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$(0,1)$ 区間上の作用素単調関数と Kwong 行列

山形大学大学院理工学研究科　森下　主理 (Juri MORISHITA)
Graduate School of Science and Engineering,
Yamagata University

山形大学理学部数理科学科　佐野　隆志 (Takashi SANO)
Department of Mathematical Sciences, Faculty of Science,
Yamagata University

We review results on operator monotone functions on $(0,1)$ and Kwong matrices. For details, we refer [10].

1 Operator monotone functions on $(0,1)$

Let f be a real-valued C^{1} function on an interval (a, b). For n distinct real numbers $t_{1}, \ldots, t_{n}\in(a, b)$ a *Loewner* (or *Pick*) *matrix* $L_{f}(t_{1}, \ldots, t_{n})$ is defined as

$$L_{f}(t_{1}, \ldots, t_{n})=\left[\frac{f(t_{i})-f(t_{j})}{t_{i}-t_{j}}\right].$$

In the case where $(a, b)\subseteqq(0, \infty)$, a *Kwong* (or an *anti-Loewner*) *matrix* $K_{f}(t_{1}, \ldots, t_{n})$ is defined by

$$K_{f}(t_{1}, \ldots, t_{n})=\left[\frac{f(t_{i})+f(t_{j})}{t_{i}+t_{j}}\right].$$

In this paper we study positive operator monotone functions on $(0,1)$ to continue our preceding studies on Loewner and Kwong matrices [3, 4, 7, 8, 11]. we show that the similar results of Loewner/Kwong matrices do not hold in general. For basic facts on operator monotone functions, we refer the reader to [2, 5, 6].

The following is useful for our study.

Lemma 1.1.

(1) $$K_{f}(t_{1}, \ldots, t_{n})+L_{f}(t_{1}, \ldots, t_{n})=2\left[\frac{t_{i}f(t_{i})-t_{j}f(t_{j})}{t_{i}^{2}-t_{j}^{2}}\right]$$
$$=2\ C\circ L_{tf(t)}(t_{1}, \ldots, t_{n})$$
$$=2\ L_{\sqrt{t}f(\sqrt{t})}(s_{1}, \ldots, s_{n}),$$

where C is given as $C = \left[\dfrac{1}{t_i + t_j}\right]$, $\circ$ stands for the Schur product and $s_i = t_i^2$.

$$\begin{aligned}(2)\quad K_f(t_1,\dots,t_n) - L_f(t_1,\dots,t_n) &= 2\left[\frac{t_i f(t_j) - t_j f(t_i)}{t_i^2 - t_j^2}\right] \\ &= 2\,D\left[\frac{t_i/f(t_i) - t_j/f(t_j)}{t_i^2 - t_j^2}\right]D \\ &= 2\,C \circ \left(DL_{t/f(t)}(t_1,\dots,t_n)D\right) \\ &= 2\,DL_{\sqrt{t}/f(\sqrt{t})}(s_1,\dots,s_n)D,\end{aligned}$$

where C and s_i are the same as in (1) and D is given as $D = \operatorname{diag}\,(f(t_1),\dots,f(t_n))$.

For our study we prepare the representation of positive operator monotone functions on $(0,1)$.

Theorem 1.2 A positive operator monotone function $f(s)$ on $(0,1)$ is of the form

$$f(s) = \int_{[0,1]} \frac{s}{s+\zeta-2s\zeta}\,dm(\zeta),$$

where m is a positive measure on $[0,1]$.

For $0 \leqq \zeta \leqq 1$, put

$$f_\zeta(s) := \frac{s}{(1-2\zeta)s+\zeta} = \frac{s}{s+\zeta-2s\zeta}. \tag{1.1}$$

Theorem 1.3 Let $f_\zeta(s)$ be the function in (1.1). Then $s/f_\zeta(s)$ is operator monotone if and only if $\zeta \leqq 1/2$.

Corollary 1.4 Let $f(s)$ be a positive operator monotone function on $(0,1)$ which is of the form

$$f(s) = \int_{[0,1/2]} f_\zeta(s)\,dm(\zeta) = \int_{[0,1/2]} \frac{s}{(1-2\zeta)s+\zeta}\,dm(\zeta), \tag{1.2}$$

where m is a positive measure on $[0,1/2]$. Then $s/f(s)$ is operator monotone on $(0,1)$.

The following corresponds to Kwong [9].

Theorem 1.5 If $f(s)$ is the operator monotone function in (1.2), then all Kwong matrices associated with f are positive semidefinite.

Theorem 1.6 Let $f_\zeta(s)$ be the function in (1.1). Then all Kwong matrices associated with f_ζ are positive semidefinite if and only if $\zeta \leqq 1/2$.

The following is a counterpart to Audenaert [1].

Theorem 1.7 Let $f(s)$ be a positive function on $(0,1)$. If $\sqrt{s}f(\sqrt{s})$ or $\sqrt{s}/f(\sqrt{s})$ is the operator monotone function in (1.2), then all Kwong matrices associated with f are positive semidefinite.

For $0 \leqq \zeta \leqq 1$, let us consider the function on $(0,1)$

$$g_\zeta(s) := \frac{f_\zeta(s^2)}{s} = \frac{s}{(1-2\zeta)s^2+\zeta}. \tag{1.3}$$

We note the following:

Theorem 1.8 Let $g_\zeta(s)$ be the function in (1.3). Then $g_\zeta(s)$ is operator monotone if and only if $1/2 \leqq \zeta$, and all Kwong matrices associated with g_ζ are positive semidefinite if and only if $\zeta \leqq 1/2$.

Proposition 1.9 Let $f(s)$ be the operator monotone function in (1.2). Then for any positive integer m,

$$\left[\frac{f(s_i)^m - f(s_j)^m}{s_i^m - s_j^m}\right]$$

are positive semidefinite for all n and $s_1, \ldots, s_n$ in $(0,1)$.

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