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Generalizations of operator Shannon inequality based on Tsallis and Rényi relative entropies

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Abstract

Furuta obtained an operator version of celebrated Shannon inequality. We call this operator Shannon inequality briefly. Extensions of operator Shannon inequality were discussed by Furuta and Yanagi-Kuriyama-Furuichi.

In this report, we shall show relations among relative operator entropies of sequences including operator Shannon inequality as follows: For relative operator entropy $S(\mathbb{A}|\mathbb{B})$, Rényi relative operator entropy $I_t(\mathbb{A}|\mathbb{B})$ and Tsallis relative operator entropy $T_t(\mathbb{A}|\mathbb{B})$ of sequences of strictly positive operators $\mathbb{A}=(A_1,\cdots,A_n)$ and $\mathbb{B}=(B_1,\cdots,B_n)$ such that $\sum_{i=1}^{n}A_i=\sum_{i=1}^{n}B_i=I$,

$$S(\mathbb{A}|\mathbb{B})\leq I_t(\mathbb{A}|\mathbb{B})\leq T_t(\mathbb{A}|\mathbb{B})\leq 0$$

holds for $0<t<1$. Moreover, we shall discuss two generalizations of this inequality by considering generalizations of relative operator entropies of sequences.

1 Introduction

This report is based on [4, 6]. In this report, an operator means a bounded linear operator on a Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in\mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T>0$) if T is positive and invertible.

In [1], for $A,B>0$, relative operator entropy was defined by

$$S(A|B)\equiv A^{\frac{1}{2}}\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}.$$

We remark that $S(A|I)=-A\log A$ is operator entropy given by Nakamura-Umegaki [7]. For $A,B>0$ and $t\in\mathbb{R}$, Furuta [2] introduced generalized relative operator entropy

$$S_t(A|B)\equiv A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}},$$

where $A \natural_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}}$ for $0 \leq t \leq 1$. We can treat the weighted geometric mean $A \natural_t B$ as a path from A to B. We remark that $S_t(A|B)$ can be considered as a tangent at t of $A \natural_t B$, and also $S_0(A|B) = S(A|B)$. Tsallis relative operator entropy was introduced by Yanagi-Kuriyama-Furuichi [8] as follows: For $A, B > 0$ and $0 < t \leq 1$,

$$T_t(A|B) \equiv \frac{A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} - A}{t} = \frac{A \natural_t B - A}{t}.$$

We remark that

$$T_0(A|B) \equiv \lim_{t \to +0} T_t(A|B) = S(A|B)$$

since $\lim\limits_{t \to +0} \frac{x^t - 1}{t} = \log x$ for $x > 0$, and also the definition of $T_t(A|B)$ can be extended for $t \in \mathbb{R}$.

Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be sequences of strictly positive operators. In [4, 6], we define relative operator entropy $S(\mathbb{A}|\mathbb{B})$, generalized relative operator entropy $S_t(\mathbb{A}|\mathbb{B})$, Tsallis relative operator entropy $T_t(\mathbb{A}|\mathbb{B})$ and Rényi relative operator entropy $I_t(\mathbb{A}|\mathbb{B})$ of two sequences $\mathbb{A}$ and $\mathbb{B}$ as follows: For $0 \leq t \leq 1$,

$$S(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} S(A_i|B_i), \quad S_t(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} S_t(A_i|B_i),$$
$$T_t(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} T_t(A_i|B_i) \quad \text{and}$$
$$I_t(\mathbb{A}|\mathbb{B}) \equiv \frac{1}{t} \log \sum_{i=1}^{n} A_i \natural_t B_i \quad (\text{if } t \neq 0).$$

In this report, we assume $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I$. We remark that

$$I_0(\mathbb{A}|\mathbb{B}) \equiv \lim_{t \to +0} I_t(\mathbb{A}|\mathbb{B}) = S(\mathbb{A}|\mathbb{B})$$

follows from (2.1) stated below.

On the other hand, for two probability distributions $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$, relative entropy is defined by $D(p|q) \equiv \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$, and also it is well known that $-\sum_{i=1}^{n} p_i \log p_i \leq -\sum_{i=1}^{n} p_i \log q_i$ holds. This inequality is called Shannon inequality, and it is equivalent to $D(p|q) = -\sum_{i=1}^{n} p_i \log \frac{q_i}{p_i} \geq 0$.

In [2], for two sequences of strictly positive operators $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ with $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I$, Furuta obtained the operator version of Shannon inequality (briefly, operator Shannon inequality).

$$S(\mathbb{A}|\mathbb{B}) \leq 0. \tag{1.1}$$

Yanagi-Kuriyama-Furuichi [8] obtained a generalization of (1.1) by using Tsallis relative operator entropy of sequences.

$$T_t(\mathbb{A}|\mathbb{B}) \le 0 \quad \text{for } 0 < t \le 1. \tag{1.2}$$

In this report, we shall show relations among relative operator entropies of sequences $S(\mathbb{A}|\mathbb{B})$, $S_t(\mathbb{A}|\mathbb{B})$, $T_t(\mathbb{A}|\mathbb{B})$ and $I_t(\mathbb{A}|\mathbb{B})$, which include operator Shannon inequality. Moreover, we shall discuss two generalizations of this result by considering generalizations of $S_t(\mathbb{A}|\mathbb{B})$, $T_t(\mathbb{A}|\mathbb{B})$ and $I_t(\mathbb{A}|\mathbb{B})$.

2 Relations among operator entropies of sequences

In this section, as relations among relative operator entropies of sequences, we obtain the following inequalities including (1.1) and (1.2).

Theorem 2.1 ([4]). *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be sequences of strictly positive operators with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. Then*

$$S(\mathbb{A}|\mathbb{B}) \le I_t(\mathbb{A}|\mathbb{B}) \le T_t(\mathbb{A}|\mathbb{B}) \le 0, \tag{2.1}$$

$$0 \le -T_{1-t}(\mathbb{B}|\mathbb{A}) \le -I_{1-t}(\mathbb{B}|\mathbb{A}) \le S_1(\mathbb{A}|\mathbb{B}) \tag{2.2}$$

and

$$T_t(\mathbb{A}|\mathbb{B}) \le S_t(\mathbb{A}|\mathbb{B}) \le -T_{1-t}(\mathbb{B}|\mathbb{A}) \tag{2.3}$$

hold for $0 < t < 1$.

In order to prove Theorem 2.1, we use the following lemma.

Lemma 2.2 ([4]). *Let $A, B > 0$. Then the following properties hold:*

(i) $S(A|B) \le T_t(A|B) \le S_t(A|B)$ *for $t > 0$.*

(ii) $S_t(A|B) = -S_{1-t}(B|A)$ *for $t \in \mathbb{R}$.*

(iii) $S_1(A|B) = -S(B|A)$.

Proof. We have (i) since $\log x \le \frac{x^t-1}{t} \le x^t \log x$ for $x > 0$.

We have (ii) since

$$\begin{aligned}
-S_{1-t}(B|A) &= -B^{\frac{1}{2}}(B^{\frac{-1}{2}}AB^{\frac{-1}{2}})^{1-t}\log(B^{\frac{-1}{2}}AB^{\frac{-1}{2}})B^{\frac{1}{2}} \\
&= A^{\frac{1}{2}}A^{\frac{-1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{t-1}\log(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})B^{\frac{1}{2}}A^{\frac{-1}{2}}A^{\frac{1}{2}} \\
&= A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{t-1}\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \\
&= A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{t}\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \\
&= S_t(A|B).
\end{aligned}$$

We have (iii) by putting $t = 1$. Hence the proof is complete. □

Jensen's operator inequality [3] plays an important role to prove results in this report.

Theorem 2.A (Jensen's operator inequality [3]). *Let $f(x)$ be an operator concave function on an interval J. Let $\{C_i\}_{i=1}^n$ be operators with $\sum_{i=1}^n C_i^*C_i = I$. Then*

$$f\left(\sum_{i=1}^n C_i^*A_iC_i\right) \ge \sum_{i=1}^n C_i^*f(A_i)C_i$$

holds for every selfadjoint operators $\{A_i\}_{i=1}^n$ whose spectra are contained in J.

Proof of Theorems 2.1. Since $f(x) = \log x$ is operator concave for $x > 0$, by using Theorem 2.A, we have that

$$\begin{aligned}
I_t(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\log\sum_{i=1}^n A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^tA_i^{\frac{1}{2}} \\
&\ge \frac{1}{t}\sum_{i=1}^n A_i^{\frac{1}{2}}\log(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^tA_i^{\frac{1}{2}} \\
&= \sum_{i=1}^n A_i^{\frac{1}{2}}\log(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})A_i^{\frac{1}{2}} = S(\mathbb{A}|\mathbb{B}).
\end{aligned}$$

Since $\log x \le x - 1$ for $x > 0$, we have

$$\begin{aligned}
I_t(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\log\sum_{i=1}^n A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^tA_i^{\frac{1}{2}} \\
&\le \frac{1}{t}\left[\sum_{i=1}^n A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^tA_i^{\frac{1}{2}} - I\right] = T_t(\mathbb{A}|\mathbb{B}).
\end{aligned}$$

Since $\frac{x^t-1}{t} \leq x-1$ for $x>0$ and $0<t<1$, we have

$$\begin{aligned} T_t(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\left[\sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}} B_i A_i^{\frac{-1}{2}})^t A_i^{\frac{1}{2}} - I\right] \\ &= \sum_{i=1}^{n} A_i^{\frac{1}{2}} \frac{(A_i^{\frac{-1}{2}} B_i A_i^{\frac{-1}{2}})^t - I}{t} A_i^{\frac{1}{2}} \\ &\leq \sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}} B_i A_i^{\frac{-1}{2}} - I) A_i^{\frac{1}{2}} \\ &= \sum_{i=1}^{n} (B_i - A_i) = 0. \end{aligned}$$

Therefore we obtain (2.1).

By (2.1),

$$S(\mathbb{B}|\mathbb{A}) \leq I_{1-t}(\mathbb{B}|\mathbb{A}) \leq T_{1-t}(\mathbb{B}|\mathbb{A}) \leq 0$$

holds for $0<t<1$, so that we have (2.2) by (iii) in Lemma 2.2.

We also have (2.3) since

$$T_t(\mathbb{A}|\mathbb{B}) \leq S_t(\mathbb{A}|\mathbb{B}) = -S_{1-t}(\mathbb{B}|\mathbb{A}) \leq -T_{1-t}(\mathbb{B}|\mathbb{A})$$

by (i) and (ii) in Lemma 2.2. Hence the proof is complete. □

3 A generalization of operator Shannon inequality

Next, we discuss a generalization of Theorem 2.1. For $A, B > 0$, $0 \leq t \leq 1$ and $-1 \leq r \leq 1$, power mean

$$A \natural_{t,r} B = A^{\frac{1}{2}}\{(1-t)I + t(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^r\}^{\frac{1}{r}} A^{\frac{1}{2}}$$

is well known as a path of operator means from harmonic mean to arithmetic mean on r. In fact,

$$\begin{aligned} &A \natural_{t,-1} B = \{(1-t)A^{-1} + tB^{-1}\}^{-1} = A \,\Delta_t\, B, \\ &A \natural_{t,0} B \equiv \lim_{r\to 0} A \natural_{t,r} B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}} = A \natural_t B, \\ &A \natural_{t,1} B = (1-t)A + tB = A \,\nabla_t\, B. \end{aligned}$$

In [5], we introduce generalizations of $S_t(A|B)$ and $T_t(A|B)$ as follows: For $A, B > 0$, $0 \leq t \leq 1$ and $-1 \leq r \leq 1$,

$$S_{t,r}(A|B) \equiv A^{\frac{1}{2}}\left(\{(1-t)I + t(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^r\}^{\frac{1}{r}-1} \cdot \frac{(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^r - I}{r}\right) A^{\frac{1}{2}},$$

$$T_{t,r}(A|B) \equiv \frac{A^{\frac{1}{2}}\{(1-t)I + t(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^r\}^{\frac{1}{r}} A^{\frac{1}{2}} - A}{t} = \frac{A \natural_{t,r} B - A}{t}.$$

Similarly to $S_t(A|B)$, we can treat $A \natural_{t,r} B$ as a path from A to B on t, and also $S_{t,r}(A|B)$ can be considered as a tangent at t of $A \natural_{t,r} B$. We remark that the following properties hold (see [5]).

$$S_{0,r}(A|B) = T_r(A|B), \quad S_{t,0}(A|B) \equiv \lim_{r \to 0} S_{t,r}(A|B) = S_t(A|B),$$
$$T_{0,r}(A|B) \equiv \lim_{t \to +0} T_{t,r}(A|B) = T_r(A|B) \quad \text{and} \quad T_{t,0}(A|B) = T_t(A|B).$$

Then we can generalize $S_t(\mathbb{A}|\mathbb{B})$, $T_t(\mathbb{A}|\mathbb{B})$ and $I_t(\mathbb{A}|\mathbb{B})$ as follows:

Definition 1 ([6]). *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be sequences of strictly positive operators with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. For $0 \le t \le 1$ and $-1 \le r \le 1$,*

$$S_{t,r}(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n S_{t,r}(A_i|B_i), \quad T_{t,r}(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n T_{t,r}(A_i|B_i),$$
$$I_{t,r}(\mathbb{A}|\mathbb{B}) \equiv \frac{1}{t} \log \sum_{i=1}^n A_i \natural_{t,r} B_i \quad (\text{if } t, r \neq 0),$$
$$I_{0,r}(\mathbb{A}|\mathbb{B}) \equiv \lim_{t \to +0} I_{t,r}(\mathbb{A}|\mathbb{B}) \quad \text{and} \quad I_{t,0}(\mathbb{A}|\mathbb{B}) \equiv \lim_{r \to 0} I_{t,r}(\mathbb{A}|\mathbb{B}) = I_t(\mathbb{A}|\mathbb{B}).$$

In this section, we obtain the following relations among $S(\mathbb{A}|\mathbb{B})$, $T_{t,r}(\mathbb{A}|\mathbb{B})$ and $I_{t,r}(\mathbb{A}|\mathbb{B})$. (3.1) and (3.2) in Theorem 3.1 imply (2.1) and (2.2) in Theorem 2.1 by letting $r \to +0$, respectively.

Theorem 3.1 ([6]). *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be sequences of strictly positive operators with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. Then*

$$S(\mathbb{A}|\mathbb{B}) \le I_{t,r}(\mathbb{A}|\mathbb{B}) \le T_{t,r}(\mathbb{A}|\mathbb{B}) \le 0 \tag{3.1}$$

and

$$0 \le -T_{1-t,r}(\mathbb{B}|\mathbb{A}) \le -I_{1-t,r}(\mathbb{B}|\mathbb{A}) \le S_1(\mathbb{A}|\mathbb{B}) \tag{3.2}$$

hold for $0 < t < 1$ and $0 < r \le 1$.

The inequalities (3.1) and (3.2) hold partially even in the case $-1 \le r < 0$.

Theorem 3.2 ([6]). *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be sequences of strictly positive operators with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. Then*

$$I_{t,r}(\mathbb{A}|\mathbb{B}) \le T_{t,r}(\mathbb{A}|\mathbb{B}) \le 0 \tag{3.3}$$

and

$$0 \le -T_{1-t,r}(\mathbb{B}|\mathbb{A}) \le -I_{1-t,r}(\mathbb{B}|\mathbb{A}) \tag{3.4}$$

hold for $0 < t < 1$ and $-1 \le r < 0$.

By the proof of Theorems 3.1 and 3.2, we get the following result on $I_{0,r}(\mathbb{A}|\mathbb{B})$.

Proposition 3.3 ([6]). *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be sequences of strictly positive operators with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. For each $-1 \le r \le 1$ such that $r \ne 0$,*

$$I_{0,r}(\mathbb{A}|\mathbb{B}) = T_r(\mathbb{A}|\mathbb{B}).$$

Proof of Theorems 3.1 and 3.2. Since $f(x) = \log x$ is operator concave for $x > 0$, by using Theorem 2.A, we have that

$$\begin{aligned} I_{t,r}(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\log\sum_{i=1}^n A_i^{\frac{1}{2}}\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}}A_i^{\frac{1}{2}} \\ &\ge \frac{1}{t}\sum_{i=1}^n A_i^{\frac{1}{2}}\log\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}}A_i^{\frac{1}{2}} \\ &= \frac{1}{tr}\sum_{i=1}^n A_i^{\frac{1}{2}}\log\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}A_i^{\frac{1}{2}} \end{aligned} \tag{3.5}$$

for $-1 \le r \le 1$ and

$$\begin{aligned} &\frac{1}{tr}\sum_{i=1}^n A_i^{\frac{1}{2}}\log\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}A_i^{\frac{1}{2}} \\ &\ge \frac{1}{tr}\sum_{i=1}^n A_i^{\frac{1}{2}}\{(1-t)\log I + t\log(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}A_i^{\frac{1}{2}} \\ &= \sum_{i=1}^n A_i^{\frac{1}{2}}\log(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})A_i^{\frac{1}{2}} = S(\mathbb{A}|\mathbb{B}) \end{aligned}$$

for $0 < r \le 1$.

Since $\log x \le x - 1$ for $x > 0$, we have

$$\begin{aligned} I_{t,r}(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\log\sum_{i=1}^n A_i^{\frac{1}{2}}\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}}A_i^{\frac{1}{2}} \\ &\le \frac{1}{t}\left[\sum_{i=1}^n A_i^{\frac{1}{2}}\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}}A_i^{\frac{1}{2}} - I\right] = T_{t,r}(\mathbb{A}|\mathbb{B}) \end{aligned} \tag{3.6}$$

for $-1 \le r \le 1$.

Since $\frac{(1-t+tx^r)^{\frac{1}{r}}-1}{t} \leq x-1$ for $x>0$ and $-1 \leq r \leq 1$, we have

$$\begin{aligned}
T_{t,r}(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\left[\sum_{i=1}^{n} A_i^{\frac{1}{2}}\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}}A_i^{\frac{1}{2}} - I\right] \\
&= \sum_{i=1}^{n} A_i^{\frac{1}{2}}\frac{\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}} - I}{t}A_i^{\frac{1}{2}} \\
&\leq \sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}} - I)A_i^{\frac{1}{2}} \\
&= \sum_{i=1}^{n}(B_i - A_i) = 0.
\end{aligned}$$

Therefore we obtain (3.1) and (3.3).

By (3.1),

$$S(\mathbb{B}|\mathbb{A}) \leq I_{1-t,r}(\mathbb{B}|\mathbb{A}) \leq T_{1-t,r}(\mathbb{B}|\mathbb{A}) \leq 0$$

holds for $0<t<1$ and $0<r\leq 1$, so that we have (3.2) by (iii) in Lemma 2.2. We also have (3.4) similarly. Hence the proof is complete. □

Proof of Proposition 3.3. By (3.5) and (3.6) in the proof of Theorems 3.1 and 3.2, we have

$$\frac{1}{tr}\sum_{i=1}^{n} A_i^{\frac{1}{2}}\log\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}A_i^{\frac{1}{2}} \leq I_{t,r}(\mathbb{A}|\mathbb{B}) \leq T_{t,r}(\mathbb{A}|\mathbb{B})$$

for $-1 \leq r \leq 1$. Therefore we get $I_{0,r}(\mathbb{A}|\mathbb{B}) = T_r(\mathbb{A}|\mathbb{B})$ since $T_{0,r}(\mathbb{A}|\mathbb{B}) = T_r(\mathbb{A}|\mathbb{B})$ and

$$\lim_{t\to+0}\frac{\log(1-t+tx^r)}{tr} = \frac{x^r-1}{r}$$

for $x>0$. □

4 Another generalization

In this section, we discuss another generalization of Theorem 2.1. We introduced $S_{t,r}(\mathbb{A}|\mathbb{B})$ as a generalizaton of $S_t(\mathbb{A}|\mathbb{B})$ in the previous section, but $S_{t,r}(\mathbb{A}|\mathbb{B})$ does not appear in Theorem 3.1. Then we expect that we can generalize (2.1) to

$$S_{0,r}(\mathbb{A}|\mathbb{B}) = T_r(\mathbb{A}|\mathbb{B}) \leq J_{t,r}(\mathbb{A}|\mathbb{B}) \leq T_{t,r}(\mathbb{A}|\mathbb{B}) \leq 0 \qquad (4.1)$$

for a suitable generalized entropy $J_{t,r}(\mathbb{A}|\mathbb{B})$ such that $J_{t,0}(\mathbb{A}|\mathbb{B}) = I_t(\mathbb{A}|\mathbb{B})$. From this viewpoint, we introduce another generalization of $I_t(\mathbb{A}|\mathbb{B})$.

Definition 2 ([6]). *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, \cdots, B_n)$ *be sequences of strictly positive operators with* $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. *For* $0 < t \leq 1$ *and* $-1 \leq r \leq 1$ *such that* $r \neq 0$,

$$J_{t,r}(\mathbb{A}|\mathbb{B}) \equiv \frac{(\sum_{i=1}^n A_i \natural_{t,r} B_i)^r - I}{tr},$$
$$J_{0,r}(\mathbb{A}|\mathbb{B}) \equiv \lim_{t\to+0} J_{t,r}(\mathbb{A}|\mathbb{B}) \quad and \quad J_{t,0}(\mathbb{A}|\mathbb{B}) \equiv \lim_{r\to 0} J_{t,r}(\mathbb{A}|\mathbb{B}).$$

Firstly we show a relation between $I_{t,r}(\mathbb{A}|\mathbb{B})$ and $J_{t,r}(\mathbb{A}|\mathbb{B})$, two generalizations of $I_t(\mathbb{A}|\mathbb{B})$.

Proposition 4.1 ([6]). *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, \cdots, B_n)$ *be sequences of strictly positive operators with* $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. *Then for each* $0 < t < 1$,

(i) $I_{t,r}(\mathbb{A}|\mathbb{B}) \leq J_{t,r}(\mathbb{A}|\mathbb{B})$ *for* $0 < r \leq 1$.

(ii) $J_{t,r}(\mathbb{A}|\mathbb{B}) \leq I_{t,r}(\mathbb{A}|\mathbb{B})$ *for* $-1 \leq r < 0$.

Proof of Proposition 4.1. Since $\log x \leq \frac{x^r-1}{r}$ for $x > 0$ and $0 < r \leq 1$, we have

$$\begin{aligned} I_{t,r}(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\log\sum_{i=1}^n A_i^{\frac{1}{2}}\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}}A_i^{\frac{1}{2}} \\ &\leq \frac{1}{t}\frac{\left[\sum_{i=1}^n A_i^{\frac{1}{2}}\{(1-t)I + t(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^r\}^{\frac{1}{r}}A_i^{\frac{1}{2}}\right]^r - I}{r} = J_{t,r}(\mathbb{A}|\mathbb{B}), \end{aligned}$$

so that we obtain (i). (ii) is also obtained similarly. □

Next we obtain the following results among $S_{t,r}(\mathbb{A}|\mathbb{B})$, $J_{t,r}(\mathbb{A}|\mathbb{B})$ and $T_{t,r}(\mathbb{A}|\mathbb{B})$. By (ii) in Proposition 4.3, we recognize that Theorem 4.2 implies Theorem 2.1 by letting $r \to 0$.

Theorem 4.2 ([6]). *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, \cdots, B_n)$ *be sequences of strictly positive operators with* $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. *Then*

$$S_{0,r}(\mathbb{A}|\mathbb{B}) = T_r(\mathbb{A}|\mathbb{B}) \leq J_{t,r}(\mathbb{A}|\mathbb{B}) \leq T_{t,r}(\mathbb{A}|\mathbb{B}) \leq 0, \tag{4.1}$$

$$0 \leq -T_{1-t,r}(\mathbb{B}|\mathbb{A}) \leq -J_{1-t,r}(\mathbb{B}|\mathbb{A}) \leq S_{1,r}(\mathbb{A}|\mathbb{B}) \tag{4.2}$$

and

$$T_{t,r}(\mathbb{A}|\mathbb{B}) \leq S_{t,r}(\mathbb{A}|\mathbb{B}) \leq -T_{1-t,r}(\mathbb{B}|\mathbb{A}) \tag{4.3}$$

hold for $0 < t < 1$ *and* $-1 \leq r \leq 1$ *such that* $r \neq 0$.

Proposition 4.3 ([6]). *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, \cdots, B_n)$ *be sequences of strictly positive operators with* $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$.

(i) *For each* $-1 \leq r \leq 1$ *such that* $r \neq 0$, $J_{0,r}(\mathbb{A}|\mathbb{B}) = T_r(\mathbb{A}|\mathbb{B})$.

(ii) *For each* $0 < t \leq 1$, $J_{t,0}(\mathbb{A}|\mathbb{B}) = I_t(\mathbb{A}|\mathbb{B})$.

The following lemma is an extension of Lemma 2.2. Lemma 4.4 leads Lemma 2.2 by letting $r \to 0$.

Lemma 4.4 ([5]). *Let* $A, B > 0$. *Then the following properties hold:*

(i) $S_{0,r}(A|B) \leq T_{t,r}(A|B) \leq S_{t,r}(A|B)$ *for* $0 < t < 1$ *and* $-1 \leq r \leq 1$.

(ii) $S_{t,r}(A|B) = -S_{1-t,r}(B|A)$ *for* $0 < t < 1$ *and* $-1 \leq r \leq 1$

(iii) $S_{1,r}(A|B) = -S_{0,r}(B|A)$ *for* $-1 \leq r \leq 1$.

We can give proofs of Lemma 4.4 and Theorem 4.2 by the similar way to those of Lemma 2.2 and Theorem 2.1. We omit these proofs.

Lastly we get the following result by combining Theorem 3.1, Theorem 3.2, Proposition 4.1 and Theorem 4.2.

Corollary 4.5 ([6]). *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, \cdots, B_n)$ *be sequences of strictly positive operators with* $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. *Then for* $0 < t < 1$,

$$S(\mathbb{A}|\mathbb{B}) \leq I_{t,r}(\mathbb{A}|\mathbb{B}) \leq J_{t,r}(\mathbb{A}|\mathbb{B}) \leq T_{t,r}(\mathbb{A}|\mathbb{B}) \leq 0$$

holds if $0 < r \leq 1$, *and also*

$$S_{0,r}(\mathbb{A}|\mathbb{B}) = T_r(\mathbb{A}|\mathbb{B}) \leq J_{t,r}(\mathbb{A}|\mathbb{B}) \leq I_{t,r}(\mathbb{A}|\mathbb{B}) \leq T_{t,r}(\mathbb{A}|\mathbb{B}) \leq 0$$

holds if $-1 \leq r < 0$.

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