数理解析研究所講究録
第 1896 巻 2014 年 53-63



# Smallness of the volume growth and the singularity of a space for the conservation property of symmetric Markov processes

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## 1. INTRODUCTION

A Markov process $\{X_t\}_{t>0}$ living on the state space $X$ is called *conservative*[1] if

$$P_t 1(x) \equiv 1, \quad \text{for all } t>0 \text{ and any } x \in X,$$

where $\{P_t\}_{t>0}$ is the transition function of the process. Namely, the conservation property means that the process stays in the space forever and the total amount of the Brownian particles will be preserved. For example, Brownian motion with no distortion on any Euclidean space $\mathbb{R}^n$ is conservative since the heat kernel $k$, which serves as the transition function of the Brownian motion, satisfies

$$P_t 1(x) = \int_{\mathbb{R}^n} k(t,x,y)dy \equiv 1, \quad \text{for all } t>0 \text{ and any } x \in \mathbb{R}^n.$$

The Brownian motion in a domain $\Omega \subset \mathbb{R}^n$ is not conservative (conservative, respectively) if we impose absorbing (reflecting, respectively) boundary condition on $\partial\Omega$. The same is true for the Brownian motion $X_t$ in the Euclidean space punctured a closed set $\Gamma$ large enough so that $X_t$ will hit $\Gamma$, namely, $\Gamma$ is not *polar*. A striking fact is that the Brownian motion of a complete manifold may fail to be conservative if the curvature rapidly goes to negative infinity [2] or the volume of the concentric ball $B(x_0,r)$ rapidly increases as $r \to \infty$, see, e.g., [16]. On the other hand, an upper bound on $m(B(x_0,r))$ will imply the conservation property [14, 15, 26, 6, 25]. In particular, Grigor'yan [15] obtained a sharp condition for a geodesically complete Riemannian manifold:

$$\int^{\infty} \frac{rdr}{\ln m(B(x_0,r))} = \infty \quad \Rightarrow \quad \text{conservativeness.} \tag{1}$$

[1]It is also called non-explosive or stochastically complete.

For example, $m(B(x_0, r)) \le \exp(cr^2)$ will imply (1). This result was extended to a strongly local Dirichlet form by Sturm [25], where he used the Carnot-Carathéodory distance associated to the form (see Subsection 3.1). The investigation of this problem for a strongly local Dirichlet form has been quite successful; however, it seems that until recently, there has been no such result for more general Markov processes including jump processes.

On the other hand, it is well-known that a set $\Gamma \subset \mathbb{R}^n$ which will not be hit by the Brownian motion should have the Hausdorff codimension at least 2, see, e.g., [1]. In more general setting of a distance space $(X, d)$, the set $\Gamma$ should be replaced by the Cauchy boundary

$$\partial_C X = \overline{X} \setminus X$$

where $\overline{X}$ is the completion of $X$ with respect to the distance $d$. Of course, in this case, there is no point in asking whether the Markov process $X_t$ hits $\partial_C X$ or not since we don't know if $X_t$ can be extended to $\overline{X}$. Indeed, the topology of $\overline{X}$ can be quite rough. However, we may reformulate the question of "$X_t$ hitting $\partial_C X$" to "$W_0^{1,2}(X) \neq W^{1,2}(X)$" or, by extending the capacity of $X$ to $\overline{X}$ and ask when does $\partial_C X$ have capacity 0. Here, a natural question is:

*If $\partial_C X$ has capacity 0, namely, is polar, then should it have codimension at least 2?*

In this note, we will survey the recent development in the research of the conservation property of a Markov process $\{X_t\}_{t>0}$ along these two directions; namely, how small should be the volume growth and the singularity of the space so that a symmetric Markov process is conservative.

The structure of the note is the following. Section 2 will be devoted for the preliminary. In particular, we will first recall Einstein's original idea about the relationships between the random walk and the diffusion equation. His simple and beautiful observation will transparent our argumentation because our approach will be based on the strong relationships between the stochastic processes, the associated diffusion (or, heat) equations and its abstraction, the Dirichlet form. We then proceed to our framework, the Dirichlet form theory. For further study about the Dirichlet form theory, we refer the reader to [12]. In Chapter 3, we will discuss about three recent developments:

- A distance associated to non local Dirichlet forms;
- Volume growth conditions; and,
- New examples of polar Cauchy boundaries.

## 2. Preliminaries

The main approach taken in the recent developments regarding to the volume growth condition for the conservation property is based on the strong relationships between the stochastic processes, the associated diffusion (or, heat) equations, and the theory of Dirichlet forms.

In order to illustrate these relationships, let us start off from reviewing the Einstein's original idea on the Brownian motion. Einstein discovered two different methods to relate the Brownian motion and the associated equation. In 1905 [7], he succeeded to identify the Brownian motion with the irregular movements which arise from thermal molecular movements, and proved that the distribution of the Brownian motion solves the diffusion equation[2]. Of course, the classical derivation of the diffusion equation is to combine Fick's first law and the continuity equation. In 1908, Einstein [8] proved that the average of irregular movements satisfies both Fick's first law and the continuity equation, and, as a consequence, the diffusion equation. Since the argumentation is important and illuminating, we will present it below.

2.1. **Random walks, Fick's laws and diffusion equation.** We consider a random walk $\{X_n\}_{n\ge0}$ in $\mathbb{R}$ modelling the following irregular thermal motion:

- On the average, particles step to the right or to the left once every $\tau$ seconds, moving at velocity $\pm v$ a distance $\delta = \pm v\tau$. For the sake of simplicity, we assume that $\tau$ and $v$ are constants[3].

[2] He initially assumes that the distribution of the Brownian motion has compact support.

[3] In practice, they will depend on the size of particles, viscosity of the liquid, and the absolute temperature. The average speed of water molecular is approximately 640 m/s.

- The chances of the particles going to the right and the left are the same; namely, 1/2. The particles forget what they did in the past.
- The particles do not interact with each other.

If we denote the position of the $i$th particle after $n$th step by $X_i(n)$, then

$$X_i(n) = X_i(n-1) \pm \delta. \tag{2}$$

Suppose there are $N$ particles in the ensemble initially concentrated at the origin. The mean of the displacement is

$$\langle X(n)\rangle = \sum_i^N X_i(n)/N = 0$$

and the mean $\langle |X(n)|^2\rangle$ of the square of the displacement is

$$\langle |X(n)|^2\rangle = \frac{1}{N}\sum_i^N X_i^2(n) = \langle |X(n-1)|^2\rangle + \delta^2 = n\delta^2.$$

Letting $t = n\tau$, the time of the particle executing $n$steps, we find that

$$\langle |X(t)|^2\rangle = 2Dt, \tag{3}$$

where $D = \delta^2/2\tau$ is called the *diffusion coefficient.* Let

- $n(t,x)$ be the number of particles at time $t$ and at position $x$.
- $\phi(t,x)$ be the flux at $(t,x)$, that is the net number of the particles crossing $x$ from left to right in the time interval $[t, t+\tau]$.

After the next step, $t+\tau$, half of the particles at $x-\delta/2$ will have stepped across $x$ from left to right, and half of the particles at $x+\delta/2$ will have stepped across $x$ from right to left. Therefore,

$$\begin{aligned}\phi(t,x) &= \frac{1}{2}\left(\frac{n(t,x-\delta/2) - n(t,x+\delta/2)}{\tau}\right)\\ &= \frac{\delta^2}{2\tau}\frac{1}{\delta}\left(\frac{n(t,x-\delta/2) - n(t,x+\delta/2)}{\delta}\right)\\ &= D\frac{1}{\delta}\left(c(t,x-\delta/2) - c(t,x+\delta/2)\right),\end{aligned}$$

where $c(t,x)$ is the concentration. By letting $\delta \to 0$, we obtain Ficks' first law:

$$\phi(t,x) = -D\frac{\partial c}{\partial x}(t,x). \tag{4}$$

Next, consider the interval $I = [x, x+\delta]$. In the time interval $[t, t+\tau]$, $\phi(t,x)\tau$ particles will enter $I$ from the left, and $\phi(t, x+\delta)\tau$ particles leave from the right. If particles are neither created nor destroyed, the difference of the number of the particles $n(x, t+\tau) - n(x,t)$ at $x$ will be

$$n(x,t+\tau) - n(x,t) = (\phi(t,x) - \phi(t,x+\delta))\,\tau.$$

Dividing the both hand sides by $\delta$ and $\tau$,

$$\frac{c(x,t+\tau) - c(x,t)}{\tau} = \frac{\phi(t,x) - \phi(t,x+\delta)}{\delta}.$$

In the limit $\tau, \delta \to 0$, we obtain the *continuity equation*:

$$\frac{\partial c}{\partial t} = -\frac{\partial \phi}{\partial x}. \tag{5}$$

If we consider a more general situation of creation (distortion, respectively) of particles, then we need to add a nonnegative (nonpositive, respectively) potential $V(x)$ as

$$\frac{\partial c}{\partial t} = -\frac{\partial \phi}{\partial x} + V\cdot c. \tag{6}$$

Combing this with (4), we get Fick's second equation:

$$\frac{\partial c}{\partial t} = \operatorname{div}(D\cdot\nabla c) + V\cdot c. \tag{7}$$

2.2. **Heat kernels, energy forms, and boundary conditions.** The associated distribution $k$, called the heat kernel, to (7) on $\mathbb{R}^n$ when $V \equiv 0$ is

$$k(t,x,y) = \frac{1}{(4\pi Dt)^{n/2}} \exp\left(-\frac{|x-y|^2}{4Dt}\right). \tag{8}$$

A direct calculation shows that for each $t > 0$ and $y \in \mathbb{R}^n$,

$$k(t,\cdot,y) \in W^{1,2}(\mathbb{R}^n),$$

where $W^{1,2}(\mathbb{R}^n) = \{u \in L^2 \mid \nabla u \in L^2\}$. If the state space $X$ has boundary $\partial X$, then the typical boundary conditions are the homogenous Dirichlet and Neumann boundary conditions. The Brownian particles associated to the Dirichlet boundary condition will be absorbed at the boundary because the associated heat kernel $k^D$ satisfies $k^D(t,x,y) = 0$ whenever $x$ or $y$ belongs to the boundary. In particular,

$$k^D(t,\cdot,y) \in W_0^{1,2}(X), \quad \text{for each } t > 0 \text{ and } y \in X,$$

where $W_0^{1,2}(X)$ is the completion of the space $C_0^\infty(X)$ of smooth functions with compact support with respect to the norm: $\|u\|_{1,2} = \|\nabla u\|_2 + \|u\|_2$, where $\|\cdot\|_2$ stands for the standard $L^2$-norm.

We can also consider the Brownian particles which will be pushed back into the space after they hit the boundary, called the *reflected Brownian motion*. More precisely, they will be reflected symmetric to the boundary, therefore, the associated heat kernel $k^N$ satisfies the Neumann boundary condition, and it satisfies

$$k^N(t,\cdot,y) \in W^{1,2}(X) \quad \text{for each } t > 0 \text{ and } y \in X.$$

Clearly, the former Brownian motion is not conservative whereas the latter is. We should also point out that the former is *regular* in the sense that $C_0(X) \cap W_0^{1,2}(X)$ is dense in $W_0^{1,2}(X)$ with respect to the $\|\cdot\|_{1,2}$ as well as dense in $C_0(X)$ with respect to the sup-norm; while, the latter is not.

The associated energy form $\mathcal{E}$ to (7) is

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} D|\nabla u|^2\,dx - \int_{\mathbb{R}^n} V(x)\cdot u(x)^2\,dx. \tag{9}$$

We will say that a process $X_t$ is associated to (9) or (7) if its distribution is the fundamental solution to (7). By (6), if $V$ is negative then the associated process is not conservative. However, we need $V \le 0$ so that[4]

$$P_t 1(x) \le 1.$$

We should mention that the condition $V \le 0$ will allow us to find the equilibrium potential for any compact sets $K \subset X$, see, e.g., [12]. Therefore, hereafter, we assume $V \equiv 0$.

2.3. **Capacity.** Let us consider the problem of determining either a compact set $K \subset \mathbb{R}^n$ will be hit by the Brownian motion $B_t$ on $\mathbb{R}^n$ or not. This is related to the conservation property because we will study the minimal Brownian motion $B_t$ and $B_t$ hitting $K$ will immediately imply that $B_t$ on the state space $X = \mathbb{R}^n \setminus K$ is not conservative. Denote by $k$ and $k_X$ the heat kernels of $\mathbb{R}^n$ and $X$ with Dirichlet boundary condition. Then, this problem reduces to the problem of determining the condition on $K$ so that

$$k = k_X,$$

or, equivalently,

$$W^{1,2}(\mathbb{R}^n) = W_0^{1,2}(X). \tag{10}$$

Of course, (10) holds true if $K = \emptyset$ and $X = \mathbb{R}^n$. We can completely characterize (10) by using the *capacity*:

$$\mathrm{Cap}(K) = \inf_{u \in \mathcal{L}} \|u\|_{1,2}, \tag{11}$$

where $\mathcal{L} = \{u \in W^{1,2}(\mathbb{R}^n) \mid u|_K \ge 1\}$. We say $K$ *is polar* if $\mathrm{Cap}(K) = 0$. We state the following well-known fact without proof:

[4] Otherwise, the probability of a particle to be found in the space may exceed 1.

**Proposition 1.** *Let $K \subset \mathbb{R}^n$ be a compact set and $X = \mathbb{R}^n \setminus K$. The following conditions are equivalent:*

(1) *$K$ is polar.*
(2) *$W^{1,2}(\mathbb{R}^n) = W_0^{1,2}(X)$.*
(3) *$B_t$ on $X$ is conservative.*

2.4. **Dirichlet forms.** We will generalize the classical Dirichlet integral on $\mathbb{R}^n$ to Dirichlet forms on more general setting. In this subsection, we collect the necessary concepts and properties regarding to the Dirichlet forms in this note without proofs.

Throughout, $X$ is a sigma finite topological space with Radon measure $m$. Let $(\mathcal{E}, \mathcal{F})$, where $\mathcal{F} = D(\mathcal{E})$ in $L^2 = L^2(X, m)$, be a densely defined, closed symmetric positive quadratic form. The form $(\mathcal{E}, \mathcal{F})$ is called a symmetric *Dirichlet form* if it satisfies the *Markov property*[5]:

$$u \in \mathcal{F} \quad \Rightarrow \quad v = (u \wedge 1)_+ \in \mathcal{F} \text{ and } \mathcal{E}(v) \leq \mathcal{E}(u), \tag{12}$$

where $\mathcal{E}(u) = \mathcal{E}(u, u)$.

The *generator* $A$ of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is the unique self-adjoint operator in $L^2$ defined as

$$\mathcal{E}(u, v) = (-Au, v)_2, \quad \text{for all } u \in \mathcal{F} \text{ and } v \in D(A). \tag{13}$$

The associated $L^2$-semigroup

$$P_t = \exp(tA) : L^2 \to L^2$$

is called *Markovian* if

$$P_t 1(x) \leq 1 \quad \text{for a.e. } x \in X \text{ and every } t > 0.$$

The semigroup $P_t$ is Markovian if and only if the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is Markovian. Due to the Markov property, this $L^2$-semigroup can be uniquely extend to a $L^\infty$-semirgroup.

We assume that our Dirichlet form is *regular*, namely, $C_0(X) \cap \mathcal{F}$ is dense in $C_0(X)$ with respect to the sup-norm and dense in $\mathcal{F}$ with respect to the $\sqrt{\mathcal{E}}_1$-norm, where

$$\mathcal{E}_1(u) = \mathcal{E}(u) + \|u\|_2^2.$$

We will denote the completion of a space $C \subset L^2$ with respect to $\sqrt{\mathcal{E}_1}$-norm by $\overline{C}^{\mathcal{E}_1}$. A regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ has the following unique decomposition:

$$\mathcal{E}(u) = \int_X \mu^c(u) + \iint_{X \times X \setminus \mathrm{diag}} (\tilde{u}(x) - \tilde{u}(y))^2 \, J(dxdy) + \int_X \tilde{u} \, dk,$$

where $\mu^c(u)$ is the strongly local measure, $J$ is the jumping measure, $k$ is the killing measure, and $\tilde{u}$ is a *q.e.*-modification of $u$. We assume[6] that $k \equiv 0$. We say that $(\mathcal{E}, \mathcal{F})$ is strongly local if $J \equiv 0$, and $(\mathcal{E}, \mathcal{F})$ is non-local or (pure) jump type if $\mu^c \equiv 0$.

Below, we present some typical examples of regular Dirichlet forms.

**Example 1** (Heat equations). The phenomena of heat flow and diffusion are basically the same. However, the heat equation has an additional parameter, namely, the heat capacity[7]. Let $X \subset \mathbb{R}^n$ be a domain. Let $D = (D_{ij})_{1 \leq ij \leq n}$ be the heat conductivity, and, $\sigma$ be the heat capacity per volume. The heat equation is

$$Au = \frac{1}{\sigma} \mathrm{div}(D \cdot \nabla u) = \frac{\partial u}{\partial t},$$

where $u$ is the temperature. The Hilbert space is $L^2(X, \sigma dx)$, and the Dirichlet form is

$$\mathcal{E}(u) = \int_X -Au \cdot u \, \sigma dx = \int_X D\nabla u \cdot \nabla u \, dx, \quad \mathcal{F} = \overline{C_0^\infty(X)}^{\mathcal{E}_1}.$$

[5]The terminology "Markov property" is often used in the sense that the process does not remember the past.
[6]For the same reason that we assumed that $V \equiv 0$ in the classical setting.
[7]The heat capacity $\sigma$ is defined as $\sigma = \kappa\rho$, where $\kappa$ is specific heat and $\rho$ density. We need $\sigma$ to convert temperature to the amount of heat per unit volume. The concentration $c$ in the diffusion equation is, by definition, the amount of diffusion substance pet unit volume so that conversion factor is needed. See, e.g., [5] for further discussion.

**Example 2** (Weighted manifolds). Let $(M, g, m)$ be a weighted manifold, that is, $(M, g)$ is a Riemannian manifold and $m$ is a measure with density function $\Psi$ against the Riemannian measure. The canonical Dirichlet form $(\mathcal{E}, \mathcal{F})$ is

$$\mathcal{E}(u) = \int_M g(\nabla u, \nabla u)\, dm, \quad \mathcal{F} = \overline{C_0^\infty(M)}^{\mathcal{E}_1}.$$

The associated Laplacian $\Delta_m$ is

$$\Delta_m u = \Delta u + \frac{g(\nabla \Psi, \nabla u)}{\Psi}.$$

**Example 3** (Weighted graphs). Let $(V, E)$ be a countably infinite connected undirected graph without loops or multiple edges. We call such a graph a *simple* graph. We furnish $(V, E)$ with weights $m$ and $b$:

$$m : V \to (0, \infty),$$

and

$$b(x, y) = b(y, x) : V \times V \to (0, \infty)$$

satisfying:

$$b(x, y) > 0 \text{ if and only if } x \sim y,$$

and the integrability condition:

$$\sum_y b(x, y) < \infty, \quad \text{for all } x \in V.$$

We will call $G = (V, E, b, m)$ a *weighted graph.* The canonical Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $G$ is

$$\mathcal{E}(u) = \sum_{x,y \in V} b(x, y)\,(u(x) - u(y))^2, \quad \mathcal{F} = \overline{C_0(V)}^{\mathcal{E}_1}.$$

**Example 4** (Quantum graphs). We follow the framework in [19]. Let $(V, E)$ be a simple graph as above furnished with

(1) An orientation map $\tau : E \to -1, 1$ satisfying

$$\tau((x, y)) = -\tau((y, x)) \quad \text{for all } (x, y) \in E.$$

We denote $E_+ = \tau^{-1}(\{1\})$.

(2) A length function $l : E_+ \to (0, +\infty]$,

(3) A family of marked intervals $\{I(e)\}_{e \in E_+}$, where $I(e) = [0, l(e)] \times \{e\}$.

For every $e = (x, y) \in E_+$, the endpoints 0 and $l(e)$ of the interval $I(e)$ will be identified with $x$ and $y$, respectively. Denoting this identification by $\sim$, we call the quotient space

$$X = \left(\cup_{e \in E_+} I(e)\right) / \sim$$

a *quantum graph*[8]. A quantum graph $X$ carries a natural distance $d$ as well as the measure $m$ via this identification. The Dirichlet form is

$$\mathcal{E}(u) = \sum_{e \in E_+} \int_0^{l(e)} (u')^2\, dm, \quad \mathcal{F} = \overline{\{u \in C_0(X) \mid \mathcal{E}(u) < \infty\}}^{\mathcal{E}_1}.$$

**Example 5** ($\alpha$-stable Levi form). Let $(X, d, m)$ be a $\beta$-regular metric measure space, that is, there is $c > 0$ such that the measure $m(B(x, r))$ of any $r$-ball at any $x \in X$ satisfies:

$$c^{-1} r^\beta < m(B(x, r)) < c r^\beta.$$

For $0 < \alpha < 2$, the $\alpha$-stable Levi form is

$$\mathcal{E}(u) = \iint_{X \times X \setminus \mathrm{diag}} (u(x) - u(y))^2 \frac{m(dx)}{(x - y)^{\beta + \alpha}} m(dy), \quad \mathcal{F} = \overline{C_0^{lip}(X)}^{\mathcal{E}_1},$$

where $C_0^{lip}(X)$ is the space of Lipschitz functions with compact support.

[8] It is also called a metric graph in the literature.

**Remark 1.** *Examples 1, 2, and 4 are strongly local Dirichlet forms and Examples 3 and 5 are non local Dirichlet forms.*

## 3. Recent developments

**3.1. Adapted distance associated to a Dirichlet form.** Biroli and Mosco [4] and Sturm [25] defined an developed the theory of Carnot-Carathéodory distance associated to a regular strongly local Dirichlet form:

$$d(x,y) = \sup\{u(x) - u(y) \mid u \in \mathcal{F}_{\text{loc}} \cap C(X),\ d\mu^c(u) \le dm\}, \tag{14}$$

where $\mathcal{F}_{\text{loc}}$ is the space of functions locally in $\mathcal{F}$. For instance, the Carnot-Carathéodory distance associated to the canonical Dirichlet form of a weighted manifold (Example 2) is independent of the density function, and it coincides with the original Riemannian distance. For Example 1, we have

**Example 6.** Let $X = (\mathbb{R}, \sigma dx)$ with a positive even function $\sigma$ and

$$\mathcal{E}(u) = \int_X (u')^2\, dx \quad \text{and} \quad \mathcal{F} = W^{1,2}(X, \sigma dx).$$

Then

$$d(x,y) = \sup\{u(x) - u(y) \mid u \in C^1(X),\ |u'| \le \sqrt{\sigma}\},$$

and the distance between the origin and $x$ is

$$r(x) = \int_0^x \sqrt{\sigma(t)}\, dt.$$

By a theorem in [25], the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative if there exists $c > 0$ such that

$$m(B(r)) = 2\int_0^{r^{-1}(r)} \sigma(s)\, ds \le \exp(cr^2 \ln r) \quad \text{for all large } r > 0.$$

A counter part of Carnot-Carathéodory distance for a non local Dirichlet form was proposed in [22] (see also [13] for a similar notion of distance).

Assume that $J(dxdy)$ has a kernel $j(x,dy)$, namely, $j(x,dy)$ is a kernel that associates for any $x \in X$ a Radon measure on the Borel $\sigma$-algebra $\mathcal{B}(X \setminus \{x\})$ that depends on $x$ in a measurable way, and

$$j(x,dy)dx = J(dxdy).$$

This assumption corresponds to that $d\mu^c(u)$ has a density against $dm$ in (14).

**Definition 1.** We say that the distance $d$ is *adapted* to $(\mathcal{E}, \mathcal{F})$ if

$$\sup_{x \in X} \int_{y \ne x} \left(1 \wedge d^2(x,y)\right) j(x,dy) < \infty. \tag{15}$$

The condition (15) is equivalent to the combination of the following:

$$\sup_{x \in X} \int_{B(x,1)\setminus\{y=x\}} d^2(x,y)\, j(x,dy) < \infty \tag{16}$$

and

$$\sup_{x \in X} \int_{B^c(x,1)} j(x,dy) < \infty. \tag{17}$$

It is easy to verify that the constant 1 in (15) may be any positive number. The condition (16) is a straightforward generalisation of (14). On the other hand, (17) will vanish if the jump rage is less than 1. (This is why we don't need (17) for a strongly local case because the process associated to a strongly local Dirichlet form has no jumps.) Of course, we need (17) so that the Euclidean distance is adapted to the classical Levi form.

**Example 7** (Standard adapted distance for a weighted graph [10, 17]). For a weighted graph, set

$$\deg(x) = \frac{1}{m(x)} \sum_{y \in V} b(x,y)$$

and

$$\sigma(x,y) = \min\{\deg(x)^{-1}, \deg(y)^{-1}, 1\} \tag{18}$$

for $x \sim y$. A subset $\gamma : x \sim y = \{x = x_0, x_1, \cdots, y = x_{n+1} \in V \mid x_l \sim x_{l+1} \text{ for all } 0 \le l \le n\} \subset V$ is called a path connecting $x, y \in V$. The length of $\gamma : x \sim y$ is $l(\gamma) = \sum_{0 \le n \le m} \sigma(x_n, x_{n+1})$. Then the *standard adapted distance* for $x \neq y$ is defined as

$$d(x,y) = \inf_{\gamma : x \sim y} l(\gamma).$$

We should point out that we don't need (17) since the associated process jumps only to the linked vertices, namely, it is essentially a "diffusion" but in a discrete state space. Indeed, in this situation, the distance is intrinsic in the sense of [13]. We refer the author to [20] for further discussions about the distance including the Hopf-Rinow type theorem.

3.2. **Volume growth conditions.** As we had mentioned in the introduction, the Brownain motion on a geodesically complete Riemannian manifold (or more generally, symmetric strongly local Dirichlet forms) is conservative if

$$m(B(x_0, r)) \le \exp(cr^2 \ln r) \quad \text{for all large } r > 0.$$

Now, let us turn to a symmetric jump process on a metric measure space $(X, d, m)$, i.e., the associated symmetric Dirichlet form is

$$\mathcal{E}(u,v) = \iint_{X \times X \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y))\, j(x, dy) m(dx).$$

We assume that any geodesic balls $B(x,r) = \{y \in X \mid d(x,y) < r\}$ are relatively compact. In particular, $(X, d)$ is locally compact and separable. Let $d$ be an adapted distance to $(\mathcal{E}, \mathcal{F})$. Then,

**Theorem 1.** *If $m(B(x_0, r)) \le \exp(cr \ln r)$, then $(\mathcal{E}, \mathcal{F})$ is conservative.*

Theorem 1 was proved in [17] for $c = 1/2$ and for general $c > 0$ in [23]. In particular, Theorem 1 holds true for more general Dirichlet forms having both strongly local part and non local part [23]:

**Example 8.** Let $X = \cup_{i \in \mathbb{Z}} X_i$, where for each $i \in \mathbb{Z}$, $X_i = \{x = (x_i, i) \in \mathbb{R}^{n+1} \mid x_i \in \mathbb{R}^n\}$. Denote the associated projections of $x$ to the first and second components, respectively, by $p : X \to \mathbb{R}^n$ and $q : X \to \mathbb{Z}$. We define the distance $d$ as

$$d(x,y) = |p(x) - p(y)| + |q(x) - q(y)|, \quad x, y \in X,$$

where $|\cdot|$ is the Euclidean distance. Let $m(dx) = \sum_{i \in \mathbb{Z}} m_i(dx_i)$ be a measure on $X$ such that for each $i \ge 1$, $m_i(dx_i) = \Psi(x_i)\, dx_i$ is a measure on $X_i$ with a positive function $\Psi \in C(\mathbb{R}^n)$, and $dx_i$ is the $n$-dimensional Lebesgue measure. Clearly, $m$ is a Radon measure on $X$. The state space is the triple $(X, d, m)$.

For any $u \in C_0^{lip}(X)$, define

$$\mathcal{E}(u) = \int_X |\nabla u|^2\, dm + \iint_{X \times X \setminus x \neq y} (u(x) - u(y))^2\, j(x,y)\, m(dx) m(dy),$$

and

$$j(x,y) \asymp \frac{d(x,y)^{-(n+\alpha)} \mathbf{1}_{\{d(x,y) < 1\}} + d(x,y)^{-(n+\beta+1)} \mathbf{1}_{\{d(x,y) \ge 1\}}}{\Psi(p(x)) + \Psi(p(y))}, \quad x, y \in X$$

with some constants $0 < \alpha < 2$ and $\beta > 0$. If $\mathcal{F} = \overline{C_0^{lip}(X)}^{\mathcal{E}_1}$ then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form, and $d$ is adapted to $(\mathcal{E}, \mathcal{F})$. The volume criterion in Theorem 1 is satisfied for example

$$\Psi(x) \le |x|^{|x|} \ln |x|$$

for all $x$ when $|x|$ is large enough.

The Markov process in this example jumps from a connected component to other competent, and it behaves as a jump-diffusion inside each component.

See also Shiozawa [24] for further extension of Theorem 1.

For a weighted graph, a sharp result has been obtained by Folz [11] and Huang [19]:

**Theorem 2.** *Assume that every metric ball of a weighted graph $X$ is compact. If*

$$\int^{\infty} \frac{rdr}{\ln m(B(x_0,r))} = \infty,$$

*then $(\mathcal{E},\mathcal{F})$ is conservative.*

**Remark 2.** (1) *We don't know either the volume growth condition in Theorem 1 is sharp or not. The idea of the proof is to develop the analysis of non local operators and to adapt the Davies method* [6]*, where he proved the conservation property for the Brownian motion on a complete weighted manifold. The obstruction in the non local case is the lack of chain rule.*

(2) *Folz* [11] *and Huang* [19] *proved Theorem 2 using the probability method and analysis, respectively. Both of them established comparison theorems for a continuous time random walk on the set of vertices and the Brownian motion on the quantum graph. The latter is a strongly local Dirichlet form so that one can apply Sturm's result to get the sharp volume growth criterion. It seems that there has been no direct proof for Theorem 2 yet.*

(3) *Sturm* [25] *assumed that every metric ball is relatively compact. It is possible to extend his result to certain geodesically incomplete quantum graphs. For example, if $\partial_C X$ is polar and there exists a relatively compact open set $O \subset \overline{X}$ such that $O \supset \partial_C X$ and $O \setminus \partial_C X$ is connected. This observation may be useful to weaken the topological assumption required in Theorem 2.*

3.3. **Polar conditions.** As we already have mentioned in the introduction, a compact polar set $K$ of a Riemannian manifold satisfies $\mathrm{codim}_H(K) \geq 2$, where $\mathrm{codim}_H(K)$ is the Hausdorff codimension of $K$. In this subsection we will show that this classical fact is not true in more general setting by present counter examples. The upper Minkowski codimension of a Borel set $K$ in a metric measure space is defined as

$$\mathrm{codim}_M(K) = \limsup_{r\to 0} \frac{\ln m(B(K,r))}{\ln r}. \tag{19}$$

It is known that these two dimensions coincide for a wide class of fractals[9]. For a Riemannian manifold $M$, the *Cauchy boundary* is defined as

$$\partial_C M = \overline{M} \setminus M,$$

where $\overline{M}$ is the completion of $M$ with respect to the Riemannian distance. We recall the 1-capacity of a set $\Sigma \subset \overline{M}$ associated to $W^{1,2}$. Let $\mathcal{O}$ denote the family of all open subsets of $\overline{M}$. First we define for $\Omega \in \mathcal{O}$:

$$\mathrm{Cap}(\Omega) := \inf_{u\in\mathcal{L}(\Omega)} \int_M u^2 + |\nabla u|^2 \, d\mu, \quad \text{if } \mathcal{L}(\Omega) \neq \phi,$$

where $\mathcal{L}(\Omega)$ is the set of functions $u \in W^{1,2}$ satisfying that $0 \leq u \leq 1$ and $u|_{\Omega\cap M} = 1$. We let $\mathrm{Cap}(\Omega) = \infty$ if $\mathcal{L}(\Omega) = \phi$, and $\mathrm{Cap}(\phi) = 0$. For arbitrary set $\Sigma \subset \overline{M}$, we let

$$\mathrm{Cap}(\Sigma) := \inf_{\Omega\in\mathcal{O},\ \Sigma\subset\Omega} \mathrm{Cap}(\Omega).$$

A set $\Sigma$ is called *polar* if $\mathrm{Cap}(\Sigma) = 0$. Clearly, $M$ is geodesically complete if and only if $\partial_C M = \phi$.

The following was proved in [18] (see also [21]).

**Theorem 3.** *The capacity defined above is a Choquet capacity*[10]*. Assume that $\partial_C M$ is compact.*

(1) *If* $\mathrm{Cap}(\partial_C M)$ *is positive, then*

(a) $W_0^{1,2}(M) \neq W^{1,2}(M)$.

[9] For example, $K$ satisfies the open set condition.

[10] A Choquet capacity is usually defined for a subset of $M$.

(b) *The Brownian motion on $M$ is not conservative.*

(2) *If* $\mathrm{Cap}(\partial_C M) = 0$, *then*
(a) $W_0^{1,2}(M) = W^{1,2}(M)$.
(b) *The Brownian motion on $M$ is conservative, provided the Grigor'yan volume condition:*

$$\int^{\infty} \frac{rdr}{\ln m(B(x_0, r))} = \infty.$$

Moreover, it was proved that

**Theorem 4.** (1) *If* $\mathit{codim}_M(\partial_C M) > 2$, *then* $\mathrm{Cap}(\partial_C M) = 0$.
(2) *For any $n \geq 2$, there exists an $n$-dimensional Riemannian manifold $M$ such that*

$$\mathit{codim}_M(\partial_C M) = 2 \text{ and } \mathrm{Cap}(\partial_C M) > 0.$$

We define the Cauchy boundary $\partial_c X$ for a graph $X$ as well. In the discrete setting, it was proved in [20] that

**Theorem 5.** *For a locally finite weighted graph $X$ with adapted distance $d$,*

$$\mathit{codim}_M(\partial_C X) > 2 \quad \Rightarrow \quad \mathrm{Cap}(\partial_C X) = 0 \quad \Rightarrow \quad W_0^{1,2}(X) = W^{1,2}(X).$$

These two theorems above agree with the classical fact about the Hausdorff dimension and the polarity, which we mentioned above. On the contrary, we have

**Example 9.** Let $X = N_0$ be a weighted graph with

$$b(x,y) = \begin{cases} 1, & |x-y| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and $m(x) = 2^{(1-2\alpha)x}$ with $\alpha > 1/2$. Consider the standard adapted distance $d$. Then, $\mathrm{Cap}(\partial X) = 0$ and

$$\mathrm{codim}_M(\partial X) = 2 - \alpha^{-1}.$$

This example is due to Mr. Y. Watanabe as a modification of Example 5.7 in [20], where the same result was obtained using an adapted distance but not the standard one. We should point out that any distance which is smaller than an adapted distance is adapted, it is natural to produce "counter examples" if we don't use the standard one.

A continuous version, also due to him, is also available:

**Example 10.** The underlying space is $X = (0, +\infty)$, $dm = x^p dx$ with $0 < p < 1$. The Dirichlet form is

$$\mathcal{E}(u) = \int_X (u')^2 \, dm$$

and $\mathcal{F} = \overline{C_0^\infty(X)}^{\mathcal{E}_1}$. A direct calculation yields: $\mathrm{codim}_M(\partial_C X) = 1 + p$. Let $0 < r < R$ and

$$u_{r,R}(x) = \left( \left( \frac{x^{1-p} - R^{1-p}}{r^{1-p} - R^{1-p}} \right) \wedge 1 \right)_+ .$$

Then, $u_{r,R}(x) = 1$ for $x \leq r$, $u_{r,R}(x) = 0$ for $x \geq R$, and

$$\mathcal{E}_1(u_{r,R}) \to 0 \quad \text{as } r < R \to 0.$$

Therefore, $\mathrm{Cap}(\partial_C X) = 0$.

**Acknowledgements.** I wish to show my gratitude toward Professor Shigeru Sakaguchi for his kind invite to the stimulating workshop *Geometry of solutions of partial differential equations* at Research Institute for Mathematical Sciences (RIMS), Kyoto. I also thank RIMS and Professor Eiji Yanagida for the financial support.

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