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Remarks on Liouville theorem for Hénon type equation on the hyperbolic space

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1 Introduction

The paper is devoted to a Hénon type equation on the hyperbolic space. In particular, we shall prove an existence of solutions to the elliptic equation. Furthermore we announce a Liouville theorem for the equation on the hyperbolic space, which is obtained in [21]. In order to state a motivation of our research, first we mention known results for semilinear elliptic equations.

To begin with, we introduce known results for the following elliptic equation in the Euclidean space:

$$\text{(E)} \qquad -\Delta u=|x|^{\alpha}|u|^{p-1}u \quad \text{in} \quad \mathbb{R}^{N},$$

where $\alpha>-2$, $N\geq 3$ and $p>1$. Here, $|x|^{\alpha}$ is called a weight. The equation (E) was posed by J.H. Lane ([27]) for the case $\alpha=0$ in 1869 and is well known as Lane-Emden-Fowler equation. The equation has been widely studied in the mathematical literature ([6, 7, 15, 19, 20, 26, 30]). Moreover, the equation was appeared in the astrophysical study of the structure of a singular star ([8, 14, 17]). In 1973, (E) for the case $\alpha>-2$ was posed by M. Hénon to study rotating stellar structures ([24]) and (E) is called Hénon equation. Although he defined the equation only in 3-dimensional unit ball with Dirichlet boundary condition, the equation has been studied for more general setting by mathematical interest ([11, 18, 28, 31, 32, 35, 36]).

Regarding the exponent p in (E), there exist certain critical exponents which characterize the structure of solutions to (E). A typical exponent is Sobolev's critical exponent:

$$p_{s}(N):=\frac{N+2}{N-2}.$$

For example, p_{s} characterizes the solution of (E) with respect to the positivity:

Theorem 1.1 (B. Gidas and J. Spruck [18, 19]). *Let* $1<p<p_{s}(N)$ *and* $p\neq(N+2+2\alpha)/(N-2)$. *If the solution* $u\in C^{2}(\mathbb{R}^{N})$ *of* (E) *is nonnegative, then* $u=0$.

Remark that Theorem 1.1 implies that there is no positive solution of (E) when $\alpha>-2$, $1<p<p_{s}(N)$ and $p\neq(N+2+2\alpha)/(N-2)$. Moreover, it is sufficient to consider only the case $\alpha>-2$ and $p\geq p_{s}(N)$, because the nonexistence of positive solution of (E) for the case $\alpha<-2$ was showed by B. Gidas and J. Spruck ([19]).

The other critical exponent, which characterizes the solution with respect to the stability, has been attracting a great interest in recent years. Indeed, the following results were proved by Farina in 2007 for $\alpha = 0$ ([15]) and by Dancer, Du and Guo in 2011 for $\alpha > -2$ ([11]).

Theorem 1.2 ([11, 15]). *Let $u \in C^2(\mathbb{R}^N)$ be a stable solution of* (E). *If $p > 1$ satisfies*

$$\begin{cases} 1 < p < +\infty & \text{if } N \le 10 + 4\alpha, \\ 1 < p < p(\alpha, N) & \text{if } N > 10 + 4\alpha, \end{cases}$$

then $u \equiv 0$ in $\mathbb{R}^N$. Here, $p(\alpha, N)$ is given by the following:

$$p(\alpha, N) := \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) + 2\sqrt{2(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)}.$$

On the other hand, if $p \ge p(\alpha, N)$, then the equation (E) *has stable, positive, and radial solutions.*

The assertion in Theorem 1.2 is called a Liouville type theorem. Remark that they proved Theorem 1.2 without any other assumption except stability, such as positivity, radial symmetry and so on. Moreover, Theorem 1.2 implies that $p(\alpha, N)$ is critical. Here, we define the stability of solutions to (E) as follows:

Definition 1.1. *A solution $u \in C^2(\mathbb{R}^N)$ of* (E) *is stable if the inequality*

$$\int_{\mathbb{R}^N} \{|\nabla\psi|^2 - p|x|^\alpha |u|^{p-1}\psi^2\}\, dx \ge 0$$

holds for any $\psi \in C_c^1(\mathbb{R}^N)$.

We mention some remark on Definition 1.1. One can observe that the equation (E) is formally derived as Euler-Lagrange equation for the functional

$$E(u) := \int_{\mathbb{R}^N} \left\{ \frac{1}{2}|\nabla u|^2 - |x|^\alpha \frac{|u|^{p+1}}{p+1} \right\} dx.$$

Recall that the stability is defined for C^2 solutions of (E) in Definition 1.1. Obviously there exist C^2 solutions with infinite energy. However Definition 1.1 is available for such solutions. Indeed, for each $R > 0$ and any C^2 solution of (E), the functional

$$E_R(u) := \int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 - |x|^\alpha \frac{|u|^{p+1}}{p+1} \right\} dx$$

is finite, where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Then, the second variational formula for E_R, which is expressed as

$$Q_R[u](\psi) := \int_{B_R} \{|\nabla\psi|^2 - p|x|^\alpha |u|^{p-1}\psi^2\}\, dx, \quad \forall \psi \in C_c^1(B_R),$$

is well-defined for any C^2 solution u of (E). Since $R > 0$ is arbitrary, Definition 1.1 is equivalent to the following: "A solution $u \in C^2(\mathbb{R}^N)$ of (E) is stable if $Q_R[u](\psi)$ is non-negative for any $\psi \in C_c^1(B_R)$." By making use of the concept of Definition 1.1, Liouville type theorems have been proved for many kinds of elliptic equations ([9, 10, 11, 12, 13, 16, 25, 36]).

On the other hand, recently semilinear parabolic and elliptic equations in the hyperbolic space have been studied ([1, 2, 3, 4, 5, 22, 29, 33, 34]). For example, the equation (E) for the case of $\alpha = 0$ can be written as

$$\text{(LH)} \qquad -\Delta_{\mathbb{H}} u = |u|^{p-1} u \quad \text{in} \quad \mathbb{B}^N,$$

where $p > 1$ and $N \geq 3$. Here, $\mathbb{B}^N$ denotes a unit ball $\{x \in \mathbb{R}^N : |x| < 1\}$ endowed with the following Riemannian metric:

$$g_{ij} = \left(\frac{2}{1 - |x|^2} \right)^2 \delta_{ij},$$

where δ_{ij} is Kronecker's delta. The geodesic distance from the origin to $x \in \mathbb{B}^N$ is given by

$$d_{\mathbb{H}}(0, x) := \int_0^{|x|} \frac{2}{1 - s^2} ds = \log \left(\frac{1 + |x|}{1 - |x|} \right).$$

Furthermore, $\Delta_{\mathbb{H}}$ is the Laplace-Beltrami operator on $\mathbb{B}^N$ and is written by

$$\Delta_{\mathbb{H}} u = \left(\frac{1 - |x|^2}{2} \right)^2 \Delta u + (N - 2) \left(\frac{1 - |x|^2}{2} \right) x \cdot \nabla u.$$

Although it is obvious that the metric affects the geodesic distance and differential operators, it might affect the structure of solutions. Indeed, [29] shows that there exists at most one positive radial $H^1(\mathbb{B}^N)$ solution for $1 < p < p_s(N)$ by using the variational method. Furthermore, Bonforte, Gazzola, Grillo, and Vázquez proved the existence of solutions with infinite energy for $1 < p < p_s(N)$:

Theorem 1.3 ([5, 29]). *Let $1 < p < p_s(N)$. Then, there exists a positive radial solution $u \in C^2(\mathbb{B}^N)$ of* (LH).

Although Theorem 1.1 showed the nonexistence of positive solution of (E) for $1 < p < p_s(N)$, Theorem 1.3 shows the existence of positive solution of (LH) for $1 < p < p_s(N)$. The difference is strongly related that Poincaré's inequality in $L^2(\mathbb{B}^N)$ holds since the first eigenvalue of $-\Delta_{\mathbb{H}}$ is $((N-1)/2)^2$, i.e., positive. Making use of the positivity, Berchio, Ferrero, and Grillo showed the following result:

Theorem 1.4 ([3]). *Let $p > 1$. Then, for each $\beta > 0$, there exists a unique radial solution u_β of* (LH) *satisfying the following conditions*:

$$u_\beta(0) = \beta, \quad u'_\beta(0) = 0.$$

Moreover, there exists some positive constant β_0 such that u_β is stable for any $\beta \leq \beta_0$.

Here, r denotes the geodesic distance $d_{\mathbb{H}}(0,x)$ from the origin to $x \in \mathbb{B}^N$. Regarding β_0, they proved that β_0 is bounded when $1 < p < p(0,N)$. In [3], the stability of solutions of (LH) is defined by the same manner as in Definition 1.1:

Definition 1.2. *The solution $u \in C^2(\mathbb{B}^N)$ of* (LH) *is stable if the inequality*

$$\int_{\mathbb{B}^N} \{|\nabla_{\mathbb{H}}\psi|_{\mathbb{H}}^2 - p\,|u|^{p-1}\,\psi^2\}\,dV_{\mathbb{H}} \geq 0$$

holds for any $\psi \in C_c^1(\mathbb{B}^N)$.

Here, $\nabla_{\mathbb{H}}$ and $dV_{\mathbb{H}}$ are the gradient operator and the volume element on the hyperbolic space, respectively. Also, $|\nabla_{\mathbb{H}}\psi|_{\mathbb{H}}^2$ denotes the inner product of $\nabla_{\mathbb{H}}\psi$ with itself, where this inner product is induced from the metric on $\mathbb{B}^N$ as follows:

$$|\nabla_{\mathbb{H}}\psi(x)|_{\mathbb{H}}^2 = \langle\nabla_{\mathbb{H}}\psi(x), \nabla_{\mathbb{H}}\psi(x)\rangle_{\mathbb{H}} := \left(\frac{2}{1-|x|^2}\right)^2 (\nabla_{\mathbb{H}}\psi(x), \nabla_{\mathbb{H}}\psi(x)). \tag{1.1}$$

Here $(\cdot,\cdot)$ denotes the usual inner product in $\mathbb{R}^N$. Theorem 1.4 implies that there is no critical exponent for (LH) such as $p(\alpha, N)$ in Theorem 1.2. This fact also arises from the structure of spectrum of $-\Delta_{\mathbb{H}}$. Indeed, letting the value of origin less than the first eigenvalue sufficiently, they first proved that the inequality in Definition 1.2 holds. Furthermore they also constructed non-trivial stable solution. Comparing Theorem 1.4 with Theorem 1.2, we are interested in the following question:

Problem 1.1. *Does Liouville theorem hold for the equation* (LH) *with some weight?*

To consider this problem, first we introduce an typical weight for (LH). From the analogue of the weight in (E), we can choose the power of geodesic distance as weight:

$$-\Delta_{\mathbb{H}}u = (d_{\mathbb{H}}(0,x))^{\alpha}\,|u|^{p-1}\,u \quad \text{in} \quad \mathbb{B}^N. \tag{1.2}$$

Actually, He and Wang proved the existence of solutions and its asymptotic behavior for (1.2) ([22, 23]). However, any Liouville type theorem with respect to the stability has not been proved yet. Indeed, we couldn't prove the Liouville type theorem for (1.2) although we make use of the same method as the proof of Theorem 1.2.

In order to give an affirmative answer to Problem 1.1, we consider the following equation:

$$-\Delta_{\mathbb{H}}u = \left(\frac{2\,|x|}{1-|x|^2}\right)^{\alpha} |u|^{p-1}\,u \quad \text{in} \quad \mathbb{B}^N, \tag{H}$$

where $\alpha > 0$, $p > 1$ and $N \geq 3$. Remark that we can write the weight as follows:

$$w(x) := \frac{2\,|x|}{1-|x|^2} = \sinh r,$$

where $r = d_{\mathbb{H}}(0,x)$. The reason why we choose this weight is that $\sinh r$, which has strong singularity in the infinity, arises in the volume element in the hyperbolic space.

By making use of the fact, we can obtain an affirmative answer to Problem 1.1. Indeed, we shall announce a Liouville theorem which is stated in concise form as follows: "For sufficiently small $p > 1$, if u is stable solution of (H), then $u = 0$." For the precise thesis, see Section 3. As a first step of our study for (H), we start with an existence of solution of (H) with small $p > 1$:

Theorem 1.5. *The equation* (H) *admits a radial positive solution in* $H^1(\mathbb{B}^N) \cap C^2(\mathbb{B}^N)$ *if*

$$p \in \left(\frac{N - 1 + 2\alpha}{N - 1}, \frac{N + 2 + 2\alpha}{N - 2} \right).$$

We shall construct this nontrivial solution by using variational methods. Moreover, Sobolev's embedding implies that the solution obtained in Theorem 1.5 has finite energy.

This paper is organized as follows. In Section 2, we shall prove Theorem 1.5. The proof is a modification of the proof of Theorem 6 in [31]. Finally, in Section 3, we state the Liouville theorem and asymptotic behavior of radial solutions of (H) for $p > 1$ big enough. We shall show you an outline of proof of the Liouville theorem. For the precise proof, see [21].

2 Existence of solution

In this section, we shall prove an existence of solution to (H) in the class $H^1(\mathbb{B}^N)$. Moreover, the following Theorem 1.5 is proved by a modification of the proof of Theorem 6 in [31]. We prove Theorem 1.5 by making use of Mountain Pass Theorem:

Proposition 2.1 (Mountain Pass Lemma). *Let E be a Banach space and let $J \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Suppose that* (A) *$J(0) = 0$ and $J(e) = 0$ for some $e \neq 0$ in E, and* (B) *there exists $\rho \in (0, |e|)$ and $\alpha > 0$ such that $J \geq \alpha$ on $S_\rho = \{u \in E : |u| = \rho\}$. Then J has a positive critical value*

$$c = \inf_{h \in \Gamma} \max_{t \in [0,1]} J(h(t)) \geq \alpha > 0$$

where $\Gamma = \{h \in C([0, 1], E) : h(0) = 0, h(1) = e\}$.

J satisfies the Palais-Smale condition if any sequence $\{u_n\} \subseteq E$ with $\{J(u_n)\}$ bounded and $J'(u_n) \to 0$ has a convergent subsequence.

Let E be the completion of radially symmetric C_0^∞ functions with respect to the norm, where

$$\|u\|_E^2 = \int_{\mathbb{B}^N} |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2 \, dV_{\mathbb{H}}.$$

Since the bottom of the spectrum of $-\Delta_{\mathbb{H}}$ is given by

$$\lambda_1(-\Delta_{\mathbb{H}}) := \inf_{u\in H^1(\mathbb{B}^N)\setminus\{0\}} \frac{\int_{\mathbb{B}^N} |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2\, dV_{\mathbb{H}}}{\int_{\mathbb{B}^N} |u|^2\, dV_{\mathbb{H}}} = \frac{(N-1)^2}{4},$$

it is easy to verify that $\|\cdot\|_E$ is equivarent to the norm of $H^1(\mathbb{B}^N)$. Indeed we observe that

$$\begin{aligned}\|u\|_E^2 &\le \int_{\mathbb{B}^N} |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2\, dV_{\mathbb{H}} + \int_{\mathbb{B}^N} |u|^2\, dV_{\mathbb{H}}\\ &\le \left(1+\frac{4}{(N-1)^2}\right)\|u\|_E^2.\end{aligned}$$

In the following, we shall prepare the proposition which we need in order to show the existence of solution of (H) in $H^1(\mathbb{B}^N)$:

Lemma 2.1. *Let $u\in E$. Then it holds that*

$$|u(x)| \le \frac{1}{\sqrt{w_N(N-2)}}\frac{\|u\|_E}{(\sinh(2\operatorname{arc\,tanh}|x|))^{\frac{N-2}{2}}}, \tag{2.1}$$

$$|u(x)| \le \frac{1}{\sqrt{w_N(N-1)}}\frac{\|u\|_E}{(\sinh(2\operatorname{arc\,tanh}|x|))^{\frac{N-1}{2}}}, \tag{2.2}$$

where w_N is the surface area of the unit ball in $\mathbb{R}^N$.

Proof. Since $u\in E$, it holds that

$$u(1)-u(|x|) = \int_{|x|}^1 u'(t)dt.$$

By Hólder's inequality, we have

(2.3)

$$\begin{aligned}|u(x)| &\le \int_{|x|}^1 |u'(t)|dt\\ &\le \left(\int_{|x|}^1 |u'(t)|^2 t^{N-1}\left(\frac{2}{1-t^2}\right)^{N-2} dt\right)^{\frac12}\left(\int_{|x|}^1 t^{-(N-1)}\left(\frac{2}{1-t^2}\right)^{-(N-2)} dt\right)^{\frac12}\\ &:= I_1 + I_2.\end{aligned}$$

First we estimate I_1 as follows:

$$\begin{aligned}I_1 &= \frac{1}{w_N}\int_{\partial B(0,1)}\left(\int_{|x|}^1 \left(\frac{1-t^2}{2}\right)^2 |u'|^2\left(\frac{2}{1-t^2}\right)^N t^{N-1}dt\right)dS\\ &= \frac{1}{w_N}\int_{|x|\le|y|\le1}\left(\frac{1-|y|^2}{2}\right)^2|\nabla u|^2\left(\frac{2}{1-|y|^2}\right)^N dy\\ &= \frac{1}{w_N}\int_{|x|\le|y|\le1}|\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2\, dV_{\mathbb{H}}(y)\\ &\le \frac{1}{w_N}\|u\|_E^2.\end{aligned}$$

Regarding I_2, we find

$$\begin{aligned}
I_2 &= \int_{2\mathrm{arc}\tanh|x|}^{\infty} \left(\tanh\frac{s}{2}\right)^{-(N-1)} \left(2\cosh^2\frac{s}{2}\right)^{-(N-2)} \left(2\cosh^2\frac{s}{2}\right)^{-1} ds \\
&= \int_{2\mathrm{arc}\tanh|x|}^{\infty} (\sinh s)^{-(N-1)}\, ds \\
&\le \int_{2\mathrm{arc}\tanh|x|}^{\infty} (\sinh s)^{-(N-1)} \cosh s\, ds \\
&= -\frac{1}{N-2}\left[(\sinh s)^{-(N-2)}\right]_{2\mathrm{arc}\tanh|x|}^{\infty} = \frac{1}{N-2}\left(\sinh\left(2\mathrm{arc}\tanh|x|\right)\right)^{-(N-2)}.
\end{aligned}$$

Then (2.1) is followed from this estimate and (2.3). Moreover, we can also estimate I_2 as follows:

$$\begin{aligned}
I_2 &= \int_{2\mathrm{arc}\tanh|x|}^{\infty} \left(\frac{1}{\sinh s}\right)^{N-1} ds \\
&\le \int_{2\mathrm{arc}\tanh|x|}^{\infty} \left(\frac{1}{\sinh s}\right)^{N-1} \frac{1}{\tanh s}\, ds \\
&= \int_{2\mathrm{arc}\tanh|x|}^{\infty} \left(\frac{1}{\sinh s}\right)^{N} \cosh s\, ds \\
&= -\frac{1}{N-1}\left[(\sinh s)^{-(N-1)}\right]_{2\mathrm{arc}\tanh|x|}^{\infty} = \frac{1}{N-1}\left(\sinh\left(2\mathrm{arc}\tanh|x|\right)\right)^{-(N-1)}.
\end{aligned}$$

Combining this estimate with (2.3), we find (2.2). □

Lemma 2.2. *Let* $0 < m < (N-1)/2$. *Then for any*

$$\tau \in \left(\frac{2(N-1)}{N-1-2m}, \hat{m}\right)$$

there exists a constant $C = C(N, \tau, m)$ *such that*

$$\|w^m u\|_{L^\tau(\mathbb{B}^N)} \le C\, \|u\|_E \tag{2.4}$$

where

$$\hat{m} = \begin{cases} \dfrac{2N}{N-2-2m} & \text{when } m < \dfrac{N-2}{2}. \\ \infty & \text{when } \dfrac{N-2}{2} \le m < \dfrac{N-1}{2}. \end{cases}$$

Proof. Let

$$0 < m < \frac{N-1}{2}.$$

To prove (2.4), we divide the integral into two parts:

$$\begin{aligned}\int_{\mathbb{B}} w^{m\tau}|u|^{\tau}\,dV_{\mathbb{H}} &= \int_{0\le|x|\le\frac12}\left(\frac{2|x|}{1-|x|^2}\right)^{m\tau}|u|^{\tau}\left(\frac{2}{1-|x|^2}\right)^{N}dx\\ &\quad+\int_{\frac12\le|x|\le1}\left(\frac{2|x|}{1-|x|^2}\right)^{m\tau}|u|^{\tau}\left(\frac{2}{1-|x|^2}\right)^{N}dx\\ &=: X+Y.\end{aligned}$$

First we estimate the term X. By (2.1), we have

$$\begin{aligned}X &\le C\|u\|_E^{\tau}\int_{0\le|x|\le\frac12}\left(\frac{2|x|}{1-|x|^2}\right)^{m\tau}(\sinh(2\operatorname{arc\,tanh}|x|))^{-\frac{N-2}{2}\tau}\left(\frac{2}{1-|x|^2}\right)^{N}dx\\ &= C\|u\|_E^{\tau}\int_0^{2\operatorname{arc\,tanh}\frac12}(\sinh s)^{m\tau+N-1-\frac{N-2}{2}\tau}\,ds\\ &\le C\|u\|_E^{\tau}\int_0^{2\operatorname{arc\,tanh}\frac12}(\sinh s)^{m\tau+N-1-\frac{N-2}{2}\tau}\cosh s\,ds\\ &= C\|u\|_E^{\tau}\int_0^{\sinh(2\operatorname{arc\,tanh}\frac12)}t^{m\tau+N-1-\frac{N-2}{2}\tau}dt.\end{aligned}$$

Since the relation

$$m\tau+N-1-\frac{N-2}{2}\tau>-1$$

holds if and only if $\tau<\hat{m}$. Thus we see that if $\tau<\hat{m}$ then it holds that

$$X\le C\|u\|_E^{\tau},$$

where C depends only on N, τ and m. On the other hand (2.2) gives us that

$$\begin{aligned}Y &\le C\|u\|_E^{\tau}\int_{\frac12\le|x|\le1}\left(\frac{2|x|}{1-|x|^2}\right)^{m\tau}(\sinh(2\operatorname{arc\,tanh}|x|))^{-\frac{N-1}{2}\tau}\left(\frac{2}{1-|x|^2}\right)^{N}dx\\ &= C\|u\|_E^{\tau}\int_{2\operatorname{arc\,tanh}\frac12}^{\infty}(\sinh s)^{m\tau+N-1-\frac{N-1}{2}\tau}\,ds\\ &\le C\|u\|_E^{\tau}\int_{2\operatorname{arc\,tanh}\frac12}^{\infty}(\sinh s)^{m\tau+N-1-\frac{N-1}{2}\tau}\frac{1}{\tanh s}ds\\ &\le C\|u\|_E^{\tau}\int_{2\operatorname{arc\,tanh}\frac12}^{\infty}(\sinh s)^{m\tau+N-2-\frac{N-1}{2}\tau}\cosh s\;ds\\ &= C\|u\|_E^{\tau}\int_{\sinh(2\operatorname{arc\,tanh}\frac12)}^{\infty}t^{m\tau+N-2-\frac{N-1}{2}\tau}dt.\end{aligned}$$

It is easy to verify that

$$m\tau+N-2-\frac{N-1}{2}\tau<-1$$

is equivalent to

$$\tag{2.5} \tau > \frac{2(N-1)}{N-1-2m}.$$

Hence we see that

$$Y \le C \left\| u \right\|_E^\tau$$

if τ satisfies (2.5). Therefore we obtain the conclusion. □

Making use of Lemma 2.2, we shall prove a compactness.

Lemma 2.3. *Let $0 < m < (N-1)/2$. Let τ satisfy the condition given in Lemma 2.2. Then the map $u \mapsto w^m u$ from E to $L^\tau(\mathbb{B}^N)$ is compact.*

Proof. Let $0 < m < (N-1)/2$ and arbitrarily fix τ satisfying the condition given in Lemma 2.2. Then Lemma 2.2 asserts that

$$\left\| w^m u \right\|_{L^\tau(\mathbb{B}^N)} \le C \left\| u \right\|_E .$$

This shows that the map $u \mapsto w^m u$ from E to $L^\tau(\mathbb{B}^N)$ is continuous. Now we shall prove that the map is compact.

We first note that the embedding $H^1_{rad}(\mathbb{B}^N) \hookrightarrow L^q(\mathbb{B}^N)$ is compact for any $q \in (2, 2N/(N-2))$(see [29], Theorem3.1). Recalling that E is equivalent to $H^1_{rad}(\mathbb{B}^N)$ with respect to the norm $\left\|\cdot\right\|_E$, we see that the embedding $E \hookrightarrow L^q(\mathbb{B}^N)$ is also compact for any $q \in (2, 2N/(N-2))$.

Let us fix $q \in (2, \min\{\tau, 2N/(N-2)\})$ arbitrarily. By Hölder's inequality, we have

$$\tag{2.6} \begin{aligned} |w^m u|_{L^\tau(\mathbb{B}^N)} &= \left(\int_{\mathbb{B}^N} |w^m u|^\tau \right)^{\frac{1}{\tau}} \\ &= \left(\int_{\mathbb{B}^N} w^{m\tau} |u|^{\tau - qa} |u|^{qa} \right)^{\frac{1}{\tau}} \\ &\le \left(\int_{\mathbb{B}^N} |u|^q \right)^{\frac{a}{\tau}} \left(\int_{\mathbb{B}^N} \left(w^{m\tau} |u|^{\tau-qa} \right)^{\frac{1}{1-a}} \right)^{\frac{1-a}{\tau}} \\ &= |u|^{\frac{aq}{\tau}}_{L^q(\mathbb{B}^N)} \left| w^{\frac{m\tau}{\tau-qa}} |u| \right|^{\frac{\tau-qa}{\tau}}_{L^{\frac{\tau-qa}{1-a}}(\mathbb{B}^N)}, \end{aligned}$$

where $a \in (0,1)$. In the following, setting

$$m^* := \frac{m\tau}{\tau - qa}, \quad \tau^* := \frac{\tau - qa}{1-a},$$

and making use of Lemma 2.2, we shall verify that

$$\tag{2.7} \left\| w^{m^*} u \right\|_{L^{\tau^*}(\mathbb{B}^N)} \le C \left\| u \right\|_E$$

holds. If $m \geq (N-2)/2$, then the relation $m < m^*$ implies $m^* \geq (N-2)/2$. Since

$$\frac{2(N-1)}{N-1-2m^*} < \tau^* \iff \frac{2(N-1)+(q-2)(N-1)a}{N-1-2m} < \tau$$

for sufficiently small $a > 0$, Lemma 2.2 asserts that (2.7) holds for each $\tau > 2(N-1)/(N-1-2m)$. Regarding the case of $0 < m < (N-2)/2$, it is sufficient to consider the case of $0 < m^* < (N-2)/2$ since the case of $m^* \geq (N-2)/2$ is contained in the above case. Recalling

$$\tau^* < \frac{2N}{N-2-2m^*} \iff \tau < \frac{qa(N-2)+2N(1-a)}{N-2-2m}$$

and $2N-(N-2)q > 0$, we observe from Lemma 2.2 that for $a \in (0,1)$ small enough (2.7) holds for each τ satisfying

$$\frac{2(N-1)}{N-1-2m} < \tau < \frac{2N}{N-2-2m}.$$

Combining (2.6) with (2.7), we obtain

$$\|w^m u\|_{L^\tau(\mathbb{B}^N)} \leq C\,\|u\|_{L^q(\mathbb{B}^N)}^{\frac{aq}{\tau}}\,\|u\|_E^{\frac{\tau-qa}{\tau}}, \tag{2.8}$$

for sufficiently small $a \in (0,1)$. Thus the map $u \mapsto w^m u$ from E to $L^\tau(\mathbb{B}^N)$ is continuous.

Finally we show that the map $u \mapsto w^m u$ is compact. Let $\{u_n\}$ be bounded sequence in E. Since $E \hookrightarrow L^q(\mathbb{B}^N)$ is compact, there exists a subsequence $\{u_{nj}\} \subset \{u_n\}$ and a function $u \in E$ such that

$$u_{nj} \to u \quad \text{in} \quad L^q(\mathbb{B}^N).$$

By (2.8), we see that

$$\begin{aligned}
|w^m(u_{nj}-u)|_{L^\tau(\mathbb{B}^N)} &\leq |u_{nj}-u|_{L^q(\mathbb{B}^N)}^{\frac{aq}{\tau}}\,|\nabla_{\mathbb{H}}(u_{nj}-u)|_{L^2(\mathbb{B}^N)}^{\frac{\tau-qa}{\tau}} \\
&\leq C\,|u_{nj}-u|_{L^q(\mathbb{B}^N)}^{\frac{aq}{\tau}}\left(|u_{nj}|_E^{\frac{\tau-qa}{\tau}} + |u|_E^{\frac{\tau-qa}{\tau}}\right) \\
&\leq C\,|u_{nj}-u|_{L^q(\mathbb{B}^N)}^{\frac{aq}{\tau}}
\end{aligned}$$

Therefore, we complete the proof. □

We are in a position to prove the following theorem by using above propositions:

Theorem 2.1. *Let*

$$p \in \left(\frac{N-1+2\alpha}{N-1}, \frac{N+2+2\alpha}{N-2}\right).$$

Then, the equation (H) *has a positive radial solution* $u \in H^1(\mathbb{B}^N) \cap C^2(\mathbb{B}^N)$.

Proof. Instead of the equation (H), we prove that

$$\begin{cases} -\Delta_{\mathbb{H}} u = w^{\alpha} (u^{+})^{p} & \text{in} \quad \mathbb{B}^N \\ \lim\limits_{|x| \to 1} u = 0 \end{cases}$$

has a nontrivial solution in $H^1(\mathbb{B}^N)$ by using Mountain Pass Theorem.
Let

$$J(u) := \frac{1}{2} \int_{\mathbb{B}^N} |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2 \, dV_{\mathbb{H}} - \int_{\mathbb{B}^N} w^{\alpha} F(u) dV_{\mathbb{H}},$$

where

$$F(u) := \frac{1}{p+1} \left(u^{+}\right)^{p+1}, \quad u^{+} := \max\{u, 0\}.$$

To begin with, we verify that the functional J is well-defined. Since

$$\frac{2N-2}{N-1-2\frac{\alpha}{p+1}} < p+1 \iff \frac{N-1+2\alpha}{N-1} < p,$$

and

$$p+1 < \frac{2N}{N-2-2\frac{\alpha}{p+1}} \iff p < \frac{N+2+2\alpha}{N-2},$$

Lemma 2.3 implies that

$$\int_{\mathbb{B}^N} w^{\alpha} F(u) dV_{\mathbb{H}} \leq C \int_{\mathbb{B}^N} \left| w^{\frac{\alpha}{p+1}} u \right|^{p+1} dV_{\mathbb{H}} \leq C \|u\|_E^{p+1} . \tag{2.9}$$

Next we show that J satisfies the hypothesis of the Proposition 2.1. The relation (2.9) yields that

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{B}^N} |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2 \, dV_{\mathbb{H}} - \int_{\mathbb{B}^N} w^{\alpha} F(u) dV_{\mathbb{H}} \\ &\geq \frac{1}{2} \|u\|_E^2 - C \|u\|_E^{p+1} \end{aligned}$$

Thus, setting

$$f(\rho) := \frac{1}{2}\rho^2 - C\rho^{p+1},$$

we see that for $\rho > 0$ sufficiently small

$$f(\rho) > f(0) = 0.$$

Therefore, (B) is fulfilled. We turn to the condition (A). It is clear that $J(0) = 0$. Since

$$J(tu) = \frac{t^2}{2}\int_{\mathbb{B}^N} |\nabla_{\mathbb{H}} u|_{\mathbb{H}}^2 \, dV_{\mathbb{H}} - t^{p+1}\int_{\mathbb{B}^N} w^\alpha F(u) dV_{\mathbb{H}} \to -\infty \quad \text{as} \quad t \to \infty$$

we observe that there exists $e \in E$ such that $J(e) = 0$. Thus, (A) is fulfilled.

Next we prove that J satisfies the Palais-Smale condition. Define a map $T : E \to E$ by

$$(Tu, v)_E = \int_{\mathbb{B}^N} w^\alpha \left(u^+\right)^p v, \quad v \in E.$$

T may be decomposed as follows:

$$\begin{aligned} T : & u \mapsto w^{\frac{\alpha}{p}} u \mapsto w^{\frac{\alpha}{p}} u^+ \mapsto \left(w^{\frac{\alpha}{p}} u^+\right)^p \mapsto w^\alpha \left(u^+\right)^p \mapsto Tu \\ & E \xrightarrow{T_1} L^{pq} \xrightarrow{T_2} L^{pq} \quad \xrightarrow{T_3} L^q \qquad\quad \xrightarrow{T_4} H^{-1} \quad \xrightarrow{T_5} E, \end{aligned}$$

where

$$q = \begin{cases} \dfrac{2N}{N+2} & \text{if} \quad p \in \left(\dfrac{2\alpha}{N-1} + \dfrac{N+2}{N}, \dfrac{N+2+2\alpha}{N-2}\right) := I_1, \\ 2 & \text{if} \quad p \in \left(\dfrac{N-1+2\alpha}{N-1}, \dfrac{N+2\alpha}{N-2}\right) := I_2. \end{cases}$$

In the following we shall show that the map T is compact. To begin with, we verify that T_1 is compact by using Lemma (2.3). To do so, setting $\tilde{m} = \alpha/p$ and $\tilde{\tau} = pq$, we check that $\tilde{m} = \alpha/p$ and $\tilde{\tau} = pq$ satisfy the condition in Lemma (2.3). Remark that $p > (N - 1 + 2\alpha)/(N - 1)$ implies $\tilde{m} < (N - 1)/2$. To begin with, we check that

$$\frac{2N-2}{N-1-2\tilde{m}} < \tilde{\tau}. \tag{2.10}$$

When $p \in I_1$, one can verify that (2.10) is equivalent to

$$\frac{2\alpha}{N-1} + \frac{N+2}{2} < p.$$

On the other hand, (2.10) is equivalent to

$$\frac{N-1+2\alpha}{N-1} < p,$$

if $p \in I_2$. Hence (2.10) is satisfied. Since $\tilde{\tau} < +\infty$, it is sufficient to show that if $\tilde{m} < (N - 2)/2$ then

$$\tilde{\tau} < \frac{2N}{N-2-2\tilde{m}}. \tag{2.11}$$

For the case of $p \in I_1$, (2.12) is equivalent to

$$p < \frac{N+2+2\alpha}{N-2},$$

and while if $p \in I_2$, then (2.12) is equivalent to

$$p < \frac{N+2\alpha}{N-2}.$$

Therefore we can apply Lemma 2.3 to the map T_1. Then Lemma 2.3 asserts that T_1 is compact. The map T_2 is clearly continuous. Regarding T_3, since the map is a Nemitski operator, we see that T_3 is continuous. Next we turn to T_4. Let us define $T_4 : L^q \to (L^{\hat{q}})^*$ by

$$(T_4(w^\alpha (u^+)^p))(v) = \int_{\mathbb{B}^N} w^\alpha (u^+)^p v, \quad v \in L^{\hat{q}},$$

where

$$\hat{q} = \begin{cases} \dfrac{2N}{N-2} & \text{if } \; p \in \left(\dfrac{2\alpha}{N-1} + \dfrac{N+2}{N}, \dfrac{N+2+2\alpha}{N-2} \right), \\ 2 & \text{if } \; p \in \left(\dfrac{N-1+2\alpha}{N-1}, \dfrac{N+2\alpha}{N-2} \right). \end{cases}$$

Hölder's inequality yields that $T_4 : L^q \to (L^{\hat{q}})^*$ is continuous. Since $H^1 \hookrightarrow L^{\hat{q}}$ implies $(L^{\hat{q}})^* \hookrightarrow H^{-1}$, we see that $T_4 : L^q \to H^{-1}$ is also continuous. Therefore, $T_4 : L^q \to H^{-1}$ is continuous. Finally we show that T_5 is continuous. Define $T_5 : H^{-1} \to H^1$ by

$$(T_5(f), v)_E = f(v) \quad \text{for} \quad f \in H^{-1} \quad \text{and} \quad v \in H^1.$$

Then we have

$$|(T_5(f), v)_E| \le \|f\|_{H^{-1}} \|v\|_{H^1} \le C \|f\|_{H^{-1}} \|v\|_E,$$

so that,

$$|T_5(f)|_{H^1} \le \hat{C} \|f\|_{H^{-1}}.$$

Therefore T_5 is continuous. In particular, we observe that

$$(T_5(T_4(w^\alpha (u^+)^p)), v)_E = (T_4(w^\alpha (u^+)^p))(v) = \int_{\mathbb{B}^N} (w^\alpha (u^+)^p) v = (Tu, v)_E.$$

Thus, $T = T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1$ is compact from E to E.

Let $\{u_n\} \subset E$ be a sequence satisfying $|J(u_n)| \le d$ and $J'(u_n) \to 0$. For $n \in \mathbb{N}$ large enough, we have

$$\begin{aligned} d + \|u_n\|_E &\ge J(u_n) - \frac{1}{\tau+1} J'(u_n)(u_n) \\ &= \left(\frac{1}{2} - \frac{1}{\tau+1} \right) \|u_n\|_E^2. \end{aligned}$$

This implies that $\|u_n\|_E^2$ is bounded. Then there exists a subsequence $u_{n_j} \subset u_n$ and a function $u \in E$ such that

$$u_{n_j} \rightharpoonup u \quad \text{in} \quad E. \tag{2.12}$$

Furthermore, since T is compact operator, it follows from (2.12) that

$$Tu_{n_j} \to \hat{u} \quad \text{in} \quad E$$

for a function $\hat{u} \in E$ up to a subsequence. Recalling that

$$(u_n - Tu_n, v)_E = J'(u_n)(v) \to 0 \quad \text{as} \quad n \to \infty$$

for any $v \in E$, it must hold $\hat{u} = u$. In the following we write u_n instead of u_{n_j} for short. By a simple calculation, we have

$$\begin{aligned} \|u_n - u\|_E &= J'(u_n)(u_n - u) - J'(u)(u_n - u) + (Tu_n - Tu, u_n - u)_E \\ &=: I_1 + I_2 + I_3, \end{aligned} \tag{2.13}$$

and then

$$\begin{aligned} I_1 &\le \|J'(u_n)\|_{E^*} \|u_n - u\|_E \le \|J'(u_n)\|_{E^*} (\|u_n\|_E + \|u\|_E) \to 0, \\ I_2 &= J'(u)(u_n - u) \to 0, \\ I_3 &= (Tu_n - u, u_n - u)_E + (u - Tu, u_n - u)_E \\ &\le \|Tu_n - u\|_E (\|u_n\|_E + \|u\|_E) + (u - Tu, u_n - u)_E \to 0. \end{aligned}$$

Therefore (2.13) yields that

$$u_n \to u \quad \text{in} \quad E.$$

This implies that $\{u_n\}$ has a convergent subsequence, i.e., J satisfies the Palais-Smale condition. Then, the Mountain Pass Lemma assures that J has a nontrivial critical value, hence, a nontrivial critical point $u \in E$. In particular, function u satisfies

$$J'(u)(v) = \int_{\mathbb{B}^N} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} v \rangle_{\mathbb{H}} \, dV_{\mathbb{H}} - \int_{\mathbb{B}^N} w^\alpha \left(u^+\right)^p v \, dV_{\mathbb{H}} = 0 \quad \text{for} \quad v \in E. \tag{2.14}$$

Taking u^- as v in (2.14), we have

$$0 = \int_{\mathbb{B}^N} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} u^- \rangle_{\mathbb{H}} \, dV_{\mathbb{H}} - \int_{\mathbb{B}^N} w^\alpha \left(u^+\right)^p u^- \, dV_{\mathbb{H}} = \|u^-\|_E ,$$

so that $u^- = 0$ a.e. in $\mathbb{B}^N$. Therefore, combining this fact with (2.14), we see that u is a nonnegative and nontrivial $H^1(\mathbb{B}^N)$ solution of (H).

By an elliptic regularity theorem, $u \in C^2$. Finally we shall prove that u is a positive solution. Suppose not, there exists $x_0 \in \mathbb{B}^N$ such that $u(x_0) = 0$. For any $r > 0$, it holds that

$$-\Delta_{\mathbb{H}} u = w^\alpha \left(u^+\right)^p \ge 0 \quad \text{in} \quad B_{\mathbb{H}}(\xi_0, r),$$

where $B_{\mathbb{H}}(x_0, r) = \{x \in \mathbb{B}^N : d_{\mathbb{H}}(x, x_0) < r\}$. Then the strong maximum principle implies that $u \equiv 0$ in $B_{\mathbb{H}}(x_0, r)$. Since $r > 0$ is arbitrary, we see that $u \equiv 0$ in $\mathbb{B}^N$. This leads a contradiction. $\square$

3 Liouville Theorem

In this section, we prove a Liouville theorem corresponding to (H). First, in order to state the result, we define the stability of solutions. The stability of solutions of (H) is defined by the same manner as in Definition 1.1:

Definition 3.1. *The solution $u \in C^2(\mathbb{B}^N)$ of* (H) *is stable if the inequality*

$$Q[u](\psi) := \int_{\mathbb{B}^N} \left\{ |\nabla_{\mathbb{H}} \psi|^2_{\mathbb{H}} - pw^{\alpha} |u|^{p-1} \psi^2 \right\} dV_{\mathbb{H}} \geq 0$$

holds for any $\psi \in C^1_c(\mathbb{B}^N)$.

Then we state the Liouville theorem corresponding to the equation (H):

Theorem 3.1 ([21]). *Let $u \in C^2(\mathbb{B}^N)$ be a stable solution of* (H). *If $p > 1$ satisfies*

$$\begin{cases} 1 < p < +\infty & \text{if} \quad N \leq 1 + 4\alpha, \\ 1 < p < p_c(\alpha, N) & \text{if} \quad N > 1 + 4\alpha, \end{cases}$$

then $u \equiv 0$ in $\mathbb{B}^N$. Here, $p_c(\alpha, N)$ is given by the following:

$$p_c(\alpha, N) := \frac{(N-1)^2 - 2\alpha(N-1) - 2\alpha^2 + 2\alpha\sqrt{2\alpha(N-1) + \alpha^2}}{(N-1)(N-4\alpha-1)}.$$

Theorem 3.1 gives us an affirmative answer to Problem 1.1. And if we find a non-trivial stable solution when $p \geq p_c$, then p_c is critical. Although we have not proved this fact yet, we obtained the following result which suggests that p_c is critical:

Theorem 3.2 ([21]). *Let $p > (N + 2 + 2\alpha) / (N - 2)$. Then, there exists a positive radial solution $u = u(r)$ of* (H) *satisfying*

$$\lim_{r \to +\infty} u(r) (\sinh r)^{\frac{\alpha}{p-1}} = \left\{ \frac{\alpha}{p-1} \left(N - 1 - \frac{\alpha}{p-1} \right) \right\}^{\frac{1}{p-1}} := L.$$

Now, using Theorem 3.2, we can give some consideration to $p_c(\alpha, N)$. Let $p \geq p_c(\alpha, N)$ and $N > 1 + 4\alpha$. We assume that there exists a radial solution $u = u(r)$ of (H) satisfying

$$u(r) (\sinh r)^{\frac{\alpha}{p-1}} \leq L \qquad (\forall r > 0). \tag{3.1}$$

Then, by some calculations, we see that the solution u satisfying (3.1) is stable. From Theorem 3.2, one can notice that the condition 3.1 is valid. Therefore we can expect that the exponent $p_c(\alpha, N)$ is critical.

Next, we state the outline of proof of Theorem 3.1. First, we prepare the following proposition:

Proposition 3.1. *Let $u \in C^2(\mathbb{B}^N)$ be a stable solution of* (H). *Then, for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and for any integer $m \geq \max\left\{\frac{p+\gamma}{p-1}, 2\right\}$, there exists some positive constant $C = C(p, m, \alpha, \gamma)$ such that for any $\psi \in C_c^2(\mathbb{B}^N)$ with $|\psi| \leq 1$,*

$$\int_{\mathbb{B}^N} w^\alpha |u|^{p+\gamma} \psi^{2m} dV_{\mathbb{H}} \leq C \int_{\mathbb{B}^N} w^{-\frac{\gamma+1}{p-1}\alpha} |\nabla_{\mathbb{H}}\psi|_{\mathbb{H}}^{2\frac{p+\gamma}{p-1}} dV_{\mathbb{H}}.$$

We can prove this assertion by a modification of the proof in Proposition 1.4 of [10] and Proposition 1.7 of [11]. In the following, we prove Theorem 3.1 by using Proposition 3.1.

Proof. Here, the essential matter of Proposition 3.1 is that one can estimate the integral of u by the integral being independent of u. Therefore, we expect that the stable solution u can be characterized by the test function. Indeed, in order to prove Theorem 3.1, we set the following test function ψ_R for each $R > 0$:

$$\psi_R(x) := \varphi\left(\frac{\sinh(d_{\mathbb{H}}(0, x))}{R}\right),$$

where $\varphi \in C_c^2(\mathbb{R})$ satisfies $0 \leq \varphi \leq 1$ and

$$\varphi(t) = \begin{cases} 1 & \text{if} \quad |t| \leq 1, \\ 0 & \text{if} \quad |t| \geq 2. \end{cases}$$

In the following, we write

$$q = \frac{p+\gamma}{p-1}, \quad \bar{q} = \frac{\gamma+1}{p-1}$$

for short and we set

$$A(R) = \operatorname{arc\,sinh} R, \quad B(R) = \operatorname{arc\,sinh} 2R.$$

Then, notice that

$$\psi_R(x) = \begin{cases} 1 & \text{if} \quad d_{\mathbb{H}}(0, x) \leq A(R), \\ 0 & \text{if} \quad d_{\mathbb{H}}(0, x) \geq B(R). \end{cases}$$

Since the change of variable $r = d_{\mathbb{H}}(0, x)$ yields $w(x) = \sinh r$ and $dV_{\mathbb{H}} = (\sinh r)^{N-1} dr$, it follows from Proposition 3.1 that

$$\begin{aligned} \text{(3.2)} \qquad \int_{d_{\mathbb{H}}(0,x) \leq A(R)} w^\alpha |u|^{p+\gamma} dV_{\mathbb{H}} &\leq C \int_{A(R) \leq d_{\mathbb{H}}(0,x) \leq B(R)} w^{-\bar{q}\alpha} |\nabla_{\mathbb{H}}\psi_R|_{\mathbb{H}}^{2q} dV_{\mathbb{H}} \\ &\leq \frac{C}{R^{2q}} \int_{A(R)}^{B(R)} (\sinh r)^{N-1-\bar{q}\alpha+2q} dr \\ &\leq C R^{N-1-\bar{q}\alpha}. \end{aligned}$$

On the other hand, $p < p_c(\alpha, N)$ if and only if there exists some $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ such that

$$N - 1 - \bar{q}\alpha < 0. \tag{3.3}$$

Hence, we can choose $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ satisfying (3.3). And then, (3.2) implies that

$$\int_{d_{\mathbb{H}}(0,x) \leq A(R)} w^\alpha |u|^{p+\gamma} \, dV_{\mathbb{H}} \to 0 \quad \text{as} \quad R \to +\infty.$$

Since $A(R) \to +\infty$ as $R \to +\infty$, we see that u must be identically equal to 0. This completes the proof of Theorem 1.2. □

Here, in order to obtain the estimate just as (3.2) in this proof, we have to select the weight w and test function ψ_R in terms of the volume element $dV_{\mathbb{H}}$. Hence, since the weight of the equation (1.2) is the power of the geodesic distance, the above argument does not work for the equation (1.2). This is the reason why we choose the weight of (H).

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