**P\_\text{\textalpha}**-FILTERS AND REGULAR EMBDDINGS OF BOOLEAN ALGEBRAS

ALEKSANDER BŁASZCZYK

Institute of Mathematics, University of Silesia

**ABSTRACT.** This is a résumé of previously published results on an analogue of Mathias forcing associated to the tree $\omega^{<\omega}$ endowed with a topology appointed by a set of filters.

1. **INTRODUCTION**

We shall consider the set

$$\text{Seq} = \omega^{<\omega} = \bigcup \{^n\omega : n < \omega\}$$

of all finite sequences of natural numbers. The set Seq is a tree with a natural order defined as follows: for $s,t \in \text{Seq}$ we set

$$s \leq t \iff t \upharpoonright \text{dom}(s) = s.$$  

The set of all immediate successors of an element $s \in \text{Seq}$ is denoted by

$$\text{succ}(s) = \{t \in \text{Seq} : t \text{ is minimal in } \{t \in \text{Seq} : t > s\}\}.$$  

Hence $\text{succ}(s) = \{s^\frown n : n \in \omega\}$, where $s^\frown k$ denotes the concatenation of sequence $s \in ^n\omega$ by a number $k$, i.e. $s^\frown k = s \cup \{(n,k)\}$ is the sequence that extends $s$ and whose last term is $k$.

**Definition 1.** Assume that $\mathcal{F} = (\mathcal{F}_t : t \in \text{Seq})$, where $\mathcal{F}_t \subseteq \mathcal{P}(\text{succ}(t))$ is a collection of free filters. A set $U \subseteq \text{Seq}$ is open in the $\mathcal{F}$-topology on $\text{Seq}$ whenever

$$(\forall s \in U)(\exists F \in \mathcal{F}_s)(F \subseteq U).$$

Then $\text{Seq}(\mathcal{F})$ denotes the set $\text{Seq}$ endowed with the $\mathcal{F}$-topology.

The idea of $\mathcal{F}$-topology on $\text{Seq}$ has been given by Szymański [10] and, independently, by Trnkova [11]. Later on it was developed by several other authors. A review of $\mathcal{F}$-topologies and their generalizations can be found in [2]. In this paper, in particular, one can find a proof of the following theorem:

**Theorem 1.** For every $\mathcal{F} = (\mathcal{F}_t : t \in \text{Seq})$ the space $\text{Seq}(\mathcal{F})$ is a zero-dimensional, nowhere compact Hausdorff space. Moreover, $\text{Seq}(\mathcal{F})$ is extremally disconnected iff all the filters in $\mathcal{F}$ are ultrafilters.

Here, nowhere compact means that every compact subset of the space is nowhere dense. In the sequel we shall discuss the Boolean algebra $\text{Clop}(\text{Seq}(\mathcal{F}))$ of all the clopen (=closed and open) subsets of the space $\text{Seq}(\mathcal{F})$. Clearly, $\text{Clop}(\text{Seq}(\mathcal{F}))$ is in fact a field of subsets of $\text{Seq}$. Theorem 1 immediately implies the following:

**Corollary 1.** The Boolean algebra $\text{Clop}(\text{Seq}(\mathcal{F}))$ is complete iff all the filters in $\mathcal{F}$ are ultrafilters.

The space $\text{Seq}(\mathcal{F})$ was used in a construction of a complete rigid Boolean algebra. Namely, it was proved by Dow, Gubi and Szymański [7] that if $\mathcal{F}$ consists of one weak $P$-ultrafilter, then the Boolean algebra $\text{Clop}(\text{Seq}(\mathcal{F}))$ is complete and rigid.

Let us recollect some well-known cardinal numbers connected to the set of all the functions from $\omega$ to $\omega$ ordered by the relation $\leq^*$ defined as follows:

$$f \leq^* g \iff (\exists n < \omega)(\forall k > n)(f(k) \leq g(k)).$$
This relation leads to the following cardinal characteristics:
1. The dominating number $\mathfrak{d}$ is defined as follows:
   \[ \mathfrak{d} = \min \{ |D| : (\forall f \in {}^\omega \omega)(\exists g \in D)(f \leq^* g) \}. \]
2. The boundedness $\mathfrak{b}$ denotes the minimal cardinality of an unbounded subset of $\omega$, i.e.
   \[ \mathfrak{b} = \min \{ |U| : (\forall f \in {}^\omega \omega)(\exists g \in U)(\{ n \in \omega : f(n) < g(n) \} \geq \omega) \}. \]
3. The cardinal number $\mathfrak{p}$ is defined as
   \[ \mathfrak{p} = \min \{ |\mathcal{U}| : \mathcal{U} \subseteq \text{Clop}(\mathcal{P}(\omega^*)) \text{ is centered and } \text{Int} \bigcap \mathcal{U} = \emptyset \}. \]
   It is well-known that: \[ \omega < \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^\omega. \]

The character of a (free) filter denotes the character of the corresponding subset of $\omega^*$, i.e. for every (free) filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ there is
\[ \chi(\mathcal{F}) = \chi(A_{\mathcal{F}}, \omega^*), \]
where $A_{\mathcal{F}} = \bigcap \{ \text{cl}_{\mathfrak{R}} U : U \in \mathcal{F} \}$.

Using the cardinal $\mathfrak{d}$ one can calculate the character of a space $\text{Seq}(\mathfrak{F})$ at every point of $\text{Seq}$.

**Proposition 1.** For every $\mathfrak{F} = (\mathcal{F}_t : t \in \text{Seq})$ and every $s \in \text{Seq}$ we have
\[ \chi(s, \text{Seq}(\mathfrak{F})) = \mathfrak{d} + \chi(\mathcal{F}_s). \]

2. $P_\lambda$-FILTERS AND $P_\lambda$-SETS

Since for every $s \in \text{Seq}$ the set $\text{succ}(s)$ of all the successors of $s$ is countable, in the definition of $\mathfrak{F}$-topology on $\text{Seq}$ instead of filters on $\mathcal{P}(\text{succ}(s))$ one can consider filters on $\mathcal{P}(\omega)$. Let us recall that all filters considered here are assumed to be free filters.

**Definition 2.** A filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a $P_\lambda$-filter whenever for every family $\mathcal{R} \subseteq \mathcal{F}$ of size less than $\lambda$ there exists $F \in \mathcal{F}$ such that $F \subseteq^* U$ for every $U \in \mathcal{F}$. A $P_{\omega_1}$-filter is simply called a $P$-filter.

Here, as usual, $F \subseteq^* U$ means that the set $F \setminus U$ is finite. In the virtue of the well-known result of Shelah, $P$-ultrafilters do not exist in ZFC. However, it is quite easy to construct a $P$-filter. In fact, if $\{ U_n : n \in \omega \}$ consists of infinite subsets of $\omega$ and $U_{n+1} \subseteq^* U_n$ for every $n \in \omega$, then there exists an infinite set $V \subseteq \omega$ such that $V \subseteq^* U_n$ for every $n < \omega$. Hence, by transfinite induction one can construct a sequence $\{ U_\alpha : \alpha < \omega_1 \} \subseteq \mathcal{P}(\omega)$ of infinite sets such that $U_\alpha \subseteq^* U_\beta$ for all $\beta < \alpha < \omega_1$. Clearly, the family $\{ U_\alpha : \alpha < \omega_1 \}$ generates a $P$-filter.

For topological spaces we have an analogous definition.

**Definition 3.** For $\lambda \geq \omega$ and a topological space $X$, a set $S \subseteq X$ is called a $P_\lambda$-set provided that $S$ is contained in the interior of the intersection of every family of size less than $\lambda$ consisting of open neighborhoods of $S$. Also, a $P$-set is just a $P_{\omega_1}$-set.

Let us note that if a Tychonoff space $X$ is nowhere compact then $X$ is simultaneously dense and boundary in the Čech–Stone compactification $\beta X$. Hence, by Theorem 1, $\text{Seq}(\mathfrak{F})$ is dense and boundary in $\beta \text{Seq}(\mathfrak{F})$. The next theorem has been recently proved in [3].

**Theorem 2** (Blaszczyk and Brzeska [3]). If $\mathfrak{F} = (\mathcal{F}_s : s \in \text{Seq})$ is a collection of $P_\lambda$-filters and $\omega < \lambda \leq \mathfrak{b}$ then the space $\text{Seq}(\mathfrak{F})$ is a $P_\lambda$-set in $\beta \text{Seq}(\mathfrak{F})$.

From Theorem 2 we immediately obtain the following:

**Corollary 2** (Simon [9]). If every filter in $\mathfrak{F} = (\mathcal{F}_s : s \in \text{Seq})$ is a $P$-filter, then $\text{Seq}(\mathfrak{F})$ is a $P$-set in $\beta \text{Seq}(\mathfrak{F})$. 

Corollary 3 (Juhász and Szymański [8]). If \( \omega < \lambda \leq b \) and \( \mathcal{F}_s = \mathcal{F} \) for every \( s \in \text{Seq} \), where \( \mathcal{F} \) is a \( \mathcal{P}_\lambda \)-ultrafilter, then \( \text{Seq}(\mathfrak{F}) \) is a \( \mathcal{P}_\lambda \)-set in \( \beta \text{Seq}(\mathfrak{F}) \).

The theorem of Simon answers a question of Arhangel’skiǐ whereas Juhász–Szymański’s theorem leads to some constructions in the theory of calibers and tightness in compact spaces.

3. Regular embeddings

In this section we shall outline some further applications of Theorem 2 in the theory of Boolean algebras. First, we shall recall some definitions. Let \( \mathbb{B} \) be a Boolean algebra. A subalgebra \( \mathbb{A} \subseteq \mathbb{B} \) is regular if for every set \( X \subseteq A \) there is

\[
\sup_{A} X = 1 \implies \sup_{B} X = 1.
\]

A Boolean algebra \( \mathbb{A} \) is regularly embedded in a Boolean algebra \( \mathbb{B} \) provided that there exists a monomorphism of \( \mathbb{A} \) into \( \mathbb{B} \) such that the image of \( \mathbb{A} \) is a regular subalgebra of \( \mathbb{B} \).

The next theorem gives us an unexpected interrelation between nowhere dense ultrafilters and Boolean algebras. Let us recall that a Boolean algebra \( \mathbb{B} \) is \( \sigma \)-centered if it is the union of countably many ultrafilters. An element \( b \in \mathbb{B} \setminus \{0\} \) is an atom if there is no \( a \in \mathbb{B} \) such that \( 0 < a < b \). A Boolean algebra is atomless if it has no atoms and it is atomic whenever below every element of \( \mathbb{B} \setminus \{0\} \) there is an atom.

Theorem 4 (Blaszczyk and Shelah [4]). There exists a \( \sigma \)-centered, atomless, complete Boolean algebra which does not contain any atomless, countable, regular subalgebra iff there exists a nowhere dense ultrafilter.

Easier part of Theorem 3 can be derived from the next theorem. A proof of this theorem can be obtained by some modification of Theorem 17 from [5].

Theorem 5. Assume that \( \mathfrak{F} = (\mathcal{F}_s : s \in \text{Seq}) \) is a collection of filters. If \( \mathcal{F}_s = \mathcal{F} \) for every \( s \in \text{Seq} \), then the countable free algebra \( \mathfrak{F}(\omega) \) can be embedded in \( \text{Clop}(\text{Seq}(\mathfrak{F})) \) as a regular subalgebra iff the filter \( \mathcal{F} \) is not nowhere dense.

The next theorem says that from the point of view of regular embeddings atomlessness is a very essential requirement. For a Boolean algebra \( \mathbb{B} \), \( \pi(\mathbb{B}) \) denotes the minimal size of a dense subset of \( \mathbb{B} \).

Theorem 6. Assume that \( \mathfrak{F} = (\mathcal{F}_s : s \in \text{Seq}) \) is a collection of \( \mathcal{P}_\lambda \)-filters where \( \omega < \lambda \leq b \). If \( \mathbb{B} \subseteq \text{Clop}(\text{Seq}(\mathfrak{F})) \) is a regular subalgebra and \( \pi(\mathbb{B}) < \lambda \), then \( \mathbb{B} \) is an atomic algebra.

A topological version of this theorem has been proved in [3]; see Theorem 3.6.

4. Skeletal mappings

A continuous mapping \( f : X \to Y \) is called skeletal whenever for every open and dense set \( G \subseteq Y \) the set \( f^{-1}[G] \) is dense in \( X \). Skeletal maps are also known as semi-open mappings. The mapping \( f \) is semi-open whenever for every non-empty open set \( U \subseteq X \), the image \( f[U] \) has a non-empty interior. It appears that skeletal mappings of topological spaces correspond to regular embeddings of Boolean algebras.

Proposition 2. An embedding of Boolean algebras is regular iff the continuous surjection of the corresponding mapping of Stone spaces is skeletal.

The following example shows that skeletal mappings are easy to construct.
Example 1. If $\mathcal{U}$ is an infinite, maximal disjoint family of clopen subsets of a zero-dimensional compact space $X$, then the quotient mapping determined by the closed partition $\{\{U\}: U \in \mathcal{U}\} \cup \{X \setminus \bigcup \mathcal{U}\}$ is a skeletal mapping onto the one-point compactification of the discrete space of cardinality $|\mathcal{U}|$.

Clearly, if a continuous surjection is skeletal and the set of values is dense in itself, then the domain has to be dense in itself as well. However, the above example shows that a skeletal surjection can map a dense in itself compact space onto a space with a dense set of isolated points. The next theorem, proved in [3] says a bit more about it.

Theorem 6. Assume that $\mathcal{F} = (\mathcal{F}_s: s \in \text{Seq})$ is a collection of $P_\lambda$-filters where $\omega < \lambda \leq b$. If $X$ is a Hausdorff space with $\pi w(X) < \lambda$ and a continuous surjection $f: \beta \text{Seq}(\mathcal{F}) \to X$ is skeletal then the set of isolated points in $X$ is dense.

In connection to skeletal mappings Burke [6] has introduced the following notion:

Definition 5. A continuous mapping $f: X \to Y$ is called nowhere constant if $f^{-1}(y)$ is nowhere dense for every $y \in Y$.

Clearly, if $Y$ is a $T_1$-space, a mapping $f: X \to Y$ is skeletal and the set $f[X]$ is dense in itself, then $f$ is nowhere constant. Example 1 shows that in general the converse is not true: skeletal mappings do not have to be nowhere constant. However, the following theorem of Burke shows an interesting connection between nowhere constant and skeletal mappings.

Theorem 7 (Burke [6]). If $X$ is Tychonoff and there is a nowhere constant continuous function from $X$ into $\mathbb{R}$, and $\pi w(X) < p$, then there also exists a skeletal function from $X$ into $\mathbb{R}$.

In particular, the above theorem shows that if a compact metric space has a nowhere constant mapping into the reals, then it has also a skeletal mapping into reals. Also, Burke [2] asked whether there exists in ZFC a Tychonoff space of $\pi$-weight $p$ which has a nowhere constant mapping onto $\mathbb{R}$ but does not have a skeletal mapping onto $\mathbb{R}$. We give a partial answer to this question.

Theorem 8. If $\mathcal{F} = (\mathcal{F}_s: s \in \text{Seq})$ is a collection of $P$-filters of character $\aleph_1$, then the space $\text{Seq}(\mathcal{F})$ is a space of the $\pi$-weight equal to $\varnothing$ which has a nowhere constant mapping into $\mathbb{R}$ but does not have a skeletal mapping into $\mathbb{R}$.

The above theorem also shows that nowhere constant mapping does not have to be skeletal.

References


Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland
E-mail address: ablaszcz@ux2.math.us.edu.pl