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Kyoto University
Construction of solutions to a fourth order parabolic obstacle problem via minimizing movements

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1 Introduction

The paper is devoted to a fourth order parabolic obstacle problem. We shall announce a result ([13]) which is a joint work with M. Novaga of Pisa University.

The obstacle problem for elliptic and parabolic PDE’s is a topics which attracted a great interest in the past years, and has been widely discussed in the mathematical literature. However, even if many studies are available for second order elliptic and parabolic equations (see for instance [6, 9] and references therein), there are relatively few results for higher order obstacle problems, even in the linear fourth order case. More precisely, the elliptic obstacle problem for the biharmonic operator has been considered in [5, 7, 8, 10, 11, 14]. To the best of our knowledge, there is no analog for the corresponding parabolic obstacle problem. The purpose of this paper is to investigate the regularity of a solution to the obstacle problem for the parabolic biharmonic equation.

In the sequel we let $\Omega \subset \mathbb{R}^N$ be a bounded domain, with boundary of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$, and we let $f : \Omega \rightarrow \mathbb{R}$ be the obstacle function, satisfying

\begin{equation}
  f \in C^2(\overline{\Omega}), \quad f < 0 \quad \text{on} \quad \partial \Omega.
\end{equation}

We consider an initial datum $u_0 : \Omega \rightarrow \mathbb{R}$ such that

\begin{equation}
  u_0 \in H_0^2(\Omega), \quad u_0 \geq f \quad \text{a.e. in} \quad \Omega.
\end{equation}

We recall that $u \in H_0^2(\Omega)$ implies $u = 0$ and $\nabla u \cdot \nu^\Omega = 0$ on $\partial \Omega$ (see [7, 8]), that is, $u$ satisfies Dirichlet boundary condition on $\partial \Omega$. 

In this paper, we consider the following fourth order parabolic obstacle problem:

\[ \begin{align*}
\partial_t u(x, t) + \Delta^2 u(x, t) &\geq 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,
\partial_t u(x, t) + \Delta^2 u(x, t) & = 0 \quad \text{on} \quad \{ (x, t) \in \Omega \times \mathbb{R}_+ : u(x, t) > f(x) \},
\partial_t u(x, t) & = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+,
\nabla u(x, t) \cdot \nu^n(x) & = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+,
u^n(x) = \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}},
\nabla u(x, t) & \geq f(x) \quad \text{in} \quad \Omega \times \mathbb{R}_+,
u^n(x) = \frac{\nabla u(x, t)}{\sqrt{1 + |\nabla u(x, t)|^2}},
\nabla u(x, t) & = u_0(x) \quad \text{in} \quad \Omega.
\end{align*} \]

The main result of this paper is the following:

**Theorem 1.1.** Let \( N \leq 3 \), and let \( f \) be a function satisfying (1.1). Then, for any initial data \( u_0 \) satisfying (1.2), the problem (P) has a unique solution

\[ u \in L^\infty(\mathbb{R}_+; H_0^2(\Omega)) \cap H_1^{1}(\mathbb{R}_+; L^2(\Omega)), \quad \text{with} \quad u_t \in L^2(\mathbb{R}_+ \times \Omega). \]

Furthermore the solution \( u \) satisfies the following properties:

(i) \( u \in L^2(0, T; W^{2,\infty}(\Omega)) \) for any \( T > 0 \). In particular, if \( N = 1 \),

\[ u \in C^{0,\beta}([0, T]; C^{1,\gamma}(\Omega)) \quad \text{with} \quad 0 < \gamma < \frac{1}{2} \quad \text{and} \quad 0 < \beta < \frac{1-2\gamma}{8}, \]

if \( N = 2, 3 \),

\[ u \in C^{0,\beta}([0, T]; C^{0,\gamma}(\Omega)) \quad \text{with} \quad 0 < \gamma < \frac{4-N}{2} \quad \text{and} \quad 0 < \beta < \frac{4-N-2\gamma}{8}. \]

(ii) For a.e. \( t \in \mathbb{R}_+ \) the quantity

\[ \mu_t := u_t(\cdot, t) + \Delta^2 u(\cdot, t) \]

defines a Radon measure in \( \Omega \), and for any \( T > 0 \) there exists a constant \( C > 0 \) such that

\[ \int_0^T \mu_t(\Omega)^2 dt < C. \]

Let us point out that the problem (P) corresponds to the gradient flow of a convex functional defined on the Hilbert space \( L^2(\Omega) \), hence we can apply the general theory of maximal monotone operators developed in [4]. Indeed, given \( f \) as above, we can define the functional \( E_f(u) : L^2(\Omega) \to [0, +\infty] \) as

\[ E_f(u) = \begin{cases} 
\int_\Omega |\Delta u|^2 & \text{if} \quad u \in H^2(\Omega) \quad \text{and} \quad u \geq f,
+\infty & \text{otherwise}.
\end{cases} \]
Notice that $E_f(u)$ is convex and lower semicontinuous on $L^2(\Omega)$, and (P) corresponds to the gradient flow

\begin{equation}
 u_t + \partial E_f(u) \ni 0, \quad u(0) = u_0,
\end{equation}

where $\partial E_f$ denotes the subdifferential of $E_f$ in $L^2(\Omega)$. In particular, given an initial datum $u_0 \in H_0^2(\Omega)$ with $u_0 \geq f$, by the results in [4] it follows that the evolution problem (1.8) has a unique solution $u$ satisfying

\begin{equation}
 u \in L^\infty(\mathbb{R}_+; H_0^2(\Omega)) \cap H^1_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \quad \text{with} \quad u_t \in L^2(\mathbb{R}_+ \times \Omega).
\end{equation}

The purpose of this paper is to give an extra regularity of solution to (P). To this aim, we characterize the solution $u$ by means of an implicit variational scheme, corresponding to the minimizing movements introduced by De Giorgi (see e.g. [2]). This approach will allow us to extend some of the arguments in [7], concerning the regularity of the elliptic obstacle problem for the biharmonic operator.

We point out that the method does not rely on the linear structure of the problem and can be applied to more general fourth order parabolic equations. Indeed our motivation for this work rise from an analysis of a motion of planar closed curves which is governed by the straightening flow with obstacle. Curve straightening flow is a $L^2$ gradient flow for the total squared curvature

$$
\mathcal{E}(\gamma) := \frac{1}{2} \int_\gamma \kappa^2 ds,
$$

where $\gamma$ is a closed planar curve, $\kappa$ and $s$ denote respectively the curvature and the arc length parameter of $\gamma$. Although the flow is a fourth order quasilinear parabolic equation, we expect that the method of this paper will be available for the geometric obstacle problem.

The paper is organized as follows: in Section 2 we introduce the implicit scheme corresponding to problem (P), by means of an appropriate variational problem; in Section 3 we study the regularity of solutions to the variational problem; in Section 4 we pass to the limit in the approximating scheme and prove Theorem 1.1.

\section{Preliminary}

The equation in (P) is the $L^2$ gradient flow for the functional

$$
E(u) = \frac{1}{2} \int_\Omega |\Delta u(x)|^2 \, dx.
$$

Let $T > 0$, $n \in \mathbb{N}$, and set

$$
\tau_n = \frac{T}{n}.
$$
Let us set $u_{0,n} = u_0$. For $i = 1, \ldots, n$, we define inductively $u_{i,n}$ as a solution of the minimum problem

$$(M_{i,n}) \quad \min \{G_{i,n}(u) : u \in K\},$$

where

$$(2.1) \quad G_{i,n}(u) := E(u) + P_{i,n}(u)$$

with

$$(2.2) \quad P_{i,n}(u) := \frac{1}{2\tau_n} \int_{\Omega} (u - u_{i-1,n})^2 dx,$$

and $K$ is a convex set given by

$$K := \{u \in H^2_0(\Omega) : u(x) \geq f(x) \text{ a.e. in } \Omega\}.$$

In the following, we let

$$(2.3) \quad V_{i,n}(x) := \frac{u_{i,n}(x) - u_{i-1,n}(x)}{\tau_n}.$$

We give a definition of a piecewise linear interpolations of $\{u_{i,n}\}$:

**Definition 2.1. (Piecewise linear interpolation)** Let $f$ be a function satisfying (1.1). Let $u_0 \in H^2_0(\Omega)$ with $u_0(x) \geq f(x)$ a.e. in $\Omega$. Define $u_n : \Omega \times [0, T] \to \mathbb{R}$ as

$$(2.4) \quad u_n(x, t) := u_{i-1,n}(x) + (t - (i-1)\tau_n)V_{i,n}(x)$$

if $(x, t) \in \Omega \times [(i-1)\tau_n, i\tau_n)$ for $i = 1, \ldots, n$.

By a technical reason, additionally we need a piecewise constant interpolations of $\{u_{i,n}\}$ and $\{V_{i,n}\}$.

**Definition 2.2. (Piecewise constant interpolation)** Define $\tilde{u}_n : \Omega \times [0, T] \to \mathbb{R}$ as

$$(2.5) \quad \tilde{u}_n(x, t) := u_{i,n}(x),$$

$$(2.6) \quad V_n(x, t) := V_{i,n}(x),$$

if $(x, t) \in \Omega \times [(i-1)\tau_n, i\tau_n)$ for $i = 1, \ldots, n$.

### 3 Existence and regularity of minimizers of $(M_{i,n})$

We first mention a well-known compactness result in $H^2_0(\Omega)$ ([1]).
Proposition 3.1. The following embedding is compact:

\[
H_{0}^{2}(\Omega) \hookrightarrow \begin{cases} 
C^{1,\gamma}(\Omega) & \text{for } 0 < \gamma < \frac{1}{2} \text{ if } N = 1, \\
C^{0,\gamma}(\Omega) & \text{for } 0 < \gamma < 2 - \frac{N}{2} \text{ if } N = 2, 3, \\
L^{q}(\Omega) & \text{for } 1 \leq \forall q < +\infty \text{ if } N = 4, \\
L^{q}(\Omega) & \text{for } 1 \leq \forall q < \frac{2N}{N-4} \text{ if } N \geq 5.
\end{cases}
\]

We start with the existence of minimizers of \((M_{i,n})\).

Theorem 3.1. (Existence of minimizers) Let \(f\) be a function satisfying (1.1). Let \(u_{0} \in H_{0}^{2}(\Omega)\) with \(u_{0}(x) \geq f(x)\) a.e. in \(\Omega\). Then the problem \((M_{i,n})\) possesses a unique solution \(u_{i,n} \in H_{0}^{2}(\Omega)\) with \(u_{i,n}(x) \geq f(x)\) a.e. in \(\Omega\) for each \(i = 1, \ldots, n\).

Proof. Fix \(n \in \mathbb{N}, T > 0,\) and \(i = 1, \ldots, n\). From (2.1)-(2.2) and the minimality of a solution \(u\) to \((M_{i,n})\), we obtain that

\[
E(u) \leq G_{i,n}(u) \leq G_{i,n}(u_{i-1,n}) = E(u_{i-1,n}),
\]

and then

\[
0 \leq \inf_{H_{0}^{2}(\Omega)} G_{i,n}(u) \leq G_{i,n}(u_{i-1,n}) = E(u_{i-1,n}) \leq \cdots \leq E(u_{0}).
\]

Thus we can take a minimizing sequence \(\{u_{j}\} \subset H_{0}^{2}(\Omega)\) for \((M_{i,n})\) such that \(u_{j}(x) \geq f(x)\) a.e. in \(\Omega\) for each \(j \in \mathbb{N}\) and \(\sup_{j} G_{i,n}(u_{j}) < \infty\).

Observing that the norm \(\|\Delta u\|_{L^{2}(\Omega)}\) is equivalent to \(\|u\|_{H_{0}^{2}(\Omega)}\) (e.g., see [12]), it follows from

\[
\|\Delta u_{j}\|_{L^{2}(\Omega)} = \sqrt{2E(u_{j})} \leq \sqrt{2E(u_{0})} = \|\Delta u_{0}\|_{L^{2}(\Omega)}
\]

that \(\{u_{j}\}\) is uniformly bounded in \(H_{0}^{2}(\Omega)\). Thus there exists \(u \in H_{0}^{2}(\Omega)\) such that

\[
u_{j} \rightharpoonup u \quad \text{in} \quad H_{0}^{2}(\Omega),
\]

in particular,

\[
\Delta u_{j} \rightharpoonup \Delta u \quad \text{in} \quad L^{2}(\Omega),
\]

up to a subsequence. Thanks to Proposition 3.1, we obtain that

\[
u_{j} \rightarrow u \quad \text{in} \quad \begin{cases} 
C^{1,\gamma}(\Omega) & \text{for } 0 < \gamma < \frac{1}{2} \text{ if } N = 1, \\
C^{0,\gamma}(\Omega) & \text{for } 0 < \gamma < 2 - \frac{N}{2} \text{ if } N = 2, 3, \\
L^{q}(\Omega) & \text{for } 1 \leq \forall q < +\infty \text{ if } N = 4, \\
L^{q}(\Omega) & \text{for } 1 \leq \forall q < \frac{2N}{N-4} \text{ if } N \geq 5.
\end{cases}
\]
In particular, for the case of $N \geq 4$,

\[(3.4)\quad u_j \rightharpoonup u \quad \text{a.e. in } \Omega \quad \text{up to a subsequence.}\]

Recalling $u_j \geq f \text{ a.e. in } \Omega$ for each $j \in \mathbb{N}$, (3.4) yields that $u \geq f \text{ a.e. in } \Omega$. Making use of Fatou's Lemma, we conclude that

\[(3.5)\quad P_{i,n}(u) \leq \liminf_{j \to \infty} P_{i,n}(u_j).\]

Furthermore (3.3) implies

\[(3.6)\quad E(u) = \frac{1}{2} \|\Delta u\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \liminf_{j \to \infty} \|\Delta u_j\|_{L^2(\Omega)}^2 = \liminf_{j \to \infty} E(u_j).\]

Combining (3.5) with (3.6), we see that $u \in H^2_0(\Omega)$ is the minimizer of $(M_{i,n})$ with $u \geq f \text{ a.e. in } \Omega$. The uniqueness follows from the fact that the functional $G_{i,n}(\cdot)$ is strictly convex. \hfill \Box

Regarding the regularity of the minimizer $u_{i,n}$ obtained in Theorem 3.1, we show the following:

**Theorem 3.2.** Let $u_{i,n}$ be the solution of $(M_{i,n})$ obtained by Theorem 3.1. Then, for any $n \in \mathbb{N}$, it holds that

\[(3.7)\quad \int_0^T \int_\Omega |V_n(x, t)|^2 \, dx \, dt \leq 2E(u_0),\]

\[(3.8)\quad \sup_i \|\Delta u_{i,n}\|_{L^2(\Omega)} \leq \sqrt{2E(u_0)}.\]

**Proof.** Fix $T > 0$ and $n \in \mathbb{N}$. For each $i = 1, \cdots, n$, it follows from (2.1)-(2.2) and the minimality of $u_{i,n}$ that

\[(3.9)\quad G_{i,n}(u_{i,n}) \leq G_{i,n}(u_{i-1,n}) = E(u_{i-1,n}).\]

Hence we get

\[P_{i,n}(u_{i,n}) \leq E(u_{i-1,n}) - E(u_{i,n}),\]

i.e.,

\[(3.10)\quad \frac{1}{2\tau_n} \int_\Omega (u_{i,n} - u_{i-1,n})^2 \, dx \leq E(u_{i-1,n}) - E(u_{i,n}).\]

Combining (3.10) with definitions (2.3) and (2.6), we obtain

\[\frac{1}{2} \int_0^T \int_\Omega |V_n(x,t)|^2 \, dx \, dt = \frac{1}{2} \sum_{i=1}^n \int_{(i-1)\tau_n}^{i\tau_n} \int_\Omega |V_{i,n}(x)|^2 \, dx \, dt\]

\[\leq \sum_{i=1}^n (E(u_{i-1,n}) - E(u_{i,n})) = E(u_0) - E(u_{n,n}) \leq E(u_0),\]
i.e., (3.7).

By (3.9), we obtain that $E(u_{i,n}) \leq E(u_{i-1,n})$ for each $i = 1, \cdots, n$, and then

\begin{equation}
\frac{1}{2} \int_{\Omega} (\Delta u_{i,n})^2 \, dx = E(u_{i,n}) \leq E(u_0).
\end{equation}

It is clear that (3.11) is equivalent to (3.8).

By the definition of $u_{i,n}$, we see that

\begin{align*}
\int_{\Omega} |\Delta(u_{i,n} + \varepsilon \zeta)|^2 \, dx + \frac{1}{2\tau_n} \int_{\Omega} (u_{i,n} - u_{i-1,n} + \varepsilon \zeta)^2 \, dx \\
\geq \int_{\Omega} |\Delta u_{i,n}|^2 \, dx + \frac{1}{2\tau_n} \int_{\Omega} (u_{i,n} - u_{i-1,n})^2 \, dx
\end{align*}

for any $\varepsilon > 0$ and $\zeta \in H^2_0(\Omega)$ with $\zeta \geq 0$. This implies

\begin{equation}
\int_{\Omega} \Delta u_{i,n} \Delta \zeta \, dx + \frac{1}{\tau_n} \int_{\Omega} (u_{i,n} - u_{i-1,n}) \zeta \, dx \geq 0,
\end{equation}

so that

\begin{equation}
\mu_{i,n} := \Delta^2 u_{i,n} + V_{i,n} \geq 0
\end{equation}

in the sense of the distribution. Hence $\mu_{i,n}$ is a measure in $\Omega$ (e.g., see [15]).

When we restrict dimensions to $N \leq 3$, Proposition 3.1 implies that $u_{i,n}$ is continuous. Under such restriction, we define

\begin{align}
C_{i,n} &:= \{x \in \Omega : u_{i,n}(x) = f(x)\}, \\
\mathcal{N}_{i,n} &:= \{x \in \Omega : u_{i,n}(x) > f(x)\}.
\end{align}

It is clear that $C_{i,n} \cup \mathcal{N}_{i,n} = \Omega$. We can show a relation between the support of $\mu_{i,n}$ and the sets.

**Lemma 3.1.** Let $N \leq 3$. If $x_0 \in \mathcal{N}_{i,n}$, then there exists a neighborhood $W$ of $x_0$ such that $\mu_{i,n}(W) = 0$. Furthermore we have

\begin{equation}
\text{supp } \mu_{i,n} \subseteq C_{i,n}.
\end{equation}

**Proof.** Let $N \leq 3$ and fix $x^0 \in \mathcal{N}_{i,n}$ arbitrarily. Since $\mathcal{N}_{i,n}$ is an open set, there exist a constant $\delta > 0$ and a neighborhood $W$ of $x^0$ such that

\begin{equation*}
u_{i,n}(x) - f(x) > \delta \quad \text{for all } x \in W.
\end{equation*}

Notice that $u_{i,n}$ satisfies

\begin{equation}
\int_{\Omega} \Delta u_{i,n} \Delta (u_{i,n} - \varphi) \, dx \leq - \int_{\Omega} V_{i,n} (u_{i,n} - \varphi) \, dx
\end{equation}
for any $\varphi \in K$, for $u_{i,n}$ is a solution of $(M_{i,n})$. Then for any $\zeta \in C_0^\infty(W)$ with $0 \leq \zeta \leq \delta/2$, the function

$$\psi = u_{i,n} - \zeta$$

belongs to $K$. Taking this $\psi$ as $\varphi$ in (3.16), we have

$$\int_{\Omega} [\triangle u_{i,n} \triangle \zeta + V_{i,n} \zeta] \, dx \leq 0.$$ 

Since $\mu_{i,n} \geq 0$, this asserts that

$$\int_{\Omega} [\triangle u_{i,n} \triangle \zeta + V_{i,n} \zeta] \, dx = 0,$$

i.e., $\mu_{i,n} = 0$ in $W$. \hfill \Box

Regarding the finiteness of $\mu_{i,n}$, we have the following:

**Theorem 3.3.** ([13]) Let $u_{i,n}$ be the solution of $(M_{i,n})$ obtained by Theorem 3.1. Then $\mu_{i,n}$ defined in (3.12) is a measure in $\Omega$ for each $i = 1, \cdots, n$. Moreover there exists a positive constant $C$ being independent of $n$ such that

$$\tau_n \sum_{i=1}^{n} \mu_{i,n}(\Omega)^2 < C. \tag{3.17}$$

Regarding the regularity of $u_{i,n}$, we obtain the following result:

**Theorem 3.4.** ([13]) Let $N \leq 3$. It holds that

$$u_{i,n} \in W^{2,\infty}(\Omega) \tag{3.18}$$

for each $n \in \mathbb{N}$ and $i = 1, \cdots, n$. Moreover, for any $R > 0$ with $\overline{B}_R \subset \Omega$, there exist positive constants $C_1$ and $C_2$ being independent of $n$ such that

$$\tau_n \sum_{i=1}^{n} \|D^2 u_{i,n}\|^2_{L^\infty(\Omega)} \leq C_1 + C_2 \|\Delta f\|^2_{L^\infty(\Omega)}. \tag{3.19}$$

### 4 Existence and regularity of solutions to (P)

In this section, we start with a convergence result of the piecewise linear interpolation $u_n$. We state several results without its proof. For the precise proof, see [13]. We first show a convergence result which holds in any dimension $N \geq 1$.

**Theorem 4.1.** Let $u_n$ be the piecewise linear interpolation of $\{u_{i,n}\}$. Then there exists a function

$$u \in L^\infty([0, +\infty); H^2_0(\Omega)) \cap H^1_{loc}(0, +\infty; L^2(\Omega))$$
such that

\[
(4.1) \quad u_n \rightharpoonup u \quad \text{in} \quad L^2(0, T; H^2_0(\Omega)) \cap H^1(0, T; L^2(\Omega)) \quad \text{as} \quad n \to +\infty,
\]

up to a subsequence, for any \(0 < T < +\infty\). Moreover

\[
\int_0^T \int_\Omega u_n^2 \, dx \, dt \leq 2E(u_0),
\]

\(u(x, t) \geq f(x)\) for a.e. \(x \in \Omega\) and for every \(t \in [0, +\infty)\), and for each \(\alpha \in (0, \frac{1}{2})\) it holds

\[
(4.2) \quad u_n \to u \quad \text{in} \quad C^{0,\alpha}([0, T]; L^2(\Omega)) \quad \text{as} \quad n \to +\infty.
\]

\textbf{Proof.} Recalling that \(u_n(x, \cdot)\) is absolutely continuous on \([0, T]\), for all \(t_1, t_2 \in [0, T]\) with \(t_1 < t_2\), Hölder's inequality and Fubini's Theorem give us

\[
\|u_n(\cdot, t_2) - u_n(\cdot, t_1)\|_{L^2(\Omega)} = \left( \int_0^L \left( \int_{t_1}^{t_2} \frac{\partial u_n}{\partial t}(x, t) \, dt \right)^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{t_1}^{t_2} \left\| \frac{\partial u_n}{\partial t}(\cdot, t) \right\|_{L^2(\Omega)}^2 \, dt \right)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}}.
\]

Then it follows from (3.7) that

\[
(4.3) \quad \int_{t_1}^{t_2} \int_\Omega u_n^2 \, dx \, dt \leq 2E(u_0)
\]

and

\[
(4.4) \quad \|u_n(\cdot, t_2) - u_n(\cdot, t_1)\|_{L^2(\Omega)} \leq \sqrt{2E(u_0)}(t_2 - t_1)^{\frac{1}{2}}.
\]

Since (3.8) yields that

\[
(4.5) \quad \sup_{t \in [0, T]} \|\Delta u_n(\cdot, t)\|_{L^2(\Omega)} \leq \sup_{1 \leq i \leq n} \|\Delta u_{i,n}\|_{L^2(\Omega)} \leq \sqrt{2E(u_0)},
\]

there exists a function \(u \in L^2(0, T; H^2_0(\Omega))\) such that \(u_n \to u\) in \(L^2(0, T; H^2_0(\Omega))\) up to a subsequence. On the other hand, the estimate (3.7) implies that

\[
(4.6) \quad V_n = \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in} \quad L^2(0, T; L^2(\Omega)).
\]

This means that \(\partial u/\partial t \in L^2(0, T; L^2(\Omega))\), i.e., \(u \in H^1(0, T; L^2(\Omega))\). Combining (4.4) with Ascoli-Arzelà's Theorem (see e.g. [3, Proposition 3.3.1]), we conclude (4.2).

Since (4.5) means that \(\{u_n(t)\}\) is uniformly bounded in \(H^2_0(\Omega)\) with respect to \(t \in [0, T]\) and \(n \in \mathbb{N}\), we deduce from (4.2) that, for each \(t \in [0, T]\)

\[
(4.7) \quad u_n(t) \rightharpoonup u(t) \quad \text{in} \quad H^2_0(\Omega)
\]
up to a subsequence. This asserts that \( u \in L^\infty([0, T]; H_0^3(\Omega)) \). Moreover, Proposition 3.1 implies that for each \( t \in [0, T] \)

\[
(4.8) \quad u_n(t) \to u(t) \quad \text{in} \quad \begin{cases} C^{1,\gamma}(\Omega) & \text{for } 0 < \gamma < \frac{1}{2} \quad \text{if } N = 1, \\ C^{0,\gamma}(\Omega) & \text{for } 0 < \gamma < 2 - \frac{N}{2} \quad \text{if } N = 2, 3, \\ L^q(\Omega) & \text{for } 0 < q < +\infty \quad \text{if } N = 4, \\ L^q(\Omega) & \text{for } 0 < q < \frac{2N}{N-4} \quad \text{if } N \geq 5. 
\end{cases}
\]

In particular, if \( N \geq 4 \),

\[
(4.9) \quad u_n(t) \to u(t) \quad \text{a.e. in} \quad \Omega
\]

up to a subsequence. Since \( u_n(t) \geq f \) a.e. in \( \Omega \) for each \( n \in \mathbb{N} \) and \( t \in [0, T] \), the fact (4.8)-(4.9) yields that \( u(t) \geq f \) a.e. in \( \Omega \) for each \( t \in [0, T] \). This completes the proof. \( \square \)

When \( N = 1 \), we can improve the convergence result obtained in Theorem 4.1:

**Theorem 4.2.** ([13]) Let \( N = 1 \). Let \( u \) be the function obtained by Theorem 4.1. Then it holds that \( u \in L^2(0, T; W^{2,\infty}(\Omega)) \cap C^{0,\beta}([0, T]; C^{1,\alpha}(\Omega)) \) and

\[
(4.10) \quad u_n \to u \quad \text{weakly}^* \quad \text{in} \quad L^2(0, T; W^{2,\infty}(\Omega)) \quad \text{as} \quad n \to \infty,
\]

\[
(4.11) \quad u_n \to u \quad \text{in} \quad C^{0,\beta}([0, T]; C^{1,\alpha}(\Omega)) \quad \text{as} \quad n \to \infty
\]

for every \( \alpha \in (0, \frac{1}{2}) \) and \( \beta \in (0, \frac{1-2\alpha}{8}) \). Furthermore \( u(\cdot, t) \to u_0 \) in \( C^{1,\alpha}(\Omega) \) as \( t \downarrow 0 \).

When \( N = 2, 3 \), we can also improve the result obtained in Theorem 4.1:

**Theorem 4.3.** ([13]) Let \( N = 2, 3 \). Let \( u \) be the function obtained by Theorem 4.1. Then it holds that \( u \in L^2(0, T; W^{2,\infty}(\Omega)) \cap C^{0,\beta}([0, T]; C^{0,\gamma}(\Omega)) \) and

\[
(4.12) \quad u_n \to u \quad \text{weakly}^* \quad \text{in} \quad L^2(0, T; W^{2,\infty}(\Omega)) \quad \text{as} \quad n \to +\infty,
\]

\[
(4.13) \quad u_n \to u \quad \text{in} \quad C^{0,\beta}([0, T]; C^{0,\gamma}(\Omega)) \quad \text{as} \quad n \to +\infty
\]

for every

\[
0 < \beta < \left( \frac{1}{2} - \frac{N}{8} \right) \left( 1 - \frac{\gamma}{2 - N/2} \right), \quad 0 < \gamma < 2 - \frac{N}{2}.
\]

Furthermore \( u(\cdot, t) \to u_0 \) in \( C^{0,\gamma}(\Omega) \) as \( t \downarrow 0 \).

Regarding the piecewise constant interpolation for \( \{u_{i,n}\} \), i.e., \( \tilde{u}_n \) defined in Definition 2.2, we can verify the following:

**Lemma 4.1.** Let \( \tilde{u}_n \) be the piecewise constant interpolation of \( \{u_{i,n}\} \). If \( N = 1 \), then

\[
(4.14) \quad \tilde{u}_n \to u \quad \text{in} \quad L^\infty([0, T]; C^{1,\gamma}(\Omega)) \quad \text{as} \quad n \to +\infty
\]
for every $\gamma \in (0, 1/2)$, where $u$ is the function obtained in Theorem 4.1. If $N = 2, 3$, then
\begin{equation}
\tilde{u}_n \rightarrow u \quad \text{in} \quad L^\infty([0, T]; C^{0,\gamma}(\Omega)) \quad \text{as} \quad n \rightarrow +\infty
\end{equation}
for every $\gamma \in (0, 2 - N/2)$. Furthermore, for any $N \geq 1$, it holds that
\begin{equation}
\Delta \tilde{u}_n \rightarrow \Delta u \quad \text{in} \quad L^2(0, T; L^2(\Omega)) \quad \text{as} \quad n \rightarrow +\infty.
\end{equation}

Let us define $\mu_n$ as
\begin{equation}
\mu_n(t) = \mu_{i,n} \quad \text{if} \quad t \in [(i-1)\tau_n, i\tau_n).
\end{equation}

We close the paper with an outline of proof of Theorem 1.1:

\section*{4.1 Proof of Theorem 1.1}

Let $u$ be the function in Theorem 4.1. Fix $T > 0$ and $\varphi \in C_c^\infty(\Omega \times (0, T))$ with $\varphi \geq 0$ arbitrary. For each $\epsilon > 0$, let $w_\epsilon$ be a unique minimizer of the functional $G_{i,n}^\epsilon$ defined by
\begin{equation}
G_{i,n}^\epsilon(v) := \int_\Omega \left[ \frac{1}{2}(\Delta v)^2 + \frac{1}{2\tau_n}(v-u_{i-1,n})^2 + \gamma_\epsilon(v-f) \right] dx,
\end{equation}
where
\begin{equation}
\gamma_\epsilon(\lambda) = \begin{cases} 
\frac{\lambda^2}{\epsilon} & \text{if} \quad \lambda < 0, \\
\lambda & \text{if} \quad \lambda > 0,
\end{cases}
\end{equation}
\begin{equation}
\beta_\epsilon(\lambda) = \gamma_\epsilon'(\lambda).
\end{equation}

Since $w_\epsilon$ satisfies
\[ \int_\Omega \left[ \Delta w_\epsilon \Delta \varphi + \frac{1}{\tau_n}(w_\epsilon-u_{i-1,n})\varphi + \beta_\epsilon(w_\epsilon-f)\varphi \right] dx = 0, \]
we observe from the definition of $\beta_\epsilon$ that
\[ \int_\Omega \left[ \Delta w_\epsilon \Delta \varphi + \frac{1}{\tau_n}(w_\epsilon-u_{i-1,n})\varphi \right] dx = -\int_\Omega \beta_\epsilon(w_\epsilon-f)\varphi dx \geq 0. \]

Letting $\epsilon \rightarrow 0$, the proof of Theorem 3.3 implies that
\[ \int_\Omega \left[ \Delta u_{i,n} \Delta \varphi + \frac{1}{\tau_n}(u_{i,n}-u_{i-1,n})\varphi \right] dx \geq 0. \]

Integrating it over $[0, T]$ and using Definitions 2.1 and 2.2, we deduce that
\begin{equation}
\int_0^T \int_\Omega [\Delta \tilde{u}_n(x, t)\Delta \varphi(x, t) + V_n(x, t)\varphi(x, t)] dxdt \geq 0.
\end{equation}
It follows from (4.16) that

\[ \int_0^T \int_\Omega \Delta \tilde{u}_n(x, t) \Delta \varphi(x, t) \, dx \, dt \to \int_0^T \int_\Omega \Delta u(x, t) \Delta \varphi(x, t) \, dx \, dt \quad \text{as} \quad n \to +\infty, \]

and while (4.6) gives us

\[ \int_0^T \int_\Omega V_n(x, t) \varphi(x, t) \, dx \, dt \to \int_0^T \int_\Omega u_t(x, t) \varphi(x, t) \, dx \, dt \quad \text{as} \quad n \to +\infty. \]

Thus, letting \( n \to +\infty \) in (4.21), we observe that

(4.22) \[ \int_0^T \int_\Omega \left[ \Delta u(x, t) \Delta \varphi(x, t) + u_t(x, t) \varphi(x, t) \right] \, dx \, dt \geq 0. \]

Since \( \varphi \) is arbitrary, (4.22) implies that

(4.23) \[ \Delta^2 u(x, t) + u_t(x, t) \geq 0 \quad \text{a.e. in} \quad \Omega \times (0, T), \]

where \( \Delta^2 u \) is written in the distribution sense. Moreover, the regularity of \( u \) follows from Theorems 4.1–4.3.

We now prove (1.7). By (4.17) and Theorem 3.3, we observe that

(4.24) \[ \|\mu_n\|_{L^2(0, T; M(\Omega))} = \int_0^T \left( \int_\Omega \sum_{i=1}^n \mu_n(x, t) \right)^2 \, dt = \tau_n \sum_{i=1}^n \mu_i(\Omega)^2 < C. \]

This implies that

\[ \mu_n \to \overline{\mu} \quad \text{weakly in} \quad L^2(0, T; \mathcal{M}(\Omega)) \]

up to a subsequence. Setting

\[ \mu_t := \Delta^2 u + u_t, \]

we observe from (4.23) that \( \mu \) is a measure on \( \Omega \times (0, T) \), and it holds that \( \overline{\mu} = \mu_t \) by uniqueness of the limit. Since \( \mu_n \) converges to \( \mu_t \) weakly in \( L^2(0, T; \mathcal{M}(\Omega)) \), it follows from (4.24) that

\[ \|\mu_t\|_{L^2(0, T; \mathcal{M}(\Omega))} \leq \liminf_{n \to \infty} \|\mu_n\|_{L^2(0, T; \mathcal{M}(\Omega))} \leq C. \]

This is equivalent to (1.7), and implies that \( \mu_t \) is a positive Radon measure on \( \Omega \) for a.e. \( t \in (0, T) \).

Finally we prove that \( u \) is a solution of the problem (P). To prove this assertion, it is sufficient to show that, if \( u > f \), then \( \Delta^2 u + u_t = 0 \) holds. Let us set

\[ \mathcal{N} := \{(x, t) \in \Omega \times (0, T) : u(x, t) > f(x)\}. \]
Since $u$ is continuous in $\Omega \times (0, T)$ by Theorems 4.2 and 4.3, $\mathcal{N}$ is an open set, so that, for any $(x^0, t^0) \in \mathcal{N}$, there exist $\delta > 0$ and a neighborhood $W \times (t_1, t_2)$ of $(x^0, t^0)$ such that

\[(4.25) \quad u(x, t) - f(x) > \delta \quad \text{in} \quad W \times (t_1, t_2).\]

Lemma 4.1 implies that there exists a number $N > 0$ such that

\[\tilde{u}_n(x, t) > u(x, t) - \frac{\delta}{2} \quad \text{in} \quad W \times (t_1, t_2) \quad \text{for any} \quad n > N.\]

Combining this with (4.25), we have, for any $n > N,$

\[(4.26) \quad \tilde{u}_n(x, t) > f(x) + \frac{\delta}{2} \quad \text{in} \quad W \times (t_1, t_2).\]

Let $\zeta \in C_0^\infty(W \times (t_1, t_2))$ with $0 \leq \zeta \leq \delta/2.$ Then (4.26) asserts that

\[\psi(x, t) := \tilde{u}_n(x, t) - \zeta(x, t) \in K \quad \text{for each} \quad t \in [0, T].\]

Taking this $\psi$ as $\varphi$ in (3.16) and integrating it with respect to $t$ on $(0, T)$, we obtain

\[(4.27) \quad \int_0^T \int_\Omega \Delta u_{i,n}(x) \zeta(x, t) \, dx \, dt \leq -\int_0^T \int_\Omega V_{i,n}(x) \zeta(x, t) \, dx \, dt.\]

From the definition (4.17), the inequality can be reduced to

\[(4.28) \quad \sum_{i=1}^n \int_{(i-1)\tau_n}^{i\tau_n} \int_\Omega \zeta(x, t) \, d\mu_{n} \, dt \leq 0.\]

Since $\mu_n \geq 0$, we see that the integral in (4.28) must be equal to 0, i.e.,

\[(4.29) \quad \mu_n(W \times (t_1, t_2)) = 0.\]

It follows from (4.24) that

\[\|\mu_n\|_{\mathcal{M}(\Omega \times (0, T))} := \int_0^T \int_\Omega d\mu_n \, dt < C.\]

Thus we deduce that $\mu_n$ converges to $\mu_t$ weakly in $\mathcal{M}(\Omega \times (0, T))$, i.e.,

\[\int_0^T \int_\Omega \varphi(x, t) \, d\mu_n \, dt \to \int_0^T \int_\Omega \varphi(x, t) \, d\mu_t \, dt\]

for any $\varphi \in C_0^\infty(\Omega \times (0, T))$. This fact also yields that

\[(4.30) \quad \|\mu_t\|_{\mathcal{M}(\Omega \times (0, T))} \leq \liminf_{n \to +\infty} \|\mu_n\|_{\mathcal{M}(\Omega \times (0, T))}.\]

Combining (4.29) with (4.30), we conclude that

\[(4.31) \quad \mu_t(W \times (t_1, t_2)) = 0,\]

which completes the proof. \qed
References


