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<td>Fuchino, Sakae; Usuba, Toshimichi</td>
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Kyoto University
A reflection principle formulated in terms of games

Sakaé Fuchino
Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

Toshimichi Usuba
Organization of Advanced Science and Technology Core Research Team
Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
usuba@people.kobe-u.ac.jp

Abstract

We introduce a principle formulated in terms of the existence of a winning strategy of a game and prove that this principle is placed between the reflection principle down to internally stationary sets (RP$_S$) and the reflection principle down to internally club sets (RP$_C$). In particular, under CH this principle gives a new characterization of Fleissner’s Axiom R.

1 Introduction

For a game $\mathcal{G}$ played by Players $I$ and $II$, let $\text{WS}_{II}(\mathcal{G})$ denote the assertion “Player $II$ has a winning strategy in $\mathcal{G}$”.

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An extended version of this paper with more details and proofs can be found at:

http://kurt.scitec.kobe-u.ac.jp/~fuchino/preprints.html
In [9], we introduced a game $G_{\omega}^\downarrow(\kappa)$ for uncountable cardinals $\kappa$ (see Section 3 for the definition of this and other games mentioned here) and proved that the Rado Conjecture (RC, see Section 2 for the definition of this principle) implies the assertion

\[(G_0) \quad \text{WS}_{II}(G_{\omega}^I(\kappa)) \text{ holds for all uncountable } \kappa.\]

Further, it is proved in [9] that $(G_0)$ implies the Fodor-type Reflection Principle (FRP, see Section 2 for the definition of this principle and, [4] and [5] for basic facts of this principle).

In [1], Philipp Doebler introduced a similar game he called $G_{\omega}(\kappa^{\omega_1}, \omega_1)$ and proved that the Rado Conjecture also implies the principle

\[(G_1) \quad \text{WS}_{II}(G_{\omega}(\kappa^{\omega_1}, \omega_1)) \text{ holds for all } \kappa \geq \aleph_2.\]

He then proved that the principle $(G_1)$ implies the Semistationary Reflection (SSR).

In this paper, we introduce a game $G_{\omega}^{\downarrow\downarrow}(\kappa^{\omega_1})$ which generalizes both $G^{\downarrow}(\kappa)$ and $G_{\omega}(\kappa^{\omega_1}, \omega_1)$. Unfortunately the principle \[(G^{\downarrow\downarrow}) \quad \text{WS}_{II}(G_{\omega}^{\downarrow\downarrow}(\kappa^{\omega_1})) \text{ for all } \kappa \geq \aleph_2\]

is not a consequence of the Rado Conjecture: In Section 4, we show that the principle $(G^{\downarrow\downarrow})$ implies the reflection principle $RP_{IS}$. It is known that $RP_{IS}$ (or even $RP$) is not a consequence of $RC$ (see Sakai [14]).

## 2 Reflection Principles

Let us first review the reflection principles we mentioned in the previous section.

We shall call here a partial ordering $T = \langle T, \leq_T \rangle$ a tree if the initial segment $\{u \in T : u \leq_T t\}$ in $T$ below each $t \in T$ is well-ordered. In particular, we assume here that a tree may have multiple roots.

A tree $T$ is special if there are $T_i \subseteq T$, $i \in \omega$ such that each of $T_i$'s is pairwise incomparable and $T = \bigcup_{i \in \omega} T_i$.

Rado's Conjecture (RC) is the assertion:

\[(RC) : \text{ Any tree } T \text{ is special if and only if all substructures of } T \text{ of cardinality } \aleph_1 \text{ are special.}\]

RC is known to be consistent (modulo a large large cardinal). E.g., Todoričević showed that, if $\kappa$ is strongly compact and $\mathbb{P} = Col(\omega_1, <\kappa)$, then we have

$$\models_{\mathbb{P}} \" \text{Rado's Conjecture}\".$$  

For a cardinal $\kappa$ and a regular cardinal $\delta < \kappa$, we denote

$$E^\kappa_\delta = \{\alpha < \kappa : cf(\alpha) = \delta\}.$$
a mapping \( g : E \rightarrow \kappa \) for \( E \subseteq E_\delta^\kappa \) is called a ladder system if \( \sup g(\alpha) = \alpha \) and \( \text{otp}(g(\alpha)) = \delta \) hold for all \( \alpha \in E \).

For a regular uncountable cardinal \( \kappa \), we define the Fodor-type Reflection Principle for \( \kappa \) by

\[
\text{FRP}(\kappa): \text{ For all stationary } E \subseteq E_\omega^\kappa \text{ and for all ladder system } g : E \rightarrow [\kappa]^{\aleph_0}, \text{ there exists } \alpha^* \in E_\omega \text{ such that } \\
\{ x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E, g(\sup(x)) \subseteq x \} \\
\text{is stationary in } [\alpha^*]^{\aleph_0}.
\]

The Fodor-type Reflection Principle (FRP) is the assertion:

\[
\text{(FRP): } \text{FRP}(\kappa) \text{ holds for all regular } \kappa > \aleph_1.
\]

FRP is known to be equivalent to many mathematical reflection principles over ZFC (see [3], [4], [5], [6], [7], see also [8]).

(2.1) Any locally countably compact topological space \( X \) is metrizable if and only if all subspaces of \( X \) of cardinality \( \leq \aleph_1 \) are metrizable

is one of such assertions equivalent to FRP over ZFC (see [4] and [5]).

FRP implies Shelah’s Strong Hypothesis and hence, in particular, Singular Cardinal Hypothesis (see [7]). It also implies the total failure of square principles \( \square \kappa \) for all cardinals \( \kappa \geq \aleph_1 \).

Suppose that \( M \prec \mathcal{H}(\lambda) \) for some regular \( \lambda \geq \aleph_2 \) and \( |M| = \aleph_1 \).

\( M \) is said to be internall cofinal (abbreviation: IU)\(^{1}) if \( [M]^{\aleph_0} \cap M \) is cofinal in \( [M]^{\aleph_0} \) with respect to \( \subseteq \). \( M \) is internall stationary (abbreviation: IS) if \( [M]^{\aleph_0} \cap M \) is stationary in \( [M]^{\aleph_0} \). \( M \) is internall club (abbreviation: IC) if \( [M]^{\aleph_0} \cap M \) contains a closed unbounded set in \( [M]^{\aleph_0} \). Finally, \( M \) is internall approachable (abbreviation: IA) if \( M \) is the union of a continuously increasing sequence \( \langle M_\alpha : \alpha < \omega_1 \rangle \) countable sets such that \( \langle M_\alpha : \alpha \leq \delta \rangle \in M_{\delta+1} \) for all \( \delta < \omega_1 \).

It is clear from the definition that, for any \( M \prec \mathcal{H}(\lambda) \), we have the implication: \( M \) is IA implies IC implies IS implies IU. It is easy to see that all of these notions can be characterized in terms of filtration (see footnote 2):

**Lemma 2.1** Suppose that \( M \prec \mathcal{H}(\lambda) \) for some regular \( \lambda \geq \omega_2 \) and \( |M| = \aleph_1 \).

\(^{1}\) Internally cofinal \( M \) is also called internally unbounded in the literature (see e.g. Krueger [11]).

\(^{2}\) For a structure \( M \) of cardinality \( \aleph_1 \), we shall call a continuously increasing sequence \( \langle M_\alpha : \alpha < \omega_1 \rangle \) of countable subsets of \( M \) with \( \bigcup_{\alpha<\omega_1} M_\alpha = M \) a filtration of \( M \). By thinning out the index set \( \omega_1 \), we may assume in some cases that the filtration \( \langle M_\alpha : \alpha < \omega_1 \rangle \) consists of elementary structures.
(1) $M$ is internally cofinal if and only if there is a filtration $(a_{\alpha} : \alpha < \omega_{1})$ of $M$ such that $a_{\alpha+1} \in M$ for every $\alpha < \omega_{1}$.

(2) $M$ is internally stationary if and only if $\{\alpha < \omega_{1} : M_{\alpha} \in M\}$ is stationary for a/any filtration $(M_{\alpha} : \alpha < \omega_{1})$ of $M$.

(3) $M$ is internally club if and only if there is a filtration $(M_{\alpha} : \alpha < \omega_{1})$ of $M$ such that $M_{\alpha} \in M_{\alpha+1}$ for all $\alpha < \omega_{1}$. \qed

These notions can be different: e.g. John Krueger proved under PFA, there are stationarily many internally club but not internally approachable $M < \mathcal{H}(\lambda)$ for all regular $\lambda > \aleph_{1}$ (for this and other results of this line see Krueger [11] and [12]). However this is not the case under CH:

**Lemma 2.2** Under CH, any $M < \mathcal{H}(\lambda)$ is IU if and only if it is IS if and only if it is IC if and only if it is IA. \qed

In the following, we shall always denote one of the properties IU, IS, IC or IA with $\mathcal{P}$. "$\subset$" in connection with a cardinal, say $\lambda$, denotes a(n arbitrary) well-ordering of the set $\mathcal{H}(\lambda)$ of all sets of hereditarily of cardinality $< \lambda$. If we have to emphasize that the well-ordering $\subset$ refers to $\mathcal{H}(\lambda)$, we write $\subset_{\mathcal{H}(\lambda)}$.

For a cardinal $\lambda > \aleph_{1}$ let

$\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_{0}})$: For any stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_{0}}$ there is a $\mathcal{P}$ elementary substructure $M$ of the structure $\langle \mathcal{H}(\lambda), \in, \subset \rangle$ (of cardinality $\aleph_{1}$) such that

\begin{equation}
(2.2) \quad S \cap [M]^{\aleph_{0}} \quad \text{is stationary in } [M]^{\aleph_{0}}.
\end{equation}

We define the global version of the reflection principle $\text{RP}_{\mathcal{P}}$ down to a structure with the property $\mathcal{P}$ to be $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_{0}})$ for all cardinal $\lambda > \aleph_{1}$.

$\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_{0}})$ is equivalent with seemingly stronger variants of the assertion:

**Lemma 2.3** the following are equivalent for a regular cardinal $\lambda > \aleph_{1}$:

(a) $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_{0}})$.

(b) For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_{0}}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \subset \rangle$ in an arbitrary countable language, there is a $\mathcal{P}$ elementary substructure $M$ of $\mathcal{M}$ (of cardinality $\aleph_{1}$) with (2.2).

(c) For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_{0}}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \subset \rangle$ in an arbitrary countable language, there are stationarily many $\mathcal{P}$ elementary substructures $M$ of $\mathcal{M}$ (of cardinality $\aleph_{1}$) with (2.2). \qed

Using Lemma 2.3 we can prove the following downward transfer property of $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_{0}})$:

---

3) That is, $S$ intersection with the set of all countable subsets of the underlying set of the structure $M$. 

---
Lemma 2.4 For regular cardinals $\aleph_1 < \lambda' < \lambda$, if $\text{RP}_P([\mathcal{H}(\lambda)]^{\aleph_0})$ holds then $\text{RP}_P([\mathcal{H}(\lambda')]^{\aleph_0})$ also holds. \hfill \Box

Lemma 2.5 The following are equivalent: (a) $\text{RP}_P$.

(b) For any uncountable $X$, stationary $S \subseteq [X]^{\aleph_0}$, regular $\theta$ with $X \subseteq \mathcal{H}(\theta)$ and any expansion $M$ of $\langle \mathcal{H}(\theta), \in, \subseteq, X \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure $M$ of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [X \cap M]^{\aleph_0}$ is stationary in $[X \cap M]^{\aleph_0}$.

(c) For any uncountable cardinal $\lambda$, stationary $S \subseteq [\lambda]^{\aleph_0}$, regular $\theta \geq \lambda$ and any expansion $\mathcal{M}$ of $\langle \mathcal{H}(\theta), \in, \subseteq, \lambda \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure $M$ of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [\lambda \cap M]^{\aleph_0}$ is stationary in $[\lambda \cap M]^{\aleph_0}$.

\hfill \Box

Fleissner’s Axiom R ([2]) is equivalent to $\text{RP}_{1\cup}$ in our notation. For a any set $X$ of cardinality $\aleph_1$, let

\begin{equation}
\text{(AR}([X]^{\aleph_0})): \text{ For any stationary } S \subseteq [X]^{\aleph_0} \text{ and } \omega_1\text{-club}^4 \text{ } T \subseteq [X]^{\aleph_1}, \text{ there is } U \in T \text{ such that } S \cap [U]^{\aleph_0} \text{ is stationary in } [U]^{\aleph_0}.
\end{equation}

Then we define Axiom R to be the assertion that $\text{AR}([\lambda]^{\aleph_0})$ holds for all cardinal $\alpha > \aleph_1$. Since $\text{AR}([\lambda]^{\aleph_0})$, for cardinals $\lambda > \aleph_1$ also satisfy the downward transfer similar to Lemma 2.4, the following Lemma implies the equivalence of $\text{RP}_{1\cup}$ and Axiom R:

Lemma 2.6 For any $\lambda > \aleph_1$, we have $\text{AR}([2^{<\lambda}]^{\aleph_0})$ if and only if $\text{RP}_{1\cup}([\mathcal{H}(\lambda)]^{\aleph_0})$. \hfill \Box

\textbf{Proof.} Note that $|\mathcal{H}(\lambda)| = 2^{<\lambda}$ and hence $\text{AR}([2^{<\lambda}]^{\aleph_0})$ is equivalent to $\text{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$.

First, assume $\text{RP}_{1\cup}([\mathcal{H}(\lambda)]^{\aleph_0})$. Suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary and $T \subseteq [\mathcal{H}(\lambda)]^{\aleph_1}$ is $\omega_1$-club.

Let $\mathcal{M} = \langle \mathcal{H}(\lambda), \in, \subseteq, \lambda, T \rangle$. By Lemma 2.5, there is $M < \mathcal{M}$ such that

(2.3) \quad |M| = \aleph_1;

(2.4) \quad M \models 1\cup \text{ and }

(2.5) \quad S \cap [M]^{\aleph_0} \text{ is stationary in } [M]^{\aleph_0}.

By (2.3), (2.4) and $M < \mathcal{M}$, it is easy to see that $M$ is the union of an $\omega_1$ chain of elements of $T$. By $\omega_1$-clubness of $T$ it follows that $M \in T$. This shows that $\text{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds.

Assume now $\text{AR}([2^{<\lambda}]^{\aleph_0})$ and suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary. Let

$$
T = \{M \in [\mathcal{H}(\lambda)]^{\aleph_1} : M < \mathcal{H}(\lambda), M \models 1\cup \}.
$$

\[4\] $T \subseteq [X]^{\aleph_1}$ for an uncountable set $X$ is said to be $\omega_1$-club (or “tight and unbounded” in Fleissner’s terminology in [2]) if $T$ is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and for any increasing chain $\langle U_\alpha : \alpha < \omega_1 \rangle$ in $T$ of length $\omega_1$, we have $\bigcup_{\alpha<\omega_1} U_\alpha \in T$. 

Then $T$ is $\omega_1$-club. By $\text{AR}([2^{<\lambda}]^{\aleph_0})$ or by its equivalent $\text{AR}(\mathcal{H}(\lambda)]^{\aleph_0})$, there is $M \in T$ such that $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$. This shows that $\text{RP}_{\mathcal{U}}(\mathcal{H}(\lambda)]^{\aleph_0})$ holds.

\[\square\ (\text{Lemma 2.6})\]

## 3 Definition of the games

For a cardinal $\kappa$, let

\[(3.1) \quad \kappa^{I}_\kappa = \{f \in \kappa^\kappa : f \text{ is regressive}\}.
\]

The game $G^1_\omega(\kappa)$ for Players $I$ and $II$ is defined as follows: A match in $G^1_\omega(\kappa)$ is a sequence of the form:

\[
\begin{array}{c|cccc}
I & f_0 \in \kappa^{I}_\kappa & f_1 \in \kappa^{I}_\kappa & \cdots & f_n \in \kappa^{I}_\kappa & \cdots \\
II & \delta_0 \in \kappa & \delta_1 \in \kappa & \cdots & \delta_n \in \kappa & \\
\end{array} \quad (n < \omega)
\]

Player $II$ wins a match of $G^1_\omega(\kappa)$ as above if

\[(3.2) \quad \{ \alpha \in E^{\kappa}_\omega : f_n(\alpha) < \sup\{\delta_i : i \in \omega\} \text{ for all } n \in \omega \} \text{ is unbounded.}
\]

The game $G^1_\omega(\kappa)$ was introduced in [9]. It is used there to prove the implication of FRP from RC by showing that the assertion $(G_0)$ as in Section 1 defined in terms of this game interpolates the implication.

The following game $G_\omega([\kappa]^{\aleph_1}, \omega_1)$ for Players $I$ and $II$ for a cardinal $\kappa$ was introduced by Doebler in [1]: A match in $G_\omega([\kappa]^{\aleph_1}, \omega_1)$ is a sequence of the form:

\[
\begin{array}{c|cccc}
I & f_0 \in [\kappa]^{\aleph_1}_\omega & f_1 \in [\kappa]^{\aleph_1}_\omega & \cdots & f_n \in [\kappa]^{\aleph_1}_\omega & \cdots \\
II & \delta_0 \in \omega_1 & \delta_1 \in \omega_1 & \cdots & \delta_n \in \omega_1 & \\
\end{array} \quad (n < \omega)
\]

$II$ wins in a match of $G_\omega([\kappa]^{\aleph_1}, \omega_1)$ as above if

\[
\{a \in [\kappa]^{\aleph_1} : f_n(a) < \sup\{\delta_i : i \in \omega\} \text{ for all } n \in \omega\}
\]

is cofinal in $[\kappa]^{\aleph_1}$.

Doebler proved that the principle $(G_1)$ as defined in Section 1 in terms of this game follows also from RC and it implies SSR.

It is easy to see that both of $(G_0)$ and $(G_1)$ are consequences of RP_{IC} (this also follows from Corollary 4.4). Hence we have the diagram on the right:

Since FRP and SSR imply almost all known consequences of RC$^5$, it seems to be an interesting question what is the natural principle which is still a consequence of both RC and RP_{IC} while which implies both FRP and SSR.

---

$^5$ Perhaps with the exception of the negation of Martin's Axiom for $\aleph_1$ dense sets which is a consequence of RC while RC_p's are consistent with Martin's Axiom since they all follow from MA^+(\sigma-closed).
The assertion of the existence of the winning strategy for player II (the principle \(G^\downarrow\) introduced in Section 1) in the following game \(G^\downarrow\) \(([\kappa]^\aleph_1)\) for all \(\kappa > \aleph_1\) seemed to be a natural candidate for such an interpolant. Unfortunately, this principle turned out to be too strong to be a consequence of RC while it is still a consequence of \(\text{RP}_{\text{IC}}\) as we shall see in Section 4. In [9] we introduce a weakening of \(G^\downarrow\) which is an interpolant of RC and \(\text{RP}_{\text{IC}}\) on one side and FRP and SSR on the other.

Here is the definition of \(G^\downarrow\) \(([\kappa]^\aleph_1)\) for a cardinal \(\kappa > \aleph_1\). We call a function \(f : [\kappa]^\aleph_1 \rightarrow \kappa\) regressive if \(f(a) \in a\) holds for all \(a \in [\kappa]^\aleph_1\). Similarly to the definition (3.1), let

\[
[\kappa]^\aleph_1 \cap \kappa = \{f \in [\kappa]^\aleph_1 : f \text{ is regressive}\}.
\]

A match in \(G^\downarrow\) \(([\kappa]^\aleph_1)\) for Players I and II is a sequence of the form:

\[
\begin{array}{c|cccc}
I & f_0 \in [\kappa]^\aleph_1 \cap \kappa & f_1 \in [\kappa]^\aleph_1 \cap \kappa & \ldots & f_n \in [\kappa]^\aleph_1 \cap \kappa \\
II & d_0 \in [\kappa]^\aleph_0 & d_1 \in [\kappa]^\aleph_0 & \ldots & d_n \in [\kappa]^\aleph_0
\end{array}
\]

II wins in a match in \(G^\downarrow\) \(([\kappa]^\aleph_1)\) as above if

\[
\{a \in [\kappa]^\aleph_1 : f_n(a) \in \bigcup\{d_i : i \in \omega\} \text{ for all } n \in \omega\}
\]

is cofinal in \([\kappa]^\aleph_1\).

Note that by the definition of the games, it is clear that \(G^\downarrow\) implies both of \((G_0)\) and \((G_1)\).

### 4 Characterizations of \((G^\downarrow)\)

The following characterization of \((G^\downarrow)\) can be obtained easily by regarding the moves of Player I in \(G^\downarrow\) \(([\kappa]^\aleph_1)\) as an enumeration of Skolem functions with parameters in some model \(M\) and the moves of Player II as the gradual capturing of \(\kappa \cap M\):

**Lemma 4.1** For any cardinal \(\kappa > \aleph_1\) the following are equivalent:

(a) \(\text{WS}_I(G^\downarrow([\kappa]^\aleph_1))\).

(b) For sufficiently large regular \(\theta\) with \(\mathcal{M} = \langle \mathcal{H}(\theta), \in, \subseteq \rangle\), for any \(M \prec \mathcal{M}\) with \(|M| = \aleph_0\) and \(\kappa \in M\), we have: for any \(a \in [\kappa]^\aleph_1\), there are \(b \in [\kappa]^\aleph_1\) and countable \(N < \mathcal{M}\) such that \(a \subseteq b\), \(b \in N\), \(M \subseteq N\) and \(b \cap N = b \cap M\).

(c) For sufficiently large regular \(\theta\) with \(\mathcal{M} = \langle \mathcal{H}(\theta), \in, \subseteq \rangle\), for club many\(^6\) countable \(M < \mathcal{M}\) with \(\kappa \in M\), we have: for any \(a \in [\kappa]^\aleph_1\), there are \(b \in [\kappa]^\aleph_1\) and countable \(N < \mathcal{M}\) such that \(a \subseteq b\), \(b \in N\), \(M \subseteq N\) and \(b \cap N = b \cap M\).

\(^6\)We can also express this "club many ..." in terms of expansion of the structure \(\mathcal{M}\) similarly to Lemma 2.3 or Lemma 2.4.
By Lemma 4.1, (c), we see immediately that $WS_H(G_{\omega}^{\downarrow\downarrow}(\kappa^{\aleph_1}))$ for cardinals $\kappa > \aleph_1$ also enjoy the downward transfer property:

**Corollary 4.2** Suppose $\aleph_1 < \kappa' < \kappa$ and $WS_H(G_{\omega}^{\downarrow\downarrow}(\kappa^{\aleph_1}))$ holds. Then we also have $WS_H(G_{\omega}^{\downarrow\downarrow}(\kappa^{\aleph_1}))$.

**Theorem 4.3** The following are equivalent: (a) ($G^{11}$).

(b) For all $\kappa > \aleph_1$, for all sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \square \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $Y \in [\mathcal{H}(\kappa)]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $X \subseteq Y$, $Y \in N$, $M \subseteq N$ and $Y \cap N = Y \cap M$.

(c) For all $\kappa > \aleph_1$, for all sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \square \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $Z \prec \langle \mathcal{H}(\kappa), \in, \subseteq, \mathcal{H}(\kappa) \rangle$ of cardinality $\aleph_1$ and countable $N \prec \mathcal{M}$ such that $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and $Z \cap N = Z \cap M$.

(d) For any $\kappa > \aleph_1$ and stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$, for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ there is a $Z \prec \mathcal{H}(\kappa)$ such that $X \subseteq Z | Z | = \aleph_1$ and $S \cap Z$ is stationary in $[Z]^{\aleph_0}$.

(e) For all $\kappa > \aleph_1$, for all sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \square \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $IS Z \prec \langle \mathcal{H}(\kappa), \in, \subseteq, \mathcal{H}(\kappa) \rangle$ of cardinality $\aleph_1$ and countable $N \prec \mathcal{H}(\theta)$ such that $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and $Z \cap N = Z \cap M$.

**Proof.** (a) $\Rightarrow$ (b): Let $\lambda = 2^{\kappa} = |\mathcal{H}(\kappa)|$ and let $\varphi : \lambda \rightarrow \mathcal{H}(\kappa)$ be a bijection. Then all countable $M \prec \mathcal{M}$ with $\varphi \in M$ satisfies the condition in (b): the situation of Lemma 4.1, (b) (for $\kappa$ there $= \lambda$) is translated to the desired condition in the present (b) by $\varphi$.

(b) $\Rightarrow$ (a): The back-translation by the mapping $\varphi$ as in the proof of (a) $\Rightarrow$ (b) implies $WS_H(G_{\omega}^{\downarrow\downarrow}([2^{\kappa}]^{\aleph_1}))$ for all $\kappa > \aleph_1$. By Corollary 4.2, it follows that $WS_H(G_{\omega}^{\downarrow\downarrow}((\kappa^{\aleph_1})))$ for all $\kappa > \aleph_1$.

(b) $\Rightarrow$ (c): Suppose that $\kappa$, $\theta$, $\mathcal{M}$, $M$, $X$, $Y$, $N$ are as in (b). Then $Z = sk_{\mathcal{M}}(Y)$ witnesses (c).

(c) $\Rightarrow$ (d): Assume that (c) holds and suppose that $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ is stationary. Let $\theta$, $\mathcal{M}$, $M$ be as in (c). Since there are club many $M$'s as in (c), we may assume that

\begin{equation}
(4.1) \quad S \in M \text{ and } \mathcal{H}(\kappa) \cap M \in S.
\end{equation}

Let $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ be defined by

\begin{equation}
(4.2) \quad X = \omega_1 \cup (\mathcal{H}(\kappa) \cap M) \cup \{M \cap \mathcal{H}(\kappa)\}.
\end{equation}

Let $Z \prec \mathcal{H}(\kappa)$ and $N \prec \mathcal{H}(\theta)$ be as in (c) for this $X$. Thus we have $N$ is countable, $Z$ is of cardinality $\aleph_1$, $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and
(4.3) \[ Z \cap N = Z \cap M. \]

We are done by showing that \( S \cap Z \) is stationary in \( [Z]^{\aleph_0} \). Since \( S, Z \in N \), it is enough to show that any club \( C \subseteq [Z]^{\aleph_0} \) with \( C \in N \) intersects with \( S \): Note that we have

(4.4) \[ Z \cap M = \mathcal{H}(\kappa) \cap M \]

by (4.2). For such a club \( C \) we have

\[ S \ni M \cap \mathcal{H}(\kappa) \]

by (4.1)
\[ = Z \cap M \]

by (4.4)
\[ = Z \cap N \]

by (4.3)
\[ = \bigcup (C \cap N) \in C \]. \text{ by elementarity, } C \in N \text{ and closedness of } C

Thus \( S \cap C \neq \emptyset \) as desired.

(c) \( \Rightarrow \) (e): The proof of (c) \( \Rightarrow \) (d) above for \( S = [\mathcal{H}(\kappa)]^{\aleph_0} \) shows this.

(e) \( \Rightarrow \) (c): trivial.

(c) \( \Rightarrow \) (b): trivial.

(d) \( \Rightarrow \) (e): Assume that (d) holds. For \( \kappa > \aleph_1 \), let \( \theta \) a sufficiently large regular cardinal and \( \mathcal{M} = \langle \mathcal{H}(\theta), \in, \subset \rangle \). Let

(4.5) \[ S = \{ M \in [\mathcal{M}]^{\aleph_0} : M \prec \mathcal{M}, \kappa \in M, \text{ there is } X_M \in [\mathcal{H}(\kappa)]^{\aleph_1} \text{ such that } \]

(4.6) there are no countable \( N \prec \mathcal{M} \)

and \( Y \prec \langle \mathcal{H}(\kappa), \in, \subset \rangle \) such that

\[ M \prec N, X_M \subseteq Y, Y \text{ is IS and of size } \aleph_1, \]

\[ Y \in N \text{ and } M \cap Y = N \cap Y \]

It is enough to show that \( S \) is non-stationary. In the following we show this indirectly: We assume that \( S \) is stationary and drive a contradiction from this assumption.

For each \( M \in S \) we choose \( X_M \in [\mathcal{H}]^{\aleph_1} \) such that

(4.7) \[ X_M \supseteq M \cup \omega_1, X_M \prec \mathcal{M} \text{ and (4.6) holds for } M \text{ and } X_M. \]

Let \( \chi > 2^{<\theta} \) be regular. Note that we have \( \mathcal{H}(\theta) \in \mathcal{H}(\chi) \). Let

(4.8) \[ \tilde{S} = \{ M \in [\mathcal{H}(\chi)]^{\aleph_0} : M \prec \langle \mathcal{H}(\chi), \in, \subset \rangle, \]

\[ \kappa, \theta, \cdots \in M, M \cap \mathcal{H}(\theta) \in S \} \]

By the assumption of the stationarity of \( S \), \( \tilde{S} \) is also stationary. Thus, by (d), there is \( Z \prec \langle \mathcal{H}(\chi), \in, \subset \rangle \) such that

(4.9) \[ | Z | = \aleph_1, \omega_1 \subseteq Z, \]

(4.10) \[ \kappa, \theta, S, \langle X_M : M \in S \rangle, \subset \mathcal{H}(\kappa), \subset \mathcal{H}(\theta), \cdots \in Z \text{ and } \]
(4.11) $\tilde{S} \cap Z$ is stationary in $[Z]^{\aleph_0}$.

Let

(4.12) $Y = Z \cap \mathcal{H}(\kappa)$.

Then we have $\omega_1 \subseteq Y$ and hence $|Y| = \aleph_1$ and $Y < \langle \mathcal{H}(\kappa), \in, \subseteq \rangle$.

$Y$ is stationary in $[Y]^{\aleph_0}$: Suppose that $C \subseteq [Y]^{\aleph_0}$ is a club. Let $\tilde{C} = \{x \in [Z]^{\aleph_0} : x \cap Y \in C\}$. Then $\tilde{C}$ is a club subset of $[Z]^{\aleph_0}$. By (4.11), $Z \cap [Z]^{\aleph_0}$ is stationary in $[Z]^{\aleph_0}$. Hence there is an $x \in \tilde{C} \cap Z$. By definition of $\tilde{C}$, we have $x \cap \mathcal{H}(\kappa) \in C$.

$Y$ is stationary in $[Y]^{\aleph_0}$: Suppose that $C \subseteq [Y]^{\aleph_0}$ is a club. Let

$$ \v{C} = \{x \in [Z]^{\aleph_0} : x \cap \mathcal{H}(\kappa) \in C\}. $$

Then $\v{C}$ is a club subset of $[Z]^{\aleph_0}$. By (4.11), $Z \cap [Z]^{\aleph_0}$ is stationary in $[Z]^{\aleph_0}$. Hence there is an $x \in \tilde{C} \cap Z$. By definition of $\tilde{C}$, we have $x \cap \mathcal{H}(\kappa) \in C$.

By definition of $\v{C}$, we have $x \cap \mathcal{H}(\kappa) \in \mathcal{H}(\kappa)$. It follows that $x \cap \mathcal{H}(\kappa) \in Y$. Thus $x \cap \mathcal{H}(\kappa) \in C \cap Y$.

For each $M \in \tilde{S} \cap Z$ we have $M \cap \mathcal{H}(\theta) \in S \cap Z$ as we just saw and, by (4.10), $X_{M \cap \mathcal{H}(\theta)} \in Z \cap \mathcal{H}(\kappa) = Y$. Since $\omega_1 \subseteq Y$, it follows that

(4.13) $X_{M \cap \mathcal{H}(\theta)} \subseteq Y$.

By (4.11), there is countable $N^* < \langle \mathcal{H}(\chi), \in, \subseteq \rangle$ such that

(4.14) $N^* \cap Z \subseteq \tilde{S} \cap Z$ and

(4.15) $X, Y, Z, \cdots \in N^*$.

Let $M^* = (N^* \cap Z) \cap \mathcal{H}(\theta)$. Then we have $M^* \in S$ by (4.14). $M^* \subseteq N^* \cap \mathcal{H}(\theta)$ by the definition of $M^*$ and $X_{M^*} \subseteq Y$ by (4.13). $Y \in N^* \cap \mathcal{H}(\theta)$ by (4.12) and (4.15).

We also have

(4.16) $M^* \cap Y = ((N^* \cap Z) \cap \mathcal{H}(\theta)) \cap (Z \cap \mathcal{H}(\kappa)) = M^* \cap \mathcal{H}(\kappa) = (N^* \cap \mathcal{H}(\theta)) \cap Y$.

Thus $N^* \cap \mathcal{H}(\theta)$ and $Y$ contradict the choice of $X_{M^*}$. □ (Theorem 4.3)

**Corollary 4.4** The following implications hold:

$$ \text{RP}_{IC} \Rightarrow (G^{\uparrow\uparrow}) \Rightarrow \text{RP}_{IS}. $$

**Proof.** By Theorem 4.3, (d). The implication $\text{RP}_{IC} \Rightarrow (G^{\uparrow\uparrow})$ follows from the following trivial observation. □ (Corollary 4.4)

**Lemma 4.5** If $M < \mathcal{H}(\theta)$ is IC and $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, then $S \cap (M \cap [M]^{\aleph_0})$ is stationary in $[M]^{\aleph_0}$ as well. □

**Corollary 4.6** Under the CH, we have:

Axiom R $\iff$ RP$_{IU} \iff$ RP$_{IS} \iff (G^{\uparrow\uparrow}) \iff$ RP$_{IC} \iff$ RP$_{IA}$.

**Proof.** By Lemma 2.2, Lemma 2.6 and Corollary 4.4. □ (Corollary 4.6)
References


