数理解析研究所講究録
第 1893 巻 2014 年 159-165

Majorization, Operator inequality and Operator mean

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1 Majorization

Let I be an interval in the real axis. We denote by $\mathbf{P}(I)$ the set of all operator monotone functions on I. A constant function is here admitted to be an operator monotone function.

Definition 1.1 Let $I=[a,b)$ or $I=(a,b)$ with $-\infty\leq a<b\leq\infty$.

$$\mathbf{LP}_{+}(I):=\{h \text{ on } I|\ h(t)>0 \text{ on } (a,b),\ \log h\in \mathbf{P}(a,b)\},$$

If $-\infty<a$, identifying $h\in \mathbf{LP}_{+}(a,b)$ as its natural extension to $[a,b)$ gives

$$\mathbf{LP}_{+}(a,b)=\mathbf{LP}_{+}[a,b)$$

Example 1.1 $t^{r}\in \mathbf{LP}_{+}[0,\infty)$ *for* $r>0$.

Definition 1.2

$$\begin{aligned}\mathbf{P}_+^{-1}(a,b): &= \{h|\ h \textit{ is increasing on } (a,b),\ \textit{the range is } (0,\infty)\\ &\quad h^{-1} \in \mathbf{P}(0,\infty)\}.\\ \mathbf{P}_+^{-1}[a,b): &= \{h|\ h \textit{ is increasing on } [a,b),\ \textit{the range is } [0,\infty)\\ &\quad h^{-1} \in \mathbf{P}(0,\infty)\}.\end{aligned}$$

If $-\infty < a$,

$$\mathbf{P}_+^{-1}[a,b) = \mathbf{P}_+^{-1}(a,b).$$

Example 1.2 $t^r \in \mathbf{P}_+^{-1}[0,\infty)$ *for* $r \geq 1$.
$e^t \in \mathbf{P}_+^{-1}(-\infty,\infty)$.
$t\log t \in \mathbf{P}_+^{-1}(1,\infty) = \mathbf{P}_+^{-1}[1,\infty)$

Definition 1.3 ([8, 9]) Let $h(t)$ and $k(t)$ be non-decreasing functions on I, and, further, suppose $k(t)$ is increasing. Then h is said to be *majorized* by k, in symbol

$$h \preceq k \quad \text{on } I$$

if the composite $h \circ k^{-1}$ of h and k^{-1} belongs to $\mathbf{P}(k(I))$.

This definition is equivalent with

$$\sigma(A), \sigma(B) \subset I,\ k(A) \leq k(B) \Longrightarrow h(A) \leq h(B).$$

Example 1.3 $f(t) \preceq t$ on $I \Leftrightarrow f \in \mathbf{P}(I)$.
$t \preceq e^t$ on $(-\infty,\infty)$.
$u(t)^\alpha \preceq u(t)^\beta$ on I if $0 < u(t)$ is increasing on I and $0 < \alpha < \beta$.

Lemma 1.1 Product Lemma
Let $I = [a,b)$ or $I = (a,b)$ with $-\infty \leqq a < b \leqq \infty$, and let h, g be non-negative functions defined on I. Suppose the product hg is increasing, $(hg)(a+0) = 0$ and $(hg)(b-0) = \infty$. Then

$$g \preceq hg \text{ on } I \Longleftrightarrow h \preceq hg \text{ on } I.$$

Moreover for every ψ_1, ψ_2 in $\mathbf{P}_+[0,\infty)$

$$g \preceq hg \text{ on } I \Longrightarrow \psi_1(h)\psi_2(g) \preceq hg \text{ on } I.$$

We remark that the hypothesis

$$g(t) \preceq h(t)g(t)$$

is not necessarily valid even if $h, g \in \mathbf{P}_+[0,\infty)$; for instance, $t^{1/2},\ t+1 \in \mathbf{P}_+[0,\infty)$, but

$$t^{1/2}4 \not\preceq t^{1/2}(t+1).$$

Indeed, by putting $t^{1/2} = s$, it is equivalent to

$$s \not\preceq s(s^2+1),$$

which was shown in Example 2.1 of [6].

Theorem 1.2 Product theorem *[8, 9]*For every right open interval I,

$$\mathbf{P}_+^{-1}(I)\cdot\mathbf{P}_+^{-1}(I) \subset \mathbf{P}_+^{-1}(I), \quad \mathbf{LP}_+(I)\cdot\mathbf{P}_+^{-1}(I) \subset \mathbf{P}_+^{-1}(I).$$

Further, let $g_i(t) \in \mathbf{LP}_+(I)$ for $1 \le i \le m$ and $h_j(t) \in \mathbf{P}_+^{-1}(I)$ for $1 \le j \le n$. Then for every $\psi_i, \phi_j \in \mathbf{P}_+[0,\infty)$

$$\prod_{i=1}^{m} \psi_i(g_i) \prod_{j=1}^{n} \phi_j(h_j) \preceq \prod_{i=1}^{m} g_i \prod_{j=1}^{n} h_j \in \mathbf{P}_+^{-1}(I).$$

Definition 1.4 A real function $g(t)$ is called an *operator convex function* on I if

$$g(sA+(1-s)B) \le sg(A) + (1-s)g(B)$$

for every $0 < s < 1$ and for every pair of bounded selfadjoint operators A and B whose spectra are both in I.

Proposition 1.3 *Let $f_i(t) \in \mathbf{P}_+(0,\infty)$ for $i = 1, 2, \cdots$. Then there are operator convex functions $h_i(t) > 0$ on $0 < t < \infty$ satisfying*

$$tf_1(t)f_2(t)\cdots f_n(t) = (h_n \circ \cdots \circ h_2 \circ h_1)(t) \quad (0 < t < \infty)$$

for every n.

2 Operator Inequality

There is a function η in $\mathbf{P}_+(I) \cap \mathbf{P}_+^{-1}(I)$ except for $I = (-\infty, \infty)$; for instance,

- $\eta(t) = t$ if $I = (0, \infty)$,
- $\eta(t) = t - a$ if $I = (a, \infty)$,
- $\eta(t) = \frac{t-a}{b-t}$ if $I = (a, b)$ with $-\infty < a < b < \infty$,
- $\eta(t) = \frac{1}{b-t}$ if $I = (-\infty, b)$.

Theorem 2.1 General operator inequality Let I be a right open interval, and let $f \in \mathbf{P}_+(I)$ and $g(t) \in \mathbf{LP}_+(I)$. Put $k(t) = f(t)g(t)$. Let $h(t)$ be increasing function on I such that $f(t)h(t) \in \mathbf{P}_+^{-1}(I)$. Then the following holds.

(i) If $I \neq (-\infty, \infty)$ and $\eta \in \mathbf{P}_+(I) \cap \mathbf{P}_+^{-1}(I)$, then the function ϕ on $(0, \infty)$ defined by

$$\phi(k(t)h(t)) = k(t)\frac{\eta(t)}{f(t)} \quad (t \in I) \tag{1}$$

belongs to $\mathbf{P}_+(0, \infty)$.

(ii) If $I = (-\infty, \infty)$ and $f(t) = 1$, then the function ϕ on $(0, \infty)$ defined by

$$\phi(k(t)h(t)) = k(t) \quad (t \in I) \tag{2}$$

belongs to $\mathbf{P}_+(0, \infty)$.

(iii) Let ϕ be a function given in (i) or (ii). Then for every $\varphi \in \mathbf{P}_+(0, \infty)$ such that

$$\varphi \preceq \phi \quad (0, \infty),$$

$A \leq C \leq B$ implies

$$\begin{aligned} & \varphi(k(C)^{\frac{1}{2}}h(B)k(C)^{\frac{1}{2}}) \\ \geq\ & \varphi(k(C)^{\frac{1}{2}}h(C)k(C)^{\frac{1}{2}}) \\ \geq\ & \varphi(k(C)^{\frac{1}{2}}h(A)k(C)^{\frac{1}{2}}). \end{aligned} \tag{3}$$

Corollary 2.2 Let $f_i, g_j \in \mathbf{P}_+[a,\infty)$ for $1 \leq i \leq m,\ 1 \leq j \leq n$, and put $k(t) = (t-a)^{r_0} f_1(t)^{r_1} \cdots f_m(t)^{r_m}$ and $h(t) = (t-a)^{p_0} g_1(t)^{p_1} \cdots g_n(t)^{p_n}$ for $r_i, p_j \geq 0$. Suppose $p_0 + r_0 \geq 1$. If $a \leq A \leq C \leq B$, for α such that

$$0 < \alpha \leq \frac{\min(1,p_0) + r_0}{p + r_0} \quad (p := p_0 + \cdots + p_n),$$

$$(k(C)^{\frac{1}{2}}h(B)k(C)^{\frac{1}{2}})^\alpha \geq (k(C)^{\frac{1}{2}}h(C)k(C)^{\frac{1}{2}})^\alpha \geq (k(C)^{\frac{1}{2}}h(A)k(C)^{\frac{1}{2}})^\alpha.$$

Corollary 2.3 Let $h \in \mathbf{LP}_+[0,\infty)$ and $g \in \mathbf{LP}_+[0,\infty)$, and put $k(t) = tg(t)$. Suppose $0 \leq A \leq C \leq B$ and A is invertible. Then

$$\log(k(C)^{\frac{1}{2}}h(B)k(C)^{\frac{1}{2}}) \geq \log(k(C)^{\frac{1}{2}}h(C)k(C)^{\frac{1}{2}}) \geq \log(k(C)^{\frac{1}{2}}h(A)k(C)^{\frac{1}{2}}). \tag{4}$$

In particular, for every $r \geq 1$ and $p > 0$

$$\log(C^{\frac{r}{2}}e^{pB}C^{\frac{r}{2}}) \geq \log(C^{\frac{r}{2}}e^{pC}C^{\frac{r}{2}}) \geq \log(C^{\frac{r}{2}}e^{pA}C^{\frac{r}{2}}).$$

Example 2.1 Let $k(t) \in \mathbf{LP}_+[0,1)$. Suppose $0 \leq A \leq C \leq B \leq 1$ and $1-B$ is invertible. Then for $p \geq 1$ and for $0 < \alpha \leq \frac{1}{p}$,

$$\begin{aligned} \left(k(C)^{\frac{1}{2}}(B(1-B)^{-1})^p k(C)^{\frac{1}{2}}\right)^\alpha &\geq \left(k(C)^{\frac{1}{2}}(C(1-C)^{-1})^p k(C)^{\frac{1}{2}}\right)^\alpha \\ &\geq \left(k(C)^{\frac{1}{2}}(A(1-A)^{-1})^p k(C)^{\frac{1}{2}}\right)^\alpha. \end{aligned}$$

3 Operator Mean

From now on we assume $A, B \geq 0$ and A is invertible if A^{-1} appears.

Lemma 3.1 (i) If $A^{-1}\#B \le 1$, then for every operator convex function $h(t) \ge 0$ on $[0,\infty)$ with $h(0) = 0$,

$$1 \ge A^{-1}\#B \ge h(A)^{-1}\#h(B).$$

Moreover, for $f_i(t) \in \mathbf{P}_+[0,\infty)$ $(i = 1,2,\cdots)$ put $k_0(t) = 1$, $k_n(t) = f_1(t)\cdots f_n(t)$; then

$$(Ak_{n-1}(A))^{-1}\#Bk_{n-1}(B) \ge (Ak_n(A))^{-1}\#Bk_n(B) \quad (n = 1,2,\cdots).$$

(ii) If $A^{-1}\#B \ge 1$, then the reverse assertion holds.

Theorem 3.2 (i) If $A^{-1}\#B \le 1$, then for $a \ge 1$ and for $k(t) \in \mathbf{LP}_+[0,\infty)$

$$A^{-a}h(A)^{-1}\#B^{a}h(B) \le 1.$$

Moreover, $A^{-s}h(A)^{-t}\#B^{s}h(B)^{t}$ is non-increasing for $1 \le s < \infty$ and for $0 \le t < \infty$.

(ii) If $A^{-1}\#B \ge 1$, the reverse assertion holds.

Example 3.1 (i) If $A^{-1}\#B \le 1$, then for $s \ge 1$, $t > 0$

$$A^{-s}e^{-tA}\#B^{s}e^{tB} \le 1.$$

Moreover, $A^{-s}e^{-tA}\#B^{s}e^{tB}$ is non-increasing for $1 \le s < \infty$ and for $0 \le t < \infty$.

(ii) If $A^{-1}\#B \ge 1$, the reverse assertion holds.

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