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THE APPROXIMATION PROPERTY AND THE CHAIN CONDITION

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1. THE APPROXIMATION PROPERTY

Definition 1.1. Let \( P \) be a poset and \( \kappa \) a cardinal. We say that the poset \( P \) has the \( \kappa \)-approximation property if for every ordinal \( \tau \) and every \( f \in (\tau^2)^P \), if \( f \restriction x \in V \) for every \( x \in ([\tau]^{<\kappa})^V \), then \( f \in V \).

It is known that for an uncountable \( \kappa \), if \( P \) is an atomless poset of size \( < \kappa \) and \( \dot{Q} \) is a \( P \)-name for a \( \kappa \)-closed poset, then \( P * \dot{Q} \) has the \( \kappa \)-approximation property (e.g., see Mitchell [1]). In this note, we show that the size assumption for a poset \( P \) can be relaxed to the chain condition assumption.

Definition 1.2. Let \( \kappa \) be a regular uncountable cardinal. A poset \( P \) satisfies the strong \( \kappa \)-chain condition (strong \( \kappa \)-c.c., for short) if \( P \) satisfies the \( \kappa \)-c.c. and for every \( \kappa \)-Suslin tree \( T \), \( P \) does not add a cofinal branch of \( T \).

Note 1.3. (1) If there is no \( \kappa \)-Suslin tree, then the \( \kappa \)-c.c. is equivalent to the strong \( \kappa \)-c.c.

(2) For a poset \( P \), if \( P \times P \) satisfies the \( \kappa \)-c.c., then \( P \) satisfies the strong \( \kappa \)-c.c.

Lemma 1.4. If a poset \( P \) satisfies the \( \mu \)-c.c. for some \( \mu < \kappa \), then \( P \) satisfies the strong \( \kappa \)-c.c. In particular, every poset of size \( < \kappa \) satisfies the strong \( \kappa \)-c.c.

Proof. Suppose to the contrary that there is a \( \kappa \)-Suslin tree \( T \) such that \( \models_P \) \[ T \] has a cofinal branch \( \dot{B} \). Let \( T' = \{ t \in T : p \models_P t \in \dot{B} \text{ for some } p \in P \} \). It is easy to check that \( T' \) is a downward closed subtree of \( T \) of height \( \kappa \). Since \( P \) satisfies the \( \mu \)-c.c. and \( \mu < \kappa \), each level of \( T' \) has size \( < \mu \). Now, by Kurepa's theorem, \( T' \) has a cofinal branch. Then this branch is a cofinal branch of \( T \), this is a contradiction. \( \square \)

The following is a main result of this note:

Lemma 1.5. Let \( \kappa \) be a regular uncountable cardinal. Let \( P \) be an atomless poset which satisfies the strong \( \kappa \)-c.c. Let \( \dot{Q} \) be a \( P \)-name for a \( \kappa \)-closed poset (trivial poset is possible). Then \( P * \dot{Q} \) has the \( \kappa \)-approximation property.
Proof. Let \( \mathcal{Q} \) be a term poset of \( \mathcal{Q} \), that is, \( \mathcal{Q} \) is the set of all \( P \)-names \( \dot{q} \) with \( \vdash_{P} \dot{q} \in \mathcal{Q} \). For \( \dot{q}_{0}, \dot{q}_{1} \in \mathcal{Q} \), define \( \dot{q}_{0} \leq \dot{q}_{1} \) if \( \vdash_{P} \dot{q}_{0} \leq \dot{q}_{1} \) in \( \mathcal{Q} \). Since \( \mathcal{Q} \) is a name for a \( \kappa \)-closed poset, \( \mathcal{Q} \) is \( \kappa \)-closed.

Let \( \dot{x} \) be a \( P \times \mathcal{Q} \)-name such that \( \vdash \dot{x} \in V \). We say that a condition \( \langle p, \dot{q} \rangle \in P \times \mathcal{Q} \) decides \( \dot{x} \) if there is \( y \) with \( \langle p, \dot{q} \rangle \vdash \dot{x} = y \).

**Claim 1.6.** Let \( \tau \) be an ordinal and \( \dot{f} \) be a \( P \times \mathcal{Q} \)-name such that \( \vdash \dot{f} : \tau \to 2 \) and \( \dot{f}|x \in V \) for every \( x \in (\\lfloor \tau \rfloor ^{< \kappa})^{V} \). Let \( \langle p, \dot{q} \rangle \in P \times \mathcal{Q} \) and \( x \in \lfloor \tau \rfloor ^{< \kappa} \). Then there are \( \dot{q}^{*} \leq \dot{q} \) and \( F \subseteq \kappa 2 \) such that:

1. \( |F| < \kappa \).
2. For every \( g \in F \), there is \( p' \leq p \) such that \( \langle p', \dot{q}^{*} \rangle \vdash \dot{f}|x = g \).
3. For every \( p' \leq p \), there are \( p'' \leq p' \) and \( g \in F \) such that \( \langle p'', \dot{q}^{*} \rangle \vdash \dot{f}|x = g \).

**Proof.** It is easy to check that the set \( \{ p' \leq p : \exists \dot{q}' (\langle p', \dot{q}' \rangle \leq \langle p, \dot{q} \rangle \text{ and } \langle p', \dot{q}' \rangle \text{ decides } \dot{f}|x \} \) is predense below \( p \). Take a maximal antichain \( A \) which is contained in this set. Since \( P \) satisfies the \( \kappa \)-c.c., we know that \( |A| < \kappa \). Then for each \( r \in A \), there are \( \dot{q}_{r} \) and \( g_{r} \) such that \( \langle r, \dot{q}_{r} \rangle \leq \langle p, \dot{q} \rangle \) and \( \langle r, \dot{q}_{r} \rangle \vdash \dot{f}|x = g_{r} \). Let \( F = \{ g_{r} : r \in A \} \) and one can take \( \dot{q}^{*} \) such that \( \dot{q}^{*} \leq \dot{q} \) and \( r \vdash \dot{q}^{*} = g_{r} \) for every \( r \in A \). Then \( \dot{q}^{*} \) and \( F \) work. \( \square \) [Claim]

In order to show that \( P \times \mathcal{Q} \) has the \( \kappa \)-approximation property, take \( \langle p, \dot{q} \rangle \in P \times \mathcal{Q} \), an ordinal \( \tau \), and a name \( \dot{f} \) such that \( \langle p, \dot{q} \rangle \vdash \dot{f} : \tau \to 2 \) and \( \dot{f}|x \in V \) for every \( x \in (\lfloor \tau \rfloor ^{< \kappa})^{V} \). Suppose to the contrary that \( \langle p, \dot{q} \rangle \vdash \dot{f} \notin V \).

By induction on \( \alpha < \kappa \), we would find \( x_{\alpha}, \dot{q}_{\alpha}, F_{\alpha} (\alpha < \kappa) \) such that:

1. \( x_{\alpha} \in \lfloor \tau \rfloor ^{< \kappa} \) and \( \langle x_{\alpha} : \alpha < \kappa \rangle \) is \( \subseteq \)-increasing.
2. \( \langle \dot{q}_{\alpha} : \alpha < \kappa \rangle \) is decreasing in \( \mathcal{Q} \) and \( \dot{q}_{0} \leq \dot{q} \).
3. \( F_{\alpha} \subseteq \kappa 2 \) and \( |F_{\alpha}| < \kappa \).
4. For every \( g \in F_{\alpha} \), there is \( p' \leq p \) such that \( \langle p', \dot{q}_{\alpha} \rangle \vdash \dot{f}|x_{\alpha} = g \).
5. For every \( p' \leq p \) there are \( p'' \leq p' \) and \( g \in F_{\alpha} \) such that \( \langle p'', \dot{q}_{\alpha} \rangle \vdash \dot{f}|x_{\alpha} = g \), i.e., the set \( \{ p' \leq p : \langle p', \dot{q}_{\alpha} \rangle \vdash \dot{f}|x_{\alpha} = g \} \) is predense below \( p \).
6. For every \( g \in F_{\alpha} \), there are \( g_{0}, g_{1} \in F_{\alpha+1} \) such that \( g \subseteq g_{0}, g_{1} \) and \( g_{0} \neq g_{1} \).

When \( \alpha = 0 \), pick an arbitrary \( x_{0} \in \lfloor \tau \rfloor ^{< \kappa} \). Then we can find required \( \dot{q}_{0} \leq \dot{q} \) and \( F_{0} \) by Claim 1.6.

Let \( \alpha > 0 \) and suppose \( x_{\beta}, \dot{q}_{\beta}, F_{\beta} \) are defined for all \( \beta < \alpha \).

Case 1: \( \alpha \) is limit. We can find \( x_{\alpha} \in \lfloor \tau \rfloor ^{< \kappa} \) such that \( x_{\beta} \subseteq x_{\alpha} \) for \( \beta < \alpha \). Since \( \mathcal{Q} \) is \( \kappa \)-closed, we can find \( \dot{q}^{*} \leq \dot{q}_{\beta} \) for every \( \beta < \alpha \). Then take \( \dot{q}_{\alpha} \leq \dot{q}^{*} \) and \( F_{\alpha} \) by Claim 1.6.

Case 2: \( \alpha \) is successor, say \( \alpha = \beta + 1 \). Pick a maximal antichain \( A \subseteq P \) below \( p \) such that for every \( p' \in A \) there is \( g \in F_{\beta} \) such that \( \langle p', \dot{q}_{\beta} \rangle \vdash \dot{f}|x_{\beta} = g \). Note
that $|A| < \kappa$, and, for every $g \in F_{\beta}$, there is $p' \in A$ with $\langle p', \dot{q}_{\beta} \rangle \Vdash \langle \dot{f} | x_{\beta} = g \rangle$.

Since $|A| < \kappa$ and $\langle p, \dot{q}_{\beta} \rangle \Vdash \langle \dot{f} \notin V \rangle$, we can find $x_{\alpha} \in [\tau]^{< \kappa}$ such that $x_{\beta} \subseteq x_{\alpha}$ for $\beta < \alpha$, but $\langle p', \dot{q}_{\beta} \rangle$ does not decide $\dot{f} | x_{\alpha}$ for every $p' \in A$.

**Claim 1.7.** For each $p' \in A$, there are $p_{0}', p_{1}', g_{0}', g_{1}': x_{\alpha} \rightarrow 2$, and $\dot{r} \leq \dot{q}_{\beta}$ such that $g_{0}' \neq g_{1}'$ and $\langle p_{i}', \dot{r} \rangle \Vdash \langle \dot{f} | x_{\alpha} = g_{i}' \rangle$.

**Proof.** Since $\langle p', \dot{q}_{\beta} \rangle$ does not decide $\dot{f} | x_{\alpha}$, we can take $(p_{0}', \dot{q}_{0})$, $(p_{1}', \dot{q}_{1}) \leq (p', \dot{q}_{\beta})$, and $g_{0}', g_{1}': x_{\alpha} \rightarrow 2$ such that $g_{0}' \neq g_{1}'$ and $\langle p_{i}', \dot{q}_{i} \rangle \Vdash \langle \dot{f} | x_{\alpha} = g_{i}' \rangle$. We may assume that $p_{0}'$ is incompatible with $p_{1}'$; if $p_{0}'$ and $p_{1}'$ have a common extension $p_{2}$, take $p_{0}'' \leq p_{2}$ such that $p_{0}'' \bot p_{1}'$ and replace $p_{1}'$ by $p_{0}''$.

Now take $\dot{r} \leq \dot{q}_{\beta}$ such that $p_{i}' \Vdash \langle \dot{r} = \dot{q}_{i} \rangle$. Clearly $p_{i}'$, $g_{i}'$ and $\dot{r}$ work. □[Claim]

For each $p' \in A$, pick $\dot{r}_{p'} \leq \dot{q}_{\beta}$ such that there are $p_{0}', p_{1}' \leq p'$, $g_{0}', g_{1}': x_{\alpha} \rightarrow 2$ with $g_{0}' \neq g_{1}'$ and $\langle p_{i}', \dot{r}_{p'} \rangle \Vdash \langle \dot{f} | x_{\alpha} = g_{i}' \rangle$.

Then pick $\dot{q}^* \leq \dot{q}_{\beta}$ such that $p' \Vdash \langle \dot{q}^* = \dot{r}_{p'} \rangle$ for every $p' \in A$. Finally, take $\dot{q}_{\alpha} \leq \dot{q}^*$ and $F_{\alpha} \subseteq x^{*2}$ as in Claim 1.6. The following claim shows that $x_{\alpha}$, $\dot{q}_{\alpha}$, and $F_{\alpha}$ work well:

**Claim 1.8.** For each $g \in F_{\beta}$, there are $g_{0}, g_{1} \in F_{\alpha}$ such that $g_{0} \neq g_{1}$ and $g \subseteq g_{0}, g_{1}$.

**Proof.** Take $p' \in A$ so that $\langle p', \dot{q}_{\beta} \rangle \Vdash \langle \dot{f} | x_{\beta} = g \rangle$. Then we can take $p_{0}', p_{1}' \leq p'$ and $g_{0}', g_{1}': x_{\alpha} \rightarrow 2$ such that $g_{0}' \neq g_{1}'$ and $\langle p_{i}', \dot{r}_{p'} \rangle \Vdash \langle \dot{f} | x_{\alpha} = g_{i}' \rangle$. Clearly $g \subseteq g_{0}', g_{1}'$. By the choice of $F_{\alpha}$ and $\dot{q}_{\alpha}$, for each $i < 2$, one can take $p_{i} \leq p_{i}'$ and $g_{i} \in F_{\alpha}$ such that $\langle p_{i}, \dot{q}_{\alpha} \rangle \Vdash \langle \dot{f} | x_{\alpha} = g_{i} \rangle$. Since $\dot{q}_{\alpha} \leq \dot{q}^*$, each $\langle p_{i}, \dot{q}_{\alpha} \rangle$ is compatible with $\langle p_{i}', \dot{r}_{p'} \rangle$. This means that $g_{i}' = g_{i}$, so $g_{0}' \neq g_{1}'$ and $g \subseteq g_{0}, g_{1}$. □[Claim]

Suppose $\dot{q}_{\alpha}, x_{\alpha}, F_{\alpha}$ are defined for $\alpha < \kappa$. Note that, for every $\alpha < \beta < \kappa$ and $g \in F_{\beta}$, we have $g | x_{\alpha} \in F_{\alpha}$; take $p' \leq p$ such that $\langle p', \dot{q}_{\beta} \rangle \Vdash \langle \dot{f} | x_{\beta} = g \rangle$. Then one can pick $p'' \leq p'$ and $h \in F_{\alpha}$ such that $\langle p'', \dot{q}_{\alpha} \rangle \Vdash \langle \dot{f} | x_{\alpha} = h \rangle$. $\langle p', \dot{q}_{\beta} \rangle$ is compatible with $\langle p'', \dot{q}_{\alpha} \rangle$. So $h = g | x_{\alpha}$.

Let $T = \bigcup_{\alpha < \kappa} F_{\alpha}$. $T$ with the inclusion forms a $\kappa$-tree, and each node of $T$ has at least two immediate successors.

**Claim 1.9.** $T$ has no antichain of size $\kappa$.

**Proof.** For each $g \in T$, there are $p_{g}$ and $\alpha_{g} < \kappa$ such that $\langle p_{g}, \dot{q}_{\alpha_{g}} \rangle \Vdash \langle \dot{f} | x_{\alpha_{g}} = g \rangle$.

For $g, g'$ in $T$, if $g$ and $g'$ are incompatible in $T$, then $p_{g}$ is incompatible with $p_{g'}$ in $\mathbb{P}$. This means that if $T$ has an antichain of size $\kappa$, then $\mathbb{P}$ also has an antichain of size $\kappa$. This is impossible, hence $T$ does not have an antichain of size $\kappa$. □[Claim]

Hence $T$ is a $\kappa$-Suslin tree. We finish the proof by showing the following claim, which contradicts the strong $\kappa$-c.c. of $\mathbb{P}$:
Claim 1.10. $p \Vdash \text{"T has a cofinal branch".}$

Proof. Take a $(V, \mathbb{P})$-generic $G$ with $p \in G$ and work in $V[G]$. Let $\alpha < \kappa$. Since $\{p' \leq p : \langle p', \dot{q}_\alpha \rangle \Vdash \text{"}f|\alpha = g\text{"}\}$ for some $g \in \mathcal{F}_\alpha$ is predense below $p$, we can find $p_\alpha \in G$ and $g_\alpha \in \mathcal{F}_\alpha \subseteq T$ such that $\langle p_\alpha, \dot{q}_\alpha \rangle \Vdash \text{"}f|\alpha = g_\alpha\text{"}$. Now, for $\alpha < \beta < \kappa$, $p_\alpha$ is compatible with $p_\beta$ and $\dot{q}_\beta \leq \dot{q}_\alpha$. So $\langle p_\alpha, \dot{q}_\alpha \rangle$ is compatible with $\langle p_\beta, \dot{q}_\beta \rangle$. This means that $g_\alpha \subseteq g_\beta$, so $\{g_\alpha : \alpha < \kappa\}$ is a cofinal branch of $T$. \hfill $\Box$[Claim]

Note 1.11. If $\mathbb{P}$ satisfies the $\kappa$-c.c. but does not have the strong $\kappa$-c.c., then $\mathbb{P}$ cannot have the $\kappa$-approximation property.

2. Applications

We consider some applications of Lemma 1.5.

Definition 2.1. Let $\kappa$ be a regular uncountable cardinal and $\lambda \geq \kappa$ a cardinal. A set $X \subseteq \mathcal{P}_{\kappa}\lambda$ has the strong tree property if for every $\langle d_x : x \in X \rangle$ with $d_x \subseteq x$, if $|\{d_x \cap a : x \in X\}| < \kappa$ for every $a \in \mathcal{P}_\kappa\lambda$, then there is $D \subseteq \lambda$ such that for every $a \in \mathcal{P}_\kappa\lambda$ the set $\{x \in X : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa\lambda$.

Fact 2.2 (Viale-Weiss [3]). (1) The following are equivalent:

(a) $\mathcal{P}_\kappa\lambda$ has the strong tree property.

(b) There is some unbounded set $X \subseteq \mathcal{P}_\kappa\lambda$ such that $X$ has the strong tree property.

(c) Every unbounded subset of $\mathcal{P}_\kappa\lambda$ has the strong tree property.

(2) $\kappa$ has the tree property if and only if $\mathcal{P}_\kappa\kappa$ has the strong tree property.

(3) $\kappa$ is strongly compact if and only if $\kappa$ is inaccessible and $\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$.

(4) Suppose Proper Forcing Axiom. Then $\mathcal{P}_{\omega_2}\lambda$ has the strong tree property for every $\lambda \geq \omega_2$.

Viale-Weiss [3] showed that for an inaccessible $\kappa$, if a standard $\kappa$-stage iteration satisfying the $\kappa$-c.c. forces that "$\kappa = \omega_2$ and Proper forcing axiom"", then $\kappa$ must be strongly compact in the ground model. The following is a slight improvement of their result.

Proposition 2.3. Let $\kappa$ be a regular uncountable cardinal. Suppose that there is a poset $\mathbb{P}$ which has the strong $\kappa$-c.c. and forces that "$\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$". Then $\mathcal{P}_\kappa\lambda$ has the strong tree property for every $\lambda \geq \kappa$ in the ground model.
Proof. We check that $\mathcal{P}_\kappa \lambda$ has the strong tree property for every $\lambda \geq \kappa$. Fix $\lambda \geq \kappa$ and take $\langle d_x : x \in \mathcal{P}_\kappa \lambda \rangle$ such that $d_x \subseteq x$ and $|\{d_x \cap a : x \in \mathcal{P}_\kappa \lambda\}| < \kappa$ for every $a \in \mathcal{P}_\kappa \lambda$. Take a $(V, \mathbb{P})$-generic $G$ and work in $V[G]$. In $V[G]$, $\mathcal{P}_\kappa^V \lambda$ is unbounded in $\mathcal{P}_\kappa \lambda$ since $\mathbb{P}$ satisfies the $\kappa$-c.c. By the strong tree property of $\mathcal{P}_\kappa^V \lambda$ in $V[G]$, we can find $D \subseteq \lambda$ such that $\{x \in \mathcal{P}_\kappa^V \lambda : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa \lambda$ for every $a \in \mathcal{P}_\kappa \lambda$. We see $D \in V$, this completes the proof. For each $a \in \mathcal{P}_\kappa^V \lambda$, there is $x \in \mathcal{P}_\kappa^V \lambda$ with $D \cap a = d_x \cap a \in V$. Thus, by the $\kappa$-approximation property of $\mathbb{P}$, we have $D \in V$. \hfill \qed \\

Next we look at the indestructibility of weak compactness.

**Definition 2.4.** Let $\kappa$ be weakly compact. If every $\kappa$-directed closed forcing preserves the weak compactness of $\kappa$, then $\kappa$ is said to be *indestructibly weakly compact*.

The existence of an indestructibly weakly compact cardinal is consistent (Laver [2]). The following theorem suggests that the consistency of the existence of an indestructibly weakly compact cardinal might be at least strongly compact cardinal.

**Proposition 2.5.** Let $\kappa$ be a regular uncountable cardinal. If there is a poset which satisfies the strong $\kappa$-c.c. and forces that “$\kappa$ is indestructibly weakly compact”, then $\kappa$ is strongly compact.

Proof. Take $\lambda \geq \kappa$. We see that $\mathcal{P}_\kappa \lambda$ has the strong tree property. Take $\langle d_x : x \in \mathcal{P}_\kappa \lambda \rangle$ with $d_x \subseteq x$ and $|\{d_x \cap a : x \in \mathcal{P}_\kappa \lambda\}| < \kappa$ for every $a \in \mathcal{P}_\kappa \lambda$.

Take a $(V, \mathbb{P})$-generic $G$, and a $(V[G], \text{Col}(\kappa, \lambda))$-generic $H$. We work in $V[G][H]$. Fix a bijection $\pi : \lambda \rightarrow \kappa$. We know that $\{\pi^{-1}x : x \in \mathcal{P}_\kappa^V \lambda\}$ is unbounded in $\mathcal{P}_\kappa \kappa$. Since $\kappa$ is weakly compact in $V[G][H]$, by the tree property of $\kappa$, there is $C \subseteq \kappa$ such that $\{\pi^{-1}x \in \mathcal{P}_\kappa \kappa : \pi^{-1}(d_x) \cap a = C \cap a\}$ is unbounded for all $a \in \mathcal{P}_\kappa \kappa$. Put $D = \pi^{-1}C$. Then for every $a \in \mathcal{P}_\kappa \lambda$, the set $\{x \in \mathcal{P}_\kappa^V \lambda : d_x \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_\kappa \lambda$. We know $D \in V$ since $\mathbb{P} \ast \text{Col}(\kappa, \lambda)$ has the $\kappa$-approximation property by Lemma 1.5. \hfill \qed 

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