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Kyoto University
(0, 1) 区間上の作用素単調関数と Kwong 行列

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We review results on operator monotone functions on (0,1) and Kwong matrices. For details, we refer [10].

1 Operator monotone functions on (0,1)

Let $f$ be a real-valued $C^1$ function on an interval $(a,b)$. For $n$ distinct real numbers $t_1, \ldots, t_n \in (a,b)$ a Loewner (or Pick) matrix $L_f(t_1, \ldots, t_n)$ is defined as

$$L_f(t_1, \ldots, t_n) = \begin{bmatrix} f(t_i) - f(t_j) \end{bmatrix} \frac{1}{t_i - t_j}.$$ 

In the case where $(a, b) \subseteqq (0, \infty)$, a Kwong (or an anti-Loewner) matrix $K_f(t_1, \ldots, t_n)$ is defined by

$$K_f(t_1, \ldots, t_n) = \begin{bmatrix} f(t_i) + f(t_j) \end{bmatrix} \frac{1}{t_i + t_j}.$$ 

In this paper we study positive operator monotone functions on (0,1) to continue our preceding studies on Loewner and Kwong matrices [3, 4, 7, 8, 11]. we show that the similar results of Loewner/Kwong matrices do not hold in general. For basic facts on operator monotone functions, we refer the reader to [2, 5, 6].

The following is useful for our study.

Lemma 1.1.

(1) $K_f(t_1, \ldots, t_n) + L_f(t_1, \ldots, t_n) = 2 \begin{bmatrix} t_i f(t_i) - t_j f(t_j) \end{bmatrix} \frac{1}{t_i^2 - t_j^2}$

$= 2 C \circ L_{tf(t)}(t_1, \ldots, t_n)$

$= 2 L_{\sqrt{f}(\sqrt{t})}(s_1, \ldots, s_n),$
where $C$ is given as $C = \begin{bmatrix} \frac{1}{t_i + t_j} \end{bmatrix}$, $\circ$ stands for the Schur product and $s_i = t_i^2$.

(2) $K_f(t_1, \ldots, t_n) - L_f(t_1, \ldots, t_n) = 2 \left[ \frac{t_i f(t_j) - t_j f(t_i)}{t_i^2 - t_j^2} \right]$

$= 2 D \left[ \frac{t_i / f(t_i) - t_j / f(t_j)}{t_i^2 - t_j^2} \right] D$

$= 2 C \circ (DL_{t / f(t)}(t_1, \ldots, t_n) D)$

$= 2 DL_{\sqrt{t} / f(\sqrt{t})}(s_1, \ldots, s_n) D,$

where $C$ and $s_i$ are the same as in (1) and $D$ is given as $D = \text{diag} \left( f(t_1), \ldots, f(t_n) \right)$.

For our study we prepare the representation of positive operator monotone functions on $(0,1)$.

**Theorem 1.2** A positive operator monotone function $f(s)$ on $(0,1)$ is of the form

$$f(s) = \int_{[0,1]} \frac{s}{s + \zeta - 2s\zeta} dm(\zeta),$$

where $m$ is a positive measure on $[0,1]$.

For $0 \leq \zeta \leq 1$, put

$$f_{\zeta}(s) := \frac{s}{(1 - 2\zeta)s + \zeta} = \frac{s}{s + \zeta - 2s\zeta}.$$  \hspace{1cm} (1.1)

**Theorem 1.3** Let $f_{\zeta}(s)$ be the function in (1.1). Then $s / f_{\zeta}(s)$ is operator monotone if and only if $\zeta \leq 1/2$.

**Corollary 1.4** Let $f(s)$ be a positive operator monotone function on $(0,1)$ which is of the form

$$f(s) = \int_{[0,1/2]} f_{\zeta}(s) dm(\zeta) = \int_{[0,1/2]} \frac{s}{(1 - 2\zeta)s + \zeta} dm(\zeta),$$  \hspace{1cm} (1.2)

where $m$ is a positive measure on $[0,1/2]$. Then $s / f(s)$ is operator monotone on $(0,1)$.

The following corresponds to Kwong [9].

**Theorem 1.5** If $f(s)$ is the operator monotone function in (1.2), then all Kwong matrices associated with $f$ are positive semidefinite.
**Theorem 1.6** Let \( f_\zeta(s) \) be the function in (1.1). Then all Kwong matrices associated with \( f_\zeta \) are positive semidefinite if and only if \( \zeta \leq 1/2 \).

The following is a counterpart to Audenaert [1].

**Theorem 1.7** Let \( f(s) \) be a positive function on \((0,1)\). If \( \sqrt{s}f(\sqrt{s}) \) or \( \sqrt{s}/f(\sqrt{s}) \) is the operator monotone function in (1.2), then all Kwong matrices associated with \( f \) are positive semidefinite.

For \( 0 \leq \zeta \leq 1 \), let us consider the function on \((0,1)\)

\[
g_\zeta(s) := \frac{f_\zeta(s^2) - s}{s(1-2\zeta)s^2 + \zeta}.
\]  
(1.3)

We note the following:

**Theorem 1.8** Let \( g_\zeta(s) \) be the function in (1.3). Then \( g_\zeta(s) \) is operator monotone if and only if \( 1/2 \leq \zeta \), and all Kwong matrices associated with \( g_\zeta \) are positive semidefinite if and only if \( \zeta \leq 1/2 \).

**Proposition 1.9** Let \( f(s) \) be the operator monotone function in (1.2). Then for any positive integer \( m \),

\[
\left[ \frac{f(s_i)^m - f(s_j)^m}{s_i^m - s_j^m} \right]
\]

are positive semidefinite for all \( n \) and \( s_1, \ldots, s_n \) in \((0,1)\).

**参考文献**


