbation term $\rho\,\Delta T = -\rho\, A_N\, T$ as a small perturbation to our problem, we put the weight depending on η and ρ to the last term.

Next, as a lower semi-continuous convex function from H_η to $[0, +\infty]$, we define φ by

$$\varphi(U) = \begin{cases} \dfrac{\nu}{2}\, \|\, |\nabla \boldsymbol{u}|\, \|_{L^2}^2 + \dfrac{1}{2}\, \|\, \nabla T\, \|_{L^2}^2 + \dfrac{\eta^2}{18\rho^2}\, \|\, \nabla C\, \|_{L^2}^2 & \text{if } U \in D(\varphi), \\ +\infty & \text{if } U \in H_\eta \backslash D(\varphi), \end{cases} \tag{2.3}$$

where $D(\varphi) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ is the effective domain of φ. Then the subdifferential of φ is given by

$$\partial\varphi(U) = \begin{pmatrix} -\nu\, \mathcal{P}_\Omega\, \Delta \boldsymbol{u} \\ -\Delta\, T \\ -\Delta\, C \end{pmatrix} \quad \text{with domain } D(\partial\varphi) = (\mathbb{H}^2 \cap \mathbb{H}^1_\sigma) \times D(A_N) \times D(A_N). \tag{2.4}$$

Furthermore, we put

$$U = \begin{pmatrix} \boldsymbol{u} \\ T \\ C \end{pmatrix},\ \frac{dU}{dt} = \begin{pmatrix} \partial_t \boldsymbol{u} \\ \partial_t T \\ \partial_t C \end{pmatrix},\ B(U) = \begin{pmatrix} a\boldsymbol{u} - \mathcal{P}_\Omega\, \boldsymbol{g}\, T - \mathcal{P}_\Omega\, \boldsymbol{h}\, C \\ \boldsymbol{u}\cdot\nabla T \\ \boldsymbol{u}\cdot\nabla C - \rho\,\Delta\, T \end{pmatrix},\ F = \begin{pmatrix} \mathcal{P}_\Omega\, \boldsymbol{f}_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{2.5}$$

Then the initial boundary value problem for (2.1) is reduced to the following abstract Cauchy problem in H_η:

$$(\mathrm{CP}) \begin{cases} \dfrac{dU}{dt}(t) + \partial\varphi\,(U(t)) + B(U(t)) = F(t) \quad t \in [0, S], \\ U(0) = U_0, \end{cases} \tag{2.6}$$

and the periodic problem for (2.1) is reduced to the following abstract periodic problem in H_η:

$$(\mathrm{AP}) \begin{cases} \dfrac{dU}{dt}(t) + \partial\varphi\,(U(t)) + B(U(t)) = F(t) \quad t \in [0, S], \\ U(0) = U(S). \end{cases} \tag{2.7}$$

2.4 Known Abstract Theorem

In order to assure the existence of the solutions, we rely on abstract results given in [9] and [10]. To formulate these results, we introduce the following conditions.

Assumptions

(A1) For any $L \in (0, +\infty)$, the set $\{U \in H\,;\, \varphi(U) + \|U\|_H^2 \leqslant L\}$ is compact in H.

(A2) $B(\cdot)$ is φ-demiclosed in the following sense:
$U_n \to U$ strongly in $C([0,S];H)$, $\partial\varphi(U_n) \rightharpoonup \partial\varphi(U)$ weakly in $L^2(0,S;H)$, $B(U_n) \rightharpoonup b$ weakly in $L^2(0,S;H)$, then $b(t) = B(U(t))$ holds for a.e. $t \in [0,S]$.

$(\mathrm{A3})^0_\alpha$ For a given exponent $\alpha \in (0, 1/2)$, there exists a monotone increasing function $\ell(\cdot)$ such that

$$\|B(U)\|_H \leqslant \ell(\|U\|_H)\,\{\varepsilon\,\|\partial\varphi(U)\|_H + \frac{1}{\varepsilon}\,|\varphi(U)|^{\frac{1-\alpha}{1-2\alpha}} + 1\} \quad {}^\forall U \in D(\partial\varphi),$$

where ε is a positive constant determined by the initial data U_0 and the external force $F(t)$, more precisely, ε is a monotone decreasing function of $|U_0|_H + |U_0|_{\mathcal{B}_{\alpha,p}(\partial\varphi)} + |F|_{L^2(0,S;H)}$.

(A4) There exists a monotone increasing function $\ell(\cdot)$ and $k \in (0,1)$ such that

$$\|B(U)\|_H^2 \leq k\,\|\partial\varphi(U)\|_H^2 + \ell\,(\varphi(U) + \|U\|_H^2) \quad {}^\forall U \in D(\partial\varphi).$$

(A5) There exists a monotone increasing function $\ell(\cdot)$ and a constant $k \in [0,1)$ such that

$$\|B(U)\|_H^2 \leqslant k\,\|\partial\varphi(U)\|_H^2 + \ell\,(\|U\|_H)(\varphi(U) + 1)^2 \quad {}^\forall U \in D(\partial\varphi).$$

(A6) There exist positive constants α, K such that

$$(-\partial\varphi(U) - B(U),\, U)_H + \alpha\,\varphi(U) \leqslant K \quad {}^\forall U \in D(\partial\varphi).$$

Then the following results hold (see [9] and [10]).

Theorem 2.1. *Let $U_0 \in \mathcal{B}_{\alpha,p}(\partial\varphi)$ with $p \in [1,2]$ and $F \in L^2(0,S;H)$, and let* (A1), (A2) *and* $(\mathrm{A3})^0_\alpha$ *be satisfied. Then there exists $S_0 \in (0,S]$ depending on $|U_0|_H$ and $|U_0|_{\mathcal{B}_{\alpha,p}(\partial\varphi)}$ such that* (CP) *has a solution $U(t)$ in $[0,S_0]$ satisfying*

$$t^{1/2-\alpha}\,dU/dt,\ t^{1/2-\alpha}\,\partial\varphi(U(t)),\ t^{1/2-\alpha}\,B(U(t)) \in L^2(0,S_0\,;H),$$

$$t^{-\alpha}\,\|U(t) - U_0\|_H,\ t^{1/2-\alpha}\,|\varphi(U(t))|^{1/2} \in L^q_*(0,S_0) \quad {}^\forall q \in [2,\infty].$$

Theorem 2.2. *Let* (A1), (A2) *and* (A4) *be satisfied and let $U_0 \in D(\varphi)$ and $F \in L^2(0,S;H)$. Then there exists $S_0 \in (0,S]$ depending on $|U_0|_H$ and $\varphi(U_0)$ such that* (CP) *has a solution $U(t)$ in $[0,S_0]$ satisfying*

$$dU/dt,\ \partial\varphi(U(t)),\ B(U(t)) \in L^2(0,S_0\,;H),$$

$$\varphi(U(t)) \ \text{ is absolutely continuous on } [0,S_0].$$

Theorem 2.3. *Let* (A1), (A2), (A5) *and* (A6) *be satisfied. Then for every $F \in L^2(0,S;H)$,* (AP) *has a strong solution $U \in C_\pi([0,S];H)$ such that*

$$dU/dt,\ \partial\varphi(U),\ B(U) \in L^2(0,S;H),$$

$$\varphi(U) \ \text{ is absolutely continuous on } [0,S] \text{ and } \varphi(U(0)) = \varphi(U(S)).$$

Remark

In [9], Theorem 2.3 is actually proved under a different assumption $(A3)_\alpha$ which is slightly stronger than $(A3)^0_\alpha$. However it is easy to see that the proof of Theorem 2.3 holds true with $(A3)_\alpha$ replaced by $(A3)^0_\alpha$ (see the proof of Theorem I in [9]).

3 Main Results

Theorem 3.1. *([12] : Initial Boundary Value Problem)*
Let $N \leqslant 3$ and let $\boldsymbol{f}_1 \in L^2(0,S;\mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0,S;L^2(\Omega))$. Then for each initial data $U_0 = (\boldsymbol{u}_0, T_0, C_0)^t \in D(\mathcal{A}^\alpha) \times D(A_N^\alpha) \times D(A_N^\alpha)$ with $\alpha \in [1/4, 1/2]$, (BF) *admits a unique solution $U = (\boldsymbol{u},\ T,\ C)^t \in C([0,S];H)$ satisfying $U(0) = U_0$ and*

$$(\#)_\alpha \quad \begin{cases} t^{1/2-\alpha}\,\partial_t \boldsymbol{u},\ t^{1/2-\alpha}\mathcal{A}\,\boldsymbol{u} \in L^2(0,S\,;\,\mathbb{L}^2_\sigma(\Omega)), \\ t^{1/2-\alpha}\,\|\nabla \boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} \in L^p_*(0,S) \quad \textit{for all}\ p \in [2,\infty], \\ t^{1/2-\alpha}\,\partial_t\,T,\ t^{1/2-\alpha}\,\partial_t\,C,\ t^{1/2-\alpha}\Delta\,T,\ t^{1/2-\alpha}\Delta\,C \in L^2(0,S;L^2(\Omega)), \\ t^{1/2-\alpha}\,\|\nabla T\|_{L^2(\Omega)},\ t^{1/2-\alpha}\,\|\nabla C\|_{L^2(\Omega)} \in L^p_*(0,S) \quad \textit{for all}\ p \in [2,\infty], \end{cases}$$

Theorem 3.2. *([12] : Time Periodic Problem)*
Let $\boldsymbol{f}_1 \in L^2(0,S;\mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0,S;L^2(\Omega))$. Furthermore, we assume that f_2, f_3 satisfy

$$\int_0^S \int_\Omega f_2(x,t)\,dx\,dt = \int_0^S \int_\Omega f_3(x,t)\,dx\,dt = 0. \tag{3.1}$$

Then (BF) *admits a solution $U = (\boldsymbol{u},\ T,\ C)^t \in C_\pi([0,S]\,;\,H)$ satisfying*

$$(\#)_{1/2} \quad \begin{cases} \partial_t \boldsymbol{u},\ \mathcal{A}\,\boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\Omega)), \\ \boldsymbol{u} \in C([0,S];\mathbb{H}^1_\sigma(\Omega)), \\ \partial_t\,T,\ \partial_t\,C,\ \Delta\,T,\ \Delta\,C \in L^2(0,S;L^2(\Omega)), \\ T,\ C \in C([0,S];H^1(\Omega)). \end{cases}$$

Remarks

(1) If U_0 belongs to $D(\mathcal{A}^{1/2}) \times D(A_N^{1/2}) \times D(A_N^{1/2}) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ in Theorem 3.1, then the solution U satisfies property $(\#)_{1/2}$ given in Theorem 3.2.

(2) It can be shown that the required Condition (3.1) is also the necessary condition for the existence of the periodic solution of (BF) satisfying the homogeneous Neumann boundary condition. In fact, integrating the second and the third equations over $\Omega \times [0,S]$, we can derive (3.1).

(3) In [13], the same result as in Theorem 3.1 is given for (BF) with the homogeneous Neumann boundary condition replaced by the homogeneous Dirichlet boundary condition only for the case $\alpha = 1/2$. However, with obvious modifications, we can show that if $U_0 \in D(\mathcal{A}^\alpha) \times D(A_D^\alpha) \times D(A_D^\alpha)$ with $\alpha \in [1/4, 1/2]$, then the Dirichlet problem for (BF) admits a unique solution U satisfying $(\#)_\alpha$.

(4) The characterizations for the domains of the fractional powers of A_N and $\mathcal{A}$ can be found in [7] and [6].

4 Initial Boundary Value problem

In this section, we give an outline of our proof of Theorem 3.1. This argument is divided into three parts, i.e., the local existence, the global existence and the uniqueness.

4.1 Local Existence

In order to prove the local existence, we are going to check conditions required in Theorems 2.1 and 2.2. For this purpose, we choose $\eta = \varepsilon$, where ε is an exponent appearing in $(A3)^0_\alpha$.

First, it is easy to see that the compactness of φ-level set and the φ-demiclosedness of $B(U)$ (the required condition (A1) and (A2)) can be satisfied.

Furthermore, in spite of the lack of coercivity of the leading term, we can take almost the same procedure as that in [13] and we can derive the following estimate of the perturbation term B with $N \leqslant 3$:

$$\|B(U)\|_H \leqslant \varepsilon \, \|\partial\varphi(U)\|_{H_\eta} + \frac{\gamma}{\varepsilon} \left(\varphi^{3/2}(U) + \|U\|_{H_\eta} + 1 \right), \tag{4.1}$$

where γ is a constant which depends on some Sobolev's embedding constants, the positive coefficients and the constant vectors in (BF). This estimate (4.1) ensures the condition $(A3)^0_\alpha$ with $\alpha \in [1/4, 1/2)$ and the condition (A4).

Hence, Theorem 2.1 assures the existence of local solutions $U(t)$ on $[0, S_0]$ satisfying $(\#)_\alpha$ with S replaced by S_0 when $U_0 = (\boldsymbol{u}_0, T_0, C_0)^t \in D(\mathcal{A}^\alpha) \times D(A_N^\alpha) \times D(A_N^\alpha)$ for $\alpha \in [1/4, 1/2)$. Moreover, from Theorem 2.2, (BF) has local solutions satisfying $(\#)_{1/2}$ with S replaced by S_0 when $U_0 = (\boldsymbol{u}_0, T_0, C_0)^t \in D(\mathcal{A}^{1/2}) \times D(A_N^{1/2}) \times D(A_N^{1/2}) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$.

4.2 Global Existence

In this subsection, we show that every local solutions can be continued globally to $[0, S]$ by establishing some a priori estimates.

Although $\partial\varphi(U)$ loses the coercivity because of the Neumann boundary condition, we can obtain the boundedness of the solution. Indeed, establishing the following a priori estimate, we can derive the boundedness of $\sup_{0 \leq t \leq S} \varphi(U(t))$ (boundedness of H^1-norm) of the solutions:

$$\begin{array}{ll} (1) & \text{2nd equation } \times T, \\ (2) & \text{3rd equation } \times C, \\ (3) & \text{1st equation } \times \partial_t \boldsymbol{u}, \\ (4) & \text{2nd equation } \times -\Delta T, \ \partial_t T, \\ (5) & \text{3rd equation } \times -\Delta C, \ \partial_t C. \end{array} \tag{4.2}$$

Therefore, we can assure that the local solutions can be globally extended when $\alpha = 1/2$.

In the cases where $\alpha \in [1/4, 1/2)$, the local solutions also can be extended from the regularity property $(\#)_\alpha$, i.e., from the fact that there exist $0 < t_0 \leqslant S_0$ where the solutions satisfy $(\boldsymbol{u}(t_0), T(t_0), C(t_0))^t \in D(\varphi) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$.

4.3 Uniqueness

In this subsection, we are going to prove the uniqueness of the solution of the initial boundary value problem for (BF).

Let U^1 and U^2 be solutions of (BF) for the same initial data:

$$U^i = \begin{pmatrix} \boldsymbol{u}^i \\ T^i \\ C^i \end{pmatrix} \quad (i = 1, 2)$$

and let

$$\begin{pmatrix} \boldsymbol{w} \\ \tau \\ \theta \end{pmatrix} = U^1 - U^2.$$

Then we can obtain the following inequality of $(\boldsymbol{w}, \tau, \theta)^t$ as in [13].

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} y(t) &\leqslant \gamma\, y(t) + \frac{\gamma^2}{\nu} \|\nabla T^2(t)\|_{L^2}^4 \|\boldsymbol{w}(t)\|_{\mathbf{L}^2_\sigma}^2 + \frac{\gamma^2}{2\rho^4\nu^2} \|\nabla C^2\|_{L^2}^4 \|\boldsymbol{w}(t)\|_{\mathbf{L}^2_\sigma}^2 \\ &\leqslant \gamma\,(\, \|\nabla T^2\|_{L^2}^4 + \|\nabla C^2\|_{L^2}^4 + 1\,)\, y(t) \end{aligned}$$

where $y(t) = \|\boldsymbol{w}(t)\|_{\mathbf{L}^2_\sigma}^2 + \|\tau(t)\|_{L^2}^2 + \frac{1}{2\rho^2}\|\theta(t)\|_{L^2}^2$. Here we note that $(\#)_\alpha$ with $\alpha \in [1/4, 1/2]$ implies that

$$t^{1/2-\alpha}\,\|\nabla T^2\|_{L^2},\ t^{1/2-\alpha}\,\|\nabla C^2\|_{L^2} \in L^4_*(0, S) \quad \Rightarrow \quad \|\nabla T^2\|_{L^2},\ \|\nabla C^2\|_{L^2} \in L^4(0, S).$$

Hence, the uniqueness follows from Gronwall's inequality.

5 Periodic Problem

In this section, we give an outline of our proof of Theorem 3.2. When we try to apply the abstract result Theorem 2.3 directly to (BF), some difficulties arise. Therefore, we first introduce some approximate systems with two parameters ε, λ and we show the existence of periodic solutions for these approximate systems. We next consider the convergence of solutions of approximate equations as $\varepsilon, \lambda \to 0$. In the argument as $\lambda \to 0$, we face some difficulty which does not appear in the case where we impose Dirichlet boundary condition.

5.1 Approximate Equations

When one tries to apply Theorem 2.3 to (AP), one faces some difficulties. The most serious one arises in checking (A5). In fact, from (4.1), whose growth order for $\varphi(U)$ is cubic, it is difficult to show that $B(U)$ satisfies the required growth order in (A5). Moreover, when the constant vectors $\boldsymbol{g}$, $\boldsymbol{h}$ are very large, it is difficult to examine whether (A6) is satisfied. From these reasons, we are led to introduce the same type of relaxed approximate problems as in [11].

However, approximate problems introduced in [11] prevents establishing desirable a priori estimates under the homogeneous Neumann boundary conditions. In order to manage with this difficulty, we introduce another approximation procedure.

More precisely, we add linear coercive terms to the second and the third equations and also replace T and C by their cut-off functions $[T]_\varepsilon$ and $[C]_\varepsilon$. We consider the following approximate equations.

$$(\mathrm{BF})_{\varepsilon,\lambda}\quad \begin{cases} \partial_t \boldsymbol{u} = \nu\,\mathcal{P}_\Omega\,\Delta\,\boldsymbol{u} - a\,\boldsymbol{u} + \mathcal{P}_\Omega\,\boldsymbol{g}\,[T]_\varepsilon + \mathcal{P}_\Omega\,\boldsymbol{h}\,[C]_\varepsilon + \mathcal{P}_\Omega\,\boldsymbol{f}_1, \\ \partial_t\,T + \boldsymbol{u}\cdot\nabla T = \Delta\,T - \varepsilon\,|T|^{p-2}T - \lambda\,T + f_2, \\ \partial_t\,C + \boldsymbol{u}\cdot\nabla C = \Delta\,C + \rho\,\Delta\,T - \varepsilon\,|C|^{p-2}C - \lambda\,C + f_3, \end{cases} \tag{5.1}$$

where $\varepsilon,\ \lambda \in (0,1)$ are approximation parameters and the cut-off function $[T]_\varepsilon$ is defined by

$$[T]_\varepsilon = \begin{cases} T & \text{if } |T| \leqslant 1/\varepsilon, \\ (\mathrm{Sgn}\ T)\,1/\varepsilon & \text{if } |T| \geqslant 1/\varepsilon, \quad \varepsilon \in (0,1), \end{cases} \tag{5.2}$$

and p is some sufficiently large exponent.

We are going to reduce these approximate equations (5.1) to an abstract problem (AP). For the perturbation term, we replace it by

$$B_\varepsilon(U) = \begin{pmatrix} a\boldsymbol{u} - \mathcal{P}_\Omega\,\boldsymbol{g}\,[T]_\varepsilon - \mathcal{P}_\Omega\,\boldsymbol{h}\,[C]_\varepsilon \\ \boldsymbol{u}\cdot\nabla T \\ \boldsymbol{u}\cdot\nabla C - \rho\,\Delta\,T \end{pmatrix}. \tag{5.3}$$

We also need to replace the lower semi-continuous convex function φ by $\varphi_{\varepsilon,\lambda}$ which is given by

$$\varphi_{\varepsilon,\lambda}(U) = \varphi(U) + \psi_{\varepsilon,\lambda}(U),$$

$$\psi_{\varepsilon,\lambda}(U) = \begin{cases} \dfrac{\varepsilon}{p}\|T\|_{L^p}^p + \dfrac{\varepsilon}{9\rho^2 p}\,\|C\|_{L^p}^p + \dfrac{\lambda}{2}\|T\|_{L^2}^2 + \dfrac{\lambda}{18\rho^2}\|C\|_{L^2}^2 & \text{if } U \in D(\psi_{\varepsilon,\lambda}) \\ +\infty & \text{if } U \in H_\eta \backslash D(\psi_{\varepsilon,\lambda}), \end{cases}$$

where $D(\psi_{\varepsilon,\lambda}) = \mathbb{L}^2_\sigma(\Omega) \times L^p(\Omega) \times L^p(\Omega)$.

Here and henceforth, we choose $\eta = 1$ in (2.2), definition of the inner product of H_η. Then it is clear that $\psi_{\varepsilon,\lambda}$ is a lower semi-continuous convex function on H_η and Fréchet differentiable on $D(\psi_{\varepsilon,\lambda})$ and that the subdifferential of $\psi_{\varepsilon,\lambda}$ coincides with the sum of dissipation terms and coercive terms, i.e.,

$$\partial\psi_{\varepsilon,\lambda}(U) = (\,0,\ \varepsilon\,|T|^{p-2}T + \lambda\,T,\ \varepsilon\,|C|^{p-2}C + \lambda\,C\,)^t.$$

In general, the sum of two subdifferentials is not always maximal monotone. However, for this case, we have the following good property:

$$\begin{aligned} (\,\partial\varphi(U),\ \partial\psi_{\varepsilon,\lambda}(U))_H &= (-\Delta\,T,\ \varepsilon\,|T|^{p-2}T + \lambda\,T\,)_{L^2} + (\,-\,\Delta\,C,\ \varepsilon\,|C|^{p-2}C\ + \lambda\,C)_{L^2} \\ &= \lambda\,\|\nabla T\|_{L^2}^2 + \varepsilon\,(p-1)\int_\Omega |T|^{p-2}|\nabla T|^2 dx \\ &\quad + \lambda\,\|\nabla C\|_{L^2}^2 + \varepsilon\,(p-1)\int_\Omega |C|^{p-2}|\nabla C|^2 dx \geqslant 0. \end{aligned} \tag{5.4}$$

By virtue of (5.4), together with Proposition 2.17, Theorem 4.4 and Proposition 4.6 in Brézis [5], we can deduce that $\partial\varphi + \partial\psi_{\varepsilon,\lambda}$ becomes maximal monotone, and hence we get $\partial\,(\varphi + \psi_{\varepsilon,\lambda}) = \partial\varphi + \partial\psi_{\varepsilon,\lambda}$ with $D(\partial\,(\varphi + \psi_{\varepsilon,\lambda})) = D(\partial\varphi) \cap D(\partial\psi_{\varepsilon,\lambda})$.

Thus, we have another abstract problem associated with approximate problems:

$$(\mathrm{AP})_{\varepsilon,\lambda} \begin{cases} \dfrac{dU(t)}{dt} + \partial\varphi_{\varepsilon,\lambda}(U(t)) + B_\varepsilon(U(t)) = F(t) \quad t \in [0,S], \\ U(0) = U(S). \end{cases} \tag{5.5}$$

By almost the same arguments as that of [11], we can assure that $(\mathrm{BF})_{\varepsilon,\lambda}$ satisfies required conditions (A5) and (A6) by virtue of replacing by cut-off functions and adding dissipation terms $-\varepsilon\,|T|^{p-2}T$, $-\varepsilon\,|C|^{p-2}C$ where $p \geqslant 12$. Hence, applying the abstract result Theorem 2.3 to our approximate problems, we can show that $(\mathrm{BF})_{\varepsilon,\lambda}$ has time periodic solutions $U_{\varepsilon,\lambda} = (\boldsymbol{u}_{\varepsilon,\lambda}, T_{\varepsilon,\lambda}, C_{\varepsilon,\lambda})^t$.

5.2 Convergence as $\varepsilon \to 0$

In this subsection, we discuss the convergence of the approximate solutions of approximate equations. First, we consider the convergence as $\varepsilon \to 0$.

In spite of the lack of coercivity of $\partial\varphi$, due to relaxation terms $-\lambda T, -\lambda C$, we can use the same convergence argument as in [11]. Indeed, we can derive the following boundedness

$$\sup_{0\leqslant t\leqslant S} \|U_{\varepsilon,\lambda}(t)\|_H,\ \sup_{0\leqslant t\leqslant S} \varphi_{\varepsilon,\lambda}(U_{\varepsilon,\lambda}(t)),\ \|\partial\varphi_{\varepsilon,\lambda}(U_{\varepsilon,\lambda})\|_{L^2(0,S;H)},\ \left\|\frac{dU_{\varepsilon,\lambda}}{dt}\right\|_{L^2(0,S;H)} \leqslant \gamma_\lambda \tag{5.6}$$

from the following a priori estimates:

$$\begin{array}{ll} (1) & \text{2nd equation } \times T_{\varepsilon,\lambda}, \\ (2) & \text{3rd equation } \times C_{\varepsilon,\lambda}, \\ (3) & \text{1st equation } \times \boldsymbol{u}_{\varepsilon,\lambda}, \\ (4) & \text{1st equation } \times \partial_t\boldsymbol{u}_{\varepsilon,\lambda}, \\ (5) & \text{2nd equation } \times -\Delta T_{\varepsilon,\lambda},\ \partial_t T_{\varepsilon,\lambda}, \\ (6) & \text{3rd equation } \times -\Delta C_{\varepsilon,\lambda},\ \partial_t C_{\varepsilon,\lambda}, \end{array} \tag{5.7}$$

where γ_λ denotes the general constant depending on the external forces, positive constants, constant vectors in the system $(\mathrm{BF})_{\varepsilon,\lambda}$ and λ but not on ε.

Hence, by the standard argument, letting $\varepsilon \to 0$, we can assure that the following equations $(\mathrm{BF})_\lambda$ have time periodic solutions $U_\lambda = (\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)^t$, for each parameter λ:

$$(\mathrm{BF})_\lambda \begin{cases} \partial_t \boldsymbol{u} = \nu\,\mathcal{P}_\Omega\,\Delta\,\boldsymbol{u} - a\,\boldsymbol{u} + \mathcal{P}_\Omega\,\boldsymbol{g}\,T + \mathcal{P}_\Omega\,\boldsymbol{h}\,C + \mathcal{P}_\Omega\,\boldsymbol{f}_1, \\ \partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta\,T - \lambda\,T + f_2, \\ \partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta\,C + \rho\,\Delta\,T - \lambda\,C + f_3. \end{cases} \tag{5.8}$$

5.3 Convergence as $\lambda \to 0$

As the last step, we discuss the convergence of solutions when $\lambda \to 0$. To do this, we need to establish appropriate a priori estimates. However, because of the lack of coercivity of the leading term, it is difficult to establish appropriate a priori estimates. To cope with this difficulty, we use the assumption for external forces (3.1).

Integrating the second equation of $(\mathrm{BF})_\lambda$ over Ω, we get

$$\frac{d}{dt}\int_\Omega T_\lambda(x,t)\,dx + \lambda\int_\Omega T_\lambda(x,t)\,dx = \int_\Omega f_2(x,t)\,dx \quad {}^\forall t \in [0,S]. \tag{5.9}$$

Here we used the following facts:

$$\int_\Omega \Delta T_\lambda\,dx = \int_{\partial\Omega}\frac{\partial T_\lambda}{\partial n}\,dS = 0, \quad \int_\Omega \boldsymbol{u}_\lambda\cdot\nabla T_\lambda\,dx = \int_\Omega \mathrm{div}\,(\boldsymbol{u}_\lambda T_\lambda)\,dx = \int_{\partial\Omega}\boldsymbol{u}_\lambda T_\lambda\,dS = 0.$$

Integrating (5.9) over $(0,S)$ and using the periodic condition and (3.1), we find that

$$\lambda\int_0^S\int_\Omega T_\lambda(x,t)\,dx\,dt = 0.$$

Therefore, from the continuity of the solutions T_λ, there exist $t_0\in[0,S]$ such that $\int_\Omega T_\lambda(x,t_0)\,dx = 0$.

Hence by (5.9), we obtain

$$\int_\Omega T_\lambda(x,t)\,dx = \int_{t_0}^t e^{-\lambda(t-s)}\int_\Omega f_2(x,t)\,dx\,dt \quad {}^\forall t\in[t_0,t_0+S]. \tag{5.10}$$

Then applying Poincaré-Wirtinger's inequality

$$\|v-\overline{v}\|_{L^2} \leqslant C_W\,\|\nabla v\|_{L^2} \quad {}^\forall v\in H^1(\Omega), \quad \overline{v} = \frac{1}{|\Omega|}\int_\Omega v(x)\,dx,$$

we obtain

$$\|T_\lambda\|_{L^2(0,S;L^2(\Omega))} \leqslant C_W\,\|\nabla T_\lambda\|_{L^2(0,S;L^2(\Omega))} + S\,\|f_2\|_{L^2(0,S;L^2(\Omega))}, \tag{5.11}$$

where C_W is a suitable constant which depends only on Ω. Similarly, we can derive the following inequality from the third equation of $(\mathrm{BF})_\lambda$:

$$\|C_\lambda\|_{L^2(0,S;L^2(\Omega))} \leqslant C_W\,\|\nabla C_\lambda\|_{L^2(0,S;L^2(\Omega))} + S\,\|f_3\|_{L^2(0,S;L^2(\Omega))}. \tag{5.12}$$

Then, using these inequalities and repeating exactly the same arguments as in section 5.2, we can assure the existence of periodic solutions of the original system (BF). Indeed, (5.11) and (5.12) together with the following manipulation:

$$\begin{array}{ll} (1) & \text{2nd equation } \times T_\lambda, \\ (2) & \text{3rd equation } \times C_\lambda, \\ (3) & \text{1st equation } \times \partial_t\boldsymbol{u}_\lambda, \\ (4) & \text{2nd equation } \times -\Delta T_\lambda,\ \partial_t T_\lambda, \\ (5) & \text{3rd equation } \times -\Delta C_\lambda,\ \partial_t C_\lambda, \end{array} \tag{5.13}$$

lead us to appropriate a priori estimates. By using this uniform boundedness of U_λ derived from the above and considering the convergence of the solutions and the equations $(\mathrm{BF})_\lambda$ as $\lambda\to 0$, we can assure the existence of periodic solution of the original system (BF), whence follows our result.

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