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Pick Functions as a Tool in the Theory of Special Functions

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Abstract

The class of Pick functions is closely connected to the operator monotone functions. The class appears in other areas in mathematics, e.g. in the theory of orthogonal polynomials and Jacobi matrices, and also as a tool in the theory of special functions. In this paper we focus on the connection to special functions and give examples of their application in problems related to Euler's gamma function. Our aim is to show that quite elegant solutions to various open problems in the theory of special functions can be obtained using Pick functions.

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1 Introduction

A bounded linear operator $\mathbf{A}$ on a Hilbert space $\mathcal{H}$ is called positive definite if $\mathbf{A}$ is Hermitian and furthermore $\langle \mathbf{A}x, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For two bounded linear operators $\mathbf{A}$ and $\mathbf{B}$ we say that $\mathbf{A} \leq \mathbf{B}$ if $\mathbf{B} - \mathbf{A}$ is positive definite. The class of functions f for which $f(\mathbf{A}) \leq f(\mathbf{B})$ whenever $\mathbf{A} \leq \mathbf{B}$ is called the class of operator monotone functions. It consists of holomorphic functions in the upper half plane, and it is actually a subset of the so-called class of Pick functions.

This paper is a mainly an extended review of the talk presented at the RIMS Symposium "Operator monotone functions and related topics" on November 7, 2013 in Kyoto. However, in the last section we present a relatively elementary proof of the known result describing the conformal image of the upper half plane under the logarithm of the Gamma function as a comb like domain.

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A Pick function f is a holomorphic function in the upper half plane $\mathbb{H}$ for which $\Im f(z) \geq 0$ for every $z \in \mathbb{H}$. The class of Pick functions is denoted by $\mathcal{P}$. For a general reference to Pick functions and operator monotone functions, see [8].

It is well known that any $f \in \mathcal{P}$ admits an integral representation of the form

$$f(z) = az + b + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t),$$

where $a \geq 0$, b is real and μ is a positive measure on the real line such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty.$$

The triple (a, b, μ) is uniquely determined by f and any triple gives rise to a Pick function via this formula.

Any Pick function can be extended to the lower half plane by reflection. Let O be an open subset of the real line and let $A = \mathbb{R} \setminus O$. It is known that a Pick function f has a holomorhic extension to $\mathbb{C} \setminus A$ such that $f(z) = \overline{f(\overline{z})}$ if and only if the measure μ in the representation above is supported on $A = \mathbb{R} \setminus O$. The class of functions with this property is denoted by $\mathcal{P}_O$.

The class $\mathcal{P}_{(0,\infty)}$ is simply the operator monotone functions. We notice the following characterization of $\mathcal{P}_{(0,\infty)}$. See [4].

Proposition 1.1 *Suppose that $p : (0, \infty) \to (0, \infty)$. Then the following are equivalent:*

(a) p can be extended holomorphically to $\mathbb{C} \setminus (-\infty, 0]$ such that $p(\overline{z}) = \overline{p(z)}$.

(b) p has the representation

$$p(x) = \int_0^{\infty} \frac{x}{x+t} d\rho(t) + cx$$

where ρ is a positive measure on $[0, \infty)$ such that $\int_0^{\infty} d\rho(t)/(t+1)$ converges and $c \geq 0$.

Other important classes of holomorphic functions related to the Pick functions appear in the theory of special functions, namely the Stieltjes functions and the completely monotonic functions.

Definition 1.2 *A function $f : (0, \infty) \to \mathbb{R}$ is a Stieltjes function if*

$$f(x) = \int_0^{\infty} \frac{1}{x+t} d\rho(t) + c,$$

where $c \geq 0$, and ρ is a positive measure on $[0, \infty)$ such that $\int_0^{\infty} d\rho(t)/(1+t) < \infty$.

Definition 1.3 *A C^∞-function f defined on $(0,\infty)$ is completely monotonic if*

$$(-1)^k f^{(k)}(x) \geq 0$$

for all $x > 0$ and for all $k \geq 0$.

Bernstein's theorem characterizes the class of completely monotonic functions as the Laplace transforms of positive measures on $[0,\infty)$. See [15].

In the following sections we shall see how these classes of functions are used in solving problems related to Euler's gamma function. The techniques have also been applied successfully in connection with the so-called multiple gamma functions introduced by Barnes around 1900. See [3], and also [11] and [12].

Some of the proofs also involve conformal mappings. A conformal mapping is simply a univalent holomorphic function defined in a region.

We begin by giving a short review some fundamental properties of the famous Gamma function appearing in many areas of mathematics.

2 Euler's gamma function

Euler's gamma function Γ is usually defined as Euler's first integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \Re z > 0,$$

so Γ is holomorhic in the right half plane and by the functional equation $\Gamma(z+1) = z\Gamma(z)$ it can actually be defined as a meromorphic function in $\mathbb{C}$. The reciprocal is an entire function with Weierstraß factorization

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}.$$

(Here γ is Euler's constant.) Figure 1 shows the graph of Γ on the real line. The logarithm $\log \Gamma$ is defined initially on the positive real line but has a holomorphic extension to the cut plane $\mathcal{A} = \mathbb{C} \setminus (-\infty, 0]$, given by

$$\log \Gamma(z+1) = -\gamma z - \sum_{k=1}^{\infty} \left\{ \log\left(1 + \frac{z}{k}\right) - \frac{z}{k} \right\}, \tag{1}$$

where all logarithms are defined by analytic continuation from the positive half line.

The logarithmic derivative ψ of Γ is given by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z+k}\right) - \frac{1}{z}, \tag{2}$$

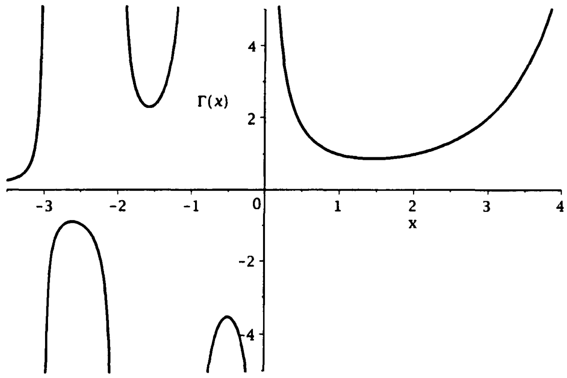

Figure 1: Γ on the real line

and is easily shown to have exactly one zero x_k in each interval $(-k, 1-k)$ for $k \geq 1$ and one zero x_0 on the positive axis. These points are the critical points of Γ.

Stirling's formula describes the asymptotic behaviour of $\Gamma(x)$ for large positive x. There is also a complex version usually written as

$$\log \Gamma(1+z) = (z+1/2)\log z - z + (1/2)\log(2\pi) + \mu(z),$$

for $z \in \mathcal{A}$. Here the remainder μ has the representation

$$\mu(z) = \frac{1}{2}\int_0^\infty \frac{Q(t)}{(z+t)^2}\,dt,$$

where Q is 1-periodic and given by $Q(t) = t - t^2$ for $t \in (0,1]$. We notice that the estimate $|\mu(z)| \leq \pi/8$ holds for $z \in \mathbb{H} \setminus \mathcal{B}$, where

$$\mathcal{B} = \{\Re z \leq 1, 0 \leq \Im z \leq 1\}.$$

The behaviour of Γ is in general very well understood, but as we shall see in the following three sections, some properties have been discovered only recently.

3 The volume of the unit ball in $\mathbb{R}^n$

The n-dimensional volume of the unit ball in $\mathbb{R}^n$ is given in terms of Γ as

$$\Omega_n = \text{Volume}(\{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| \leq 1\}) = \frac{\pi^{n/2}}{\Gamma(1+n/2)}, \quad n \geq 1.$$

From Stirling's formula it follows that $\Omega_n \to 0$ as $n \to \infty$ and even that

$$\Omega_n^{1/n\log n} = \left(\frac{\pi^{n/2}}{\Gamma(1+n/2)}\right)^{1/n\log n} \to e^{-1/2}, \quad \text{as } n \to \infty.$$

Questions concerning the detailed behaviour of the sequence $\Omega_n^{1/n \log n}$ have attracted the attention of several authors. (Logarithmic convexity is obtained in [13], see also [2]). We briefly outline the ideas behind the following result from [5].

Theorem 3.1 *Let*

$$f(x) = \left(\frac{\pi^{x/2}}{\Gamma(1 + x/2)} \right)^{1/x \log x}.$$

Then $f(x+1)$ is a completely monotonic function and in particular the sequence

$$\Omega_{n+2}^{1/(n+2)\log(n+2)}$$

is a Hausdorff moment sequence.

The proof uses properties of the auxiliary functions

$$F_a(x) = \frac{\log \Gamma(x+1)}{x \log(ax)},$$

where $x > 0$ and $a > 0$.

Proposition 3.2 *The function F_a has a holomorphic extension to $\mathcal{A} \setminus \{1/a\}$ and the integral representation*

$$F_a(z) = 1 + \frac{\log \Gamma(1 + 1/a)}{z - 1/a} - \int_0^\infty \frac{d_a(t)}{z + t}\, dt, \quad z \in \mathcal{A} \setminus \{1/a\},$$

where, for $t \in (k-1, k)$ and $k \geq 1$,

$$d_a(t) = \frac{\log |\Gamma(1-t)| + (k-1)\log(at)}{t((\log(at))^2 + \pi^2)}.$$

Using the functional equation of the Gamma function it is easy to check that $d_a(t) \geq 0$ for $a \geq 1$ and this gives part of the proof of the following corollary.

Corollary 3.3 *For $a \geq 1$, F_a is a Pick function, and for $0 < a < 1$ it is not.*

The proof of Proposition 3.2 can be based on the Residue theorem, by evaluating the integral of an auxiliary function along a curve κ of the form in Figure 2. Indeed,

$$\frac{1}{2\pi i} \int_\kappa \frac{F_a(z)}{z - w}\, dz = F_a(w) + \frac{\log \Gamma(1 + 1/a)}{1/a - w},$$

assuming both w and a are inside κ (in which the circles are of radii $k + 1/2$ and ϵ). The proof of Proposition 3.2 follows by letting ϵ tend to 0 and k tend to infinity.

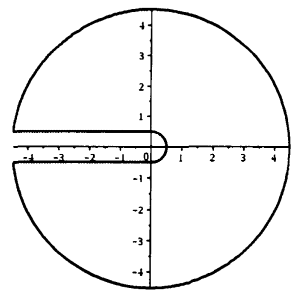

Figure 2: The curve κ

Theorem 3.1 is obtained using that the functions f and F_2 are related as

$$\log f(x+1) = \frac{\log \pi}{2\log(x+1)} - \frac{1}{2}F_2\left(\frac{x+1}{2}\right).$$

From the integral representation in Proposition 3.2 it follows that

$$\frac{1}{2} + \log f(z+1) = \frac{\log(2/\sqrt{\pi})}{z} + \frac{\log(\sqrt{\pi})}{\log(z+1)} + \frac{1}{2}\int_1^\infty \frac{d_2((t-1)/2)}{z+t}\,dt.$$

This relation states that $1/2 + \log f(z+1)$ is a Stieltjes function, and therefore the function $\sqrt{e}f(x+1)$ is a so-called logarithmically completely monotonic function, and hence completely monotonic. See [4].

It is a general fact that if g is a completely monotonic function then the sequence $a_n = g(n+1)$ is a so-called Hausdorff moment sequence, that is, is of the form

$$a_n = \int_0^1 s^n \, d\sigma(s),$$

for some positive Radon measure on $[0,1]$. (This follows from Bernstein's characterization mentioned before.)

Since moment sequences are logarithmically convex, Theorem 3.1 generalises the results on logarithmic convexity.

4 The median in the Gamma distribution

A probability measure μ on $[0,\infty)$ has median m if $\mu([0,m]) = 1/2$. Of course, a probability measure need not have a median, or it may not be uniquely determined. For probability measures on $[0,\infty)$ having a density with respect to Lebesgue measure, the median is uniquely determined. We shall focus on the median of the Gamma distribution and its dependence on the so-called shape parameter. This distribution with shape parameter $x > 0$ has the density $e^{-t}t^{x-1}/\Gamma(x)$ on the positive half line. The median $m(x)$ is thus given implicitely as

$$\int_0^{m(x)} \frac{e^{-t}t^{x-1}}{\Gamma(x)}\, dt = \frac{1}{2}.$$

In 1986, Chen and Rubin conjectured that the sequence $m(n) - n$ be decreasing. In this discrete setting the conjecture has been investigated by various authors, and the first proof was obtained by Alm. See [1]. Below we present the main ideas behind a proof treating the function $m(x) - x$ for any positive x. See [6] and [7]. Here Pick functions appear at a crucial stage.

Theorem 4.1 *The median $m(x)$ is a convex function of x.*

From Theorem 4.1 it is easy to obtain Chen and Rubin's conjecture: by convexity, $m'(x)$ is increasing, and it can be shown that $\lim_{x\to\infty} m'(x) = 1$. This means that $m'(x) \le 1$ and hence that $m(x) - x$ decreases.

The problem is attacked through the function $\varphi(x) = \log(x/m(x))$ and it can be shown that the equation defining the median can be put in the form

$$2\int_0^{\varphi(x)} e^{-x(e^{-u}+u)}\, du = \int_1^{\infty} \xi(t)e^{-xt}\, dt,$$

where ξ is a certain function, which we briefly describe below.

Let $f(x) = e^{-x} + x$ and define $v(t)$ as the inverse of $f : [0,\infty) \to [1,\infty)$, and $u(t)$ as the inverse to $f : (-\infty, 0] \to [1,\infty)$. The situation is illustrated in Figure 3. The function

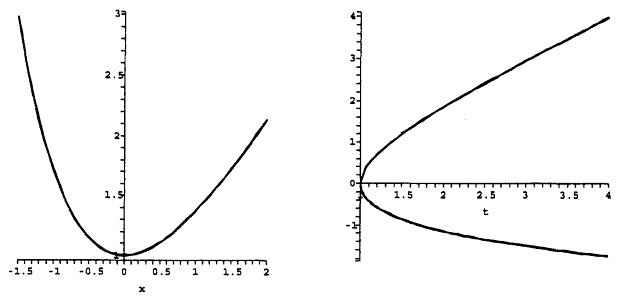

Figure 3: Graph of f and of u and v

ξ is defined as $\xi(t) = u'(t) + v'(t)$ and it is crucial for the proof of Theorem 4.1 that it is concave on $[1,\infty)$. Pick functions are used in the study ξ, in order to handle the cancellation between u and v and the concavity of ξ is deduced from a suitable integral representation.

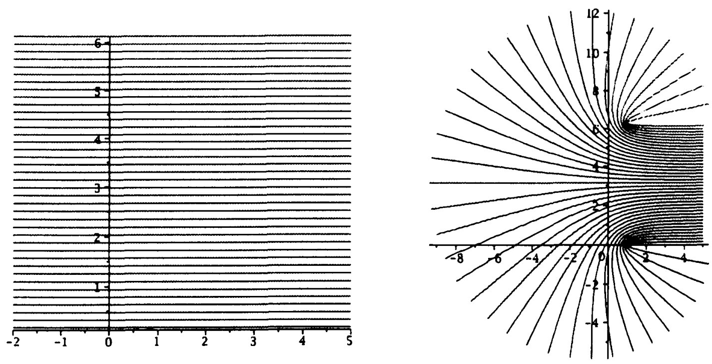

Figure 4: The conformal mapping f

When extended to the complex plane $f(z) = e^{-z} + z$ is a conformal mapping of the strip $\{0 < \Im z < 2\pi\}$ onto the two cut plane $\mathbb{C} \setminus \{[1,\infty) \cup [1,\infty) \times \{2\pi i\}\}$, as illustrated in Figure 4.

Using the maximum principle it can be shown that

$$\Psi(w) = f^{-1}(\log w) + \overline{f^{-1}(\log \overline{w})}$$

is a Pick function and for $t > e$ it is related to ξ as $\Psi'(t) = \xi(\log t)/t$. Furthermore it has the integral representation

$$\Psi(w) = \Re\Psi(i) - \int_0^\infty \left(\frac{1}{t+w} - \frac{t}{t^2+1}\right) h(t)dt,$$

where h increases on $(0,\infty)$ from 0 to 1. Consequently, $\xi(\log w)$ is holomorphic in the cut plane $\mathcal{A}$ and

$$\xi(\log w) = 1 - \int_0^\infty \frac{t}{t+w} h'(t)\, dt.$$

In particular $\xi(t)$ is defined for all real t and increases from 0 to 1 for $t \in \mathbb{R}$. Therefore ξ is increasing, but this also means that ξ cannot be concave on the whole real line. Using yet another integral representation of $\xi''(\log w)/w$ it can be showed that ξ is indeed concave on $[1,\infty)$.

5 Inverses of the Gamma function

As indicated in Figure 1, Γ increases on (x_0,∞). In [14] the inverse function of Γ defined on $(\Gamma(x_0),\infty)$ was shown to have an extension to a Pick function and hence by the reflection principle to the cut plane $\mathbb{C} \setminus (-\infty, \Gamma(x_0)]$.

The Gamma function decreases in the interval $(0, x_0)$ and [14] includes a question of how to extend the inverse on this interval. The construction relied on Löwner's theorem linking positive definite Löwner kernels with Pick functions, and would not seem to work when extending the inverse of the gamma function on this interval or for that matter on intervals of the negative line where Γ has singularities.

The answer to the question is given in [12] and uses conformal properties of the holomorphic function $\log\Gamma$, defined in (1). In fact $\log\Gamma$ is a conformal mapping of the upper half plane onto the domain $\mathcal{V}$ given by

$$\mathcal{V} = \mathbb{C} \setminus \cup_{j\geq 0}\{x - i\pi j \,|\, x \geq \log|\Gamma(x_j)|\}. \tag{3}$$

This domain is obtained by removing infinitely many half lines or "teeth" from the complex plane, and for this reason it is usually called a comb like domain. See Figure 5.

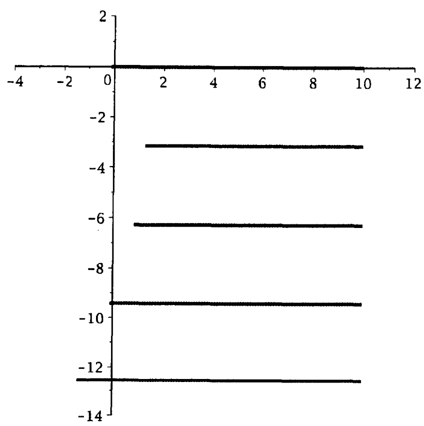

Figure 5: The conformal image $\mathcal{V} = \log\Gamma(\mathbb{H})$

For $k \in \mathbb{Z}$ we define

$$g_k(z) = (\log\Gamma)^{-1}(\log z - i(k+1)\pi), \quad z \in \mathbb{H}.$$

The logarithm maps $\mathbb{H}$ conformally onto the strip $\{z \in \mathbb{C} | \Im z \in (0,\pi)\}$, and hence $z \mapsto \log z - i(k+1)\pi$ maps $\mathbb{H}$ onto the strip

$$S_k = \{z \in \mathbb{C} | \Im z \in (-(k+1)\pi, -k\pi)\}.$$

Since $S_k \subseteq \mathcal{V}$, the holomorphic function g_k maps $\mathbb{H}$ into $\mathbb{H}$, and is thus a Pick function, for any integer k. Furthermore, it is easy to see that

$$\Gamma(g_k(z)) = (-1)^{k+1}z, \quad z \in \mathbb{H}.$$

In [12] integral representations of the functions g_k are also found. In the rest of this section we give some details when $k = -1$ or $k = 0$.

First of all, the function g_{-1} extends the inverse of $\Gamma : (x_0, \infty) \to (\Gamma(x_0), \infty)$ to the upper half plane: Suppose that $t > \Gamma(x_0)$. Choose a sequence $\{z_n\}$ in $\mathbb{H}$ such that

$z_n \to t$ as $n \to \infty$. Then $\log z_n \to \log t + i0$, and this point belongs to the upper side of the half line $(\log\Gamma(x_0), \infty)$. Thus $g_{-1}(z_n) \to (\log\Gamma)^{-1}(\log t + i0) \in (x_0, \infty)$ and then

$$t = \lim_{n\to\infty} z_n = \lim_{n\to\infty} \Gamma(g_{-1}(z_n)) = \Gamma((\log\Gamma)^{-1}(\log t + i0)).$$

In a similar way we see that g_0 is related to the inverse of Γ on the interval (x_1, x_0), and thus in particular on $(0, x_0)$: Suppose e.g. that $t > \Gamma(x_0)$. Choose $\{z_n\}$ in $\mathbb{H}$ such that $z_n \to -t$ as $n \to \infty$. Then $\log z_n - i\pi \to \log t - i0$ and this point belongs to the lower side of the half line $(\log\Gamma(x_0), \infty)$. Furthermore, $g_0(z_n) \to (\log\Gamma)^{-1}(\log t - i0)$, and hence

$$t = -\lim_{n\to\infty} z_n = \lim_{n\to\infty} \Gamma(g_0(z_n)) = \Gamma((\log\Gamma)^{-1}(\log t - i0)).$$

Figure 6 gives a hint that the curves $\{\log\Gamma(t + i\epsilon) \mid t \in \mathbb{R}\}$ approach the teeth from both upper and lower sides as $\epsilon \to 0$. The figure shows the curves corresponding to $\epsilon = 0.1$ (black) and $\epsilon = 0.01$ (grey).

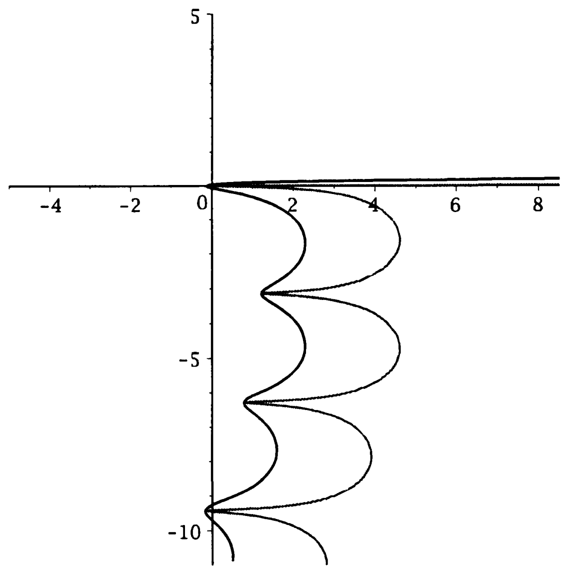

Figure 6: Images of horizontal lines under $\log\Gamma$

The representation of g_0 is given by the formula

$$g_0(z) = \frac{1}{\pi}\int_{-\Gamma(x_0)}^{-\Gamma(x_1)} \frac{d_0(t)}{t - z}dt, \quad \text{for } z \in \mathbb{C} \setminus [-\Gamma(x_0), -\Gamma(x_1)],$$

where $d_0(t) = \Im g_0(t + i0)$. The curve $g_0(t + i0)$, for $t \in (-\Gamma(x_0), 0)$ is the image of the horizontal half line $(-\infty, \log\Gamma(x_0))$ under $(\log\Gamma)^{-1}$, and is thus a smooth curve in the upper half plane. Similarly, the curve $g_0(t + i0)$, for $t \in (0, -\Gamma(x_1))$ is the image of the horizontal half line $(-\infty, \log|\Gamma(x_1)|) \times \{-i\pi\}$ under $(\log\Gamma)^{-1}$. The inverse of

$\Gamma : (x_1, 0) \cup (0, x_0) \to \mathbb{R}$ can thus be extended to a holomorphic function e_0 in $\mathbb{C} \setminus [\Gamma(x_1), \Gamma(x_0)]$, given by $e_0(z) \equiv g_0(-z)$.

We remark that the logarithm of any entire function of Laguerre-Polyá class is a conformal mapping of the upper half plane onto some comb like domain, and that any such domain is the conformal image of the logarithm of a certain Laguerre-Polyá function. These results go back to MacLane, Vinberg, Grunskii and others and can be proved using Riemann surfaces, or general results about boundary behaviour of conformal mappings. We refer to the recent paper [9].

6 The conformal image $\mathcal{V}$

The purpose of this section is to give a relatively elementary proof of the formula

$$\log \Gamma(\mathbb{H}) = \mathcal{V},$$

where $\mathcal{V}$ is given in (3). The ideas are the following: we first show that $\log \Gamma$ is conformal in $\mathbb{H}$ and maps it into $\mathcal{V}$. Then we consider the image of certain rectangular curves and show that the image curves approach infinity and the teeth of the comb $\mathcal{V}$. Then a classical result of Darboux is used to show that any point inside these image curves is contained in the image $\log \Gamma(\mathbb{H})$.

Lemma 6.1 *The map* $\log \Gamma$ *is conformal in* $\mathbb{H}$ *and* $\log \Gamma(\mathbb{H}) \subseteq \mathcal{V}$.

Proof. For different points z_0 and z_1 in $\mathbb{H}$ we write

$$\log \Gamma(z_1) - \log \Gamma(z_0) = (z_1 - z_0) \int_0^1 \psi((1-t)z_0 + tz_1)\, dt,$$

and since ψ is a Pick function the integral in the right hand side has positive imaginary part and hence does not vanish. If a point z_1 in $\mathbb{H}$ is mapped onto one of the "teeth" of the comb $\mathcal{V}$ then we construct a negative real number z_0 such that $\log \Gamma(z_0) = \log \Gamma(z_1)$. The relation above hold also for this value of z_0 and then a contradiction is reached. $\square$

Lemma 6.2

$$\inf\{|\log \Gamma(z)| \mid z \in \mathbb{H}, |z| \geq K\} \to \infty \quad as \quad K \to \infty.$$

Proof. For $z \in \mathbb{H} \setminus \mathcal{B}$ (see Section 2) the corresponding infimum tends to infinity by Stirling's formula. If $\Im z \leq 1$ and $\Re z \in [-k, -k+1]$ for some $k \geq 1$ then by the functional equation,

$$|\log \Gamma(z)| \geq \left| \sum_{l=0}^{k} \log(z+l) \right| - |\log \Gamma(z+k+1)| \geq k\pi/2 - c,$$

where $c = \sup\{|\log\Gamma(w)|\,|\,1 \leq \Re w \leq 2, < 0 \leq \Im w \leq 1\}$. We have also used that $\Re(z+l) \leq 0$ for $0 \leq l \leq k-1$, so that its argument is at least $\pi/2$. □

For $\delta > 0$ and $k > 0$ we define the simple closed curve $\kappa = \kappa_{\delta,k}$ as the rectangle

$$[k+i\delta, k+ik] \cup [-k+ik, k+ik] \cup [-k+i\delta, -k+ik] \cup [-k+i\delta, k+i\delta].$$

Lemma 6.3 *For all $R > 0$ and $\epsilon > 0$ there exist $K > 0$ and $\delta > 0$ such that*

$$\log\Gamma(\kappa_{\delta,K}) \subseteq \{w \mid |w| \geq R\} \cup \{w \mid dist(w, \partial\mathcal{V}) \leq \epsilon\}.$$

Proof. Let $R > 0$ and $\epsilon > 0$ be given. Choose, according to Lemma 6.2, a positive integer K such that $|\log\Gamma(z)| \geq R$ for all $z \in \mathbb{H}$ such that $|z| \geq K$. We consider the set of poles $\{-K, \cdots, 0\}$ of Γ in $[-K, K]$. Choose $\delta_1 > 0$ such that $|\Gamma(z)| \geq e^R$ for all z in $\cup_{l=0}^{K}\Delta(-l, \delta_1)$. (Here $\Delta(x, r)$ denotes the open square centered at x and of side length $2r$.) We can assume that $\delta_1 < 1/2$ and hence that the compact set $[-K, K] \setminus \cup_{l=0}^{K}\Delta(-l, \delta_1)$ can be written as the disjoint union $\cup_{l=0}^{K} I_l$ where the closed interval I_l contains $-l + 1/2$. On I_l, the sequence of functions $\log\Gamma(x + i/n)$ converges uniformly to $\log|\Gamma(x)| - i\pi l$, that is to points on the teeth of the comb. It follows that we can choose a positive δ such that $\mathrm{dist}(\log\Gamma(\cup_{l=0}^{K} I_l + i\delta), \partial\mathcal{V}) \leq \epsilon$. We have shown that regardless of the location of the point z on $\kappa_{\delta,K}$ we must have $|\log\Gamma(z)| \geq R$ or $\mathrm{dist}(\log\Gamma(z), \partial\mathcal{V}) \leq \epsilon$. □

Proposition 6.4 *The conformal image* $\log\Gamma(\mathbb{H})$ *is equal to* $\mathcal{V}$.

Proof. Let $w_0 \in \mathcal{V}$ be given. Choose R and ϵ such that $|w_0| < R$ and $\mathrm{dist}(w_0, \partial\mathcal{V}) > \epsilon$, and then K and δ such that the conclusion of Lemma 6.3 holds with the rectangle $\kappa = \kappa_{\delta,K}$. The simple closed curve $\iota = \log\Gamma(\kappa)$ divides the complex plane into two components and w_0 cannot belong to the unbounded component. Hence ι surrounds w_0, i.e.

$$\frac{1}{2\pi i}\int_\iota \frac{dw}{w - w_0} = 1.$$

Finally,

$$\frac{1}{2\pi i}\int_\kappa \frac{(\log\Gamma)'(z)}{\log\Gamma(z) - w_0}\,dz = \frac{1}{2\pi i}\int_\iota \frac{dw}{w - w_0} = 1,$$

and hence $\log\Gamma(z) - w_0$ has a zero inside the rectangle κ and hence in $\mathbb{H}$. (This is sometimes called Darboux's theorem, see e.g. [10]). □

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