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グラスマン束の次数公式

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1. Introduction

The purpose of our work is to give degree formulae for Grassmann bundles. This article is a summary of a joint paper [4] with Tomohide Terasoma.

Let X be a projective variety of dimension n over a field of arbitrary characteristic, let $\mathcal{E}$ be a vector bundle of rank r on X, let $\mathbb{G}_X(d,\mathcal{E})$ be the Grassmann bundle of corank d subbundles of $\mathcal{E}$ on X with projection $\pi:\mathbb{G}_X(d,\mathcal{E})\to X$, and let $\pi^*\mathcal{E}\to\mathcal{Q}$ be the universal quotient bundle of rank d. Set $\theta:=c_1(\mathcal{Q})$, the first Chern class of $\mathcal{Q}$, whose determinant bundle, det $\mathcal{Q}$, is isomorphic to the pull-back of the tautological line bundle of $\mathbb{P}_X(\wedge^d\mathcal{E})$ by the (relative) Plücker embedding over X. In this article we call θ the *Plücker class* of $\mathbb{G}_X(d,\mathcal{E})$. The theme discussed here is how to calculate the self-intersection number of the Plücker class, $\int_{\mathbb{G}_X(d,\mathcal{E})}\theta^N$, which is the degree of $\mathbb{G}_X(d,\mathcal{E})$ embedded in the projective space $\mathbb{P}(H^0(X,\wedge^d\mathcal{E}))$ via the Plücker embedding if $\wedge^d\mathcal{E}$ is very ample, where $N:=\dim\mathbb{G}_X(d,\mathcal{E})=d(r-d)+n$.

The result is

Theorem 1.1. *Let θ be the Plücker class of $\mathbb{G}_X(d,\mathcal{E})$. Then*

(1)

$$\int_{\mathbb{G}_X(d,\mathcal{E})}\theta^N=N!\sum_{|k|=n}\frac{\prod_{1\le i<j\le d}(k_i-k_j-i+j)}{\prod_{1\le i\le d}(r+k_i-i)!}\int_X\prod_{1\le i\le d}s_{k_i}(\mathcal{E}),$$

where $k=(k_1,\ldots,k_d)\in\mathbb{Z}^d_{\ge 0}$ with $|k|:=\sum_i k_i$, and $s_i(\mathcal{E})$ is the i-th Segre class of $\mathcal{E}$.

(2)

$$\int_{\mathbb{G}_X(d,\mathcal{E})}\theta^N=N!\sum_{|\lambda|=n}\frac{\prod_{1\le i<j\le d}(\lambda_i-\lambda_j-i+j)}{\prod_{1\le i\le d}(r+\lambda_i-i)!}\int_X\Delta_\lambda(s(\mathcal{E})),$$

where $\Delta_\lambda(s(\mathcal{E}))$ is the Schur polynomial of $\mathcal{E}$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$.

In fact, we give two formulae for $\pi_* \operatorname{ch}(\det \mathcal{Q})$, the push-forward of the Chern character of $\det Q$ by π, explicitly (Theorem 2.1), under the assumption that X is a scheme of finite type over a field k: The above result is a direct consequence of those formulae.

The Segre classes $s_i(\mathcal{E})$ here are the ones satisfying $s(\mathcal{E}, t)c(\mathcal{E}, -t) = 1$ as in [1], [5], where $s(\mathcal{E}, t)$ and $c(\mathcal{E}, t)$ are respectively the Segre series and the Chern polynomial of $\mathcal{E}$ in t. Note that our Segre class $s_i(\mathcal{E})$ differs by the sign $(-1)^i$ from the one in [2].

Theorem 1.1 with $n = 0$ yeilds the degree formula of Grassmann varieties, as follows:

Corollary 1.2 ([2, Example 14.7.11 (iii)]). *The degree of the Grassmann variety $\mathbb{G}(d, r)$ of codimension d subspaces of an r-dimensional vector space with respect to the Plücker embedding is given by*

$$\deg \mathbb{G}(d, r) = \frac{(d(r-d))! \prod_{1 \le k \le d-1} k!}{\prod_{1 \le k \le d} (r-k)!}.$$

2. Main Results

Theorem 1.1 follows from more general results, as follows: Setting $m! := \Gamma(m+1)$ for $m \in \mathbb{Z}$, one has $1/m! = 0$ if $m < 0$. To simplify the notation, for a finite set of integers $\{a_i\}_{0 \le i \le d-1}$, set

$$\{a_i\}! := \prod_l a_l!, \quad \Delta(a_i) := \prod_{i<j} (a_i - a_j).$$

Theorem 2.1. *Assume that X is a scheme of finite type over a field k. Let $\mathbb{G}_X(d, \mathcal{E})$ be the Grassmann bundle of corank d subbundles of a vector bundle $\mathcal{E}$ of rank r on X with projection $\pi : \mathbb{G}_X(d, \mathcal{E}) \to X$, let $\pi^*\mathcal{E} \to \mathcal{Q}$ be the universal quotient bundle of rank d, and let $\operatorname{ch}(\det \mathcal{Q})$ be the Chern character of $\det \mathcal{Q}$. Denote by $\pi_* : A^*(\mathbb{G}_X(d, \mathcal{E})) \otimes \mathbb{Q} \to A^{*-d(r-d)}(X) \otimes \mathbb{Q}$ is the push-forward by π. Then*

(1)

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_k \frac{\Delta(k_i - i)}{\{r + k_i - i\}!} \prod_{1 \le l \le d} s_{k_l}(\mathcal{E}),$$

where $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d_{\ge 0}$, and $s_i(\mathcal{E})$ is the i-th Segre class of $\mathcal{E}$.

(2)
$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_{\lambda} \frac{\Delta(\lambda_i - i)}{\{r + \lambda_i - i\}!} \Delta_\lambda(s(\mathcal{E})),$$
where $\Delta_\lambda(s(\mathcal{E})) := \det[s_{\lambda_i+j-i}(\mathcal{E})]_{1\le i,j\le d}$ *is the Schur polynomial of* $\mathcal{E}$ *for a partition* $\lambda = (\lambda_1, \ldots, \lambda_d)$.

3. (Sketch of)^2 Proof

Let X be a scheme of finite type over a field k, and let $\mathcal{E}$ be a vector bundle of rank r on X. Denote by $\mathbb{F}^d_X(\mathcal{E})$ the partial flag bundle of $\mathcal{E}$ on X, parametrising flags of subbundles of corank 1 up to d in $\mathcal{E}$. Then it is easily shown that the projection $p : \mathbb{F}^d_X(\mathcal{E}) \to X$ decomposes as a successive composition of projective space bundles, $\mathbb{P}(\mathcal{E}_i)/\mathbb{P}(\mathcal{E}_{i-1})$ $(i \ge 1)$:
$$p : \mathbb{F}^d_X(\mathcal{E}) = \mathbb{P}(\mathcal{E}_{d-1}) \to \mathbb{P}(\mathcal{E}_{d-2}) \to \cdots \to \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E}_0) \to X,$$
where $\mathcal{E}_0 := \mathcal{E}$, and $\mathcal{E}_{i+1}$ is the kernel of the canonical surjection from the pull-back of $\mathcal{E}_i$ to $\mathbb{P}(\mathcal{E}_i)$, to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)$ with $\operatorname{rk}\mathcal{E}_i = r - i$ $(i \ge 0)$: In fact, $\mathbb{P}(\mathcal{E}_i) \simeq \mathbb{F}^{i+1}_X(\mathcal{E})$ $(1 \le i \le d-1)$. Set $\xi_i := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1))$. Then, the intersection ring of $A^*(\mathbb{F}^d_X(\mathcal{E}))$ is given as follows:
$$\begin{aligned} A^*(\mathbb{F}^d_X(\mathcal{E})) &= \frac{A^*(X)[\xi_0, \xi_1, \ldots, \xi_{d-1}]}{(P_0(\xi_0), P_1(\xi_1), \ldots, P_{d-1}(\xi_{d-1}))} \\ &= \bigoplus_{\substack{0\le i_l\le r-l-1 \\ (0\le l\le d-1)}} A^*(X)\overline{\xi_0}^{i_0}\overline{\xi_1}^{i_1}\cdots\overline{\xi_{d-1}}^{i_{d-1}}, \end{aligned} \tag{3.1}$$
where $P_i(\xi_i) := \xi_i^{r-i} - c_1(\mathcal{E}_i)\xi_i^{r-i-1} + \cdots + (-1)^{r-i}c_{r-i}(\mathcal{E}_i) \in A^*(\mathbb{P}(\mathcal{E}_i))[\xi_i]$, and the symbol of pull-back to $\mathbb{F}^d_X(\mathcal{E})$ is omitted. Denote by $p_* : A^*(\mathbb{F}^d_X(\mathcal{E})) \to A^{*-c}(X)$ the push-forward by p, where $c := \sum_{0\le i\le d-1}(r-i-1)$, the relative dimension of $\mathbb{F}^d_X(\mathcal{E})/X$. Then, for $\alpha = \sum \alpha_{i_0 i_1\cdots i_{d-1}}\overline{\xi_0}^{i_0}\overline{\xi_1}^{i_1}\cdots\overline{\xi_{d-1}}^{i_{d-1}}$ in $A^*(\mathbb{F}^d_X(\mathcal{E}))$ $(\alpha_{i_0 i_1\cdots i_{d-1}} \in A^*(X))$ with respect to the decomposition in (3.1), one has
$$p_*\alpha = \alpha_{r-1,r-2,\ldots,r-d}. \tag{3.2}$$
Indeed, $\sum_l i_l \ge c$ if and only if $i_l = r - l - 1$ for each l.

Let $G := \mathbb{G}_X(d, \mathcal{E})$ be the Grassmann bundle of corank d subbundles of $\mathcal{E}$ on X, and let $\pi^*\mathcal{E} \to \mathcal{Q}$ be the universal quotient bundle of rank d. Consider the flag bundle $\mathbb{F}^{d-1}_G(\mathcal{Q})$ of $\mathcal{Q}$ on G, parametrising flags of subbundles of corank 1 up to $d-1$ in $\mathcal{Q}$. Then, as in the case of $\mathbb{F}^d_X(\mathcal{E})$, the projection $\mathbb{F}_G^{d-1}(\mathcal{Q}) \to G$ decomposes as a successive composition of projective space bundles $\mathbb{P}(\mathcal{Q}_{i+1})/\mathbb{P}(\mathcal{Q}_i)$ $(i \geq 1)$:

$$q : \mathbb{F}_G^{d-1}(\mathcal{Q}) = \mathbb{P}(\mathcal{Q}_{d-2}) \to \mathbb{P}(\mathcal{Q}_{d-2}) \to \cdots \to \mathbb{P}(\mathcal{Q}_1) \to \mathbb{P}(\mathcal{Q}_0) \to G,$$

where $\mathcal{Q}_0 := \mathcal{Q}$, and $\mathcal{Q}_{i+1}$ is the kernel of the canonical surjection from the pull-back of $\mathcal{Q}_i$ to $\mathbb{P}(\mathcal{Q}_i)$, to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{Q}_i)}(1)$ with $\operatorname{rk} \mathcal{Q}_i = d - i$ $(i \geq 0)$: In fact, $\mathbb{P}(\mathcal{Q}_i) \simeq \mathbb{F}_G^{i+1}(\mathcal{Q})$ $(1 \leq i \leq d-2)$.

It follows from the construction of vector bundles $\mathcal{E}_i$ that $\mathcal{E}_d$ is a corank d subbundle of $p^*\mathcal{E}$ on $\mathbb{F}_X^d(\mathcal{E})$, which induces a morphism, $r : \mathbb{F}_X^d(\mathcal{E}) \to G$ by the universal property of the Grassmann bundle G. Then it turns out that $\mathbb{F}_G^{d-1}(\mathcal{Q})$ is naturally isomorphic to $\mathbb{F}_X^d(\mathcal{E})$ over G via r, as is easily verified by using the universal property of flag bundles (see [5, §6], [7, §§0–1]): We identify them via the natural isomorphism $\mathbb{F}_G^{d-1}(\mathcal{Q}) \simeq \mathbb{F}_X^d(\mathcal{E})$. Under this identification, it follows that

$$\xi_i = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)) = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{Q}_i)}(1))$$

in $A^*(\mathbb{F}_X^d(\mathcal{E})) = A^*(\mathbb{F}_G^{d-1}(\mathcal{Q}))$, where the symbol of pull-back to $\mathbb{F}_X^d(\mathcal{E}) = \mathbb{F}_G^{d-1}(\mathcal{Q})$ is omitted, as before.

For the Plücker class $\theta = c_1(\mathcal{Q})$, one has

Lemma 3.1. (1) $\theta^N = q_*(\xi_0^{d-1}\xi_1^{d-2} \cdots \xi_{d-2} q^*\theta^N)$ *in* $A^*(G)$.
(2) $q^*\theta = \xi_0 + \cdots + \xi_{d-1}$ *in* $A^*(\mathbb{F}_X^d(\mathcal{E})) = A^*(\mathbb{F}_G^{d-1}(\mathcal{Q}))$.

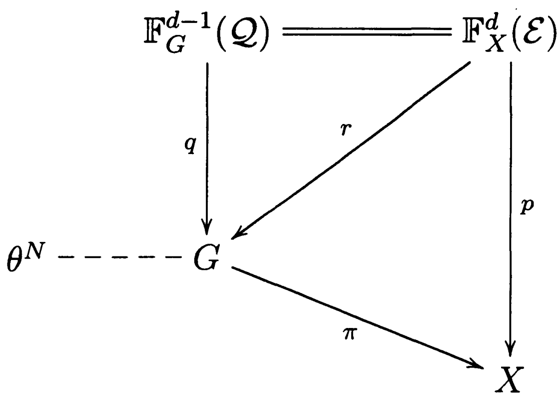

FIGURE 1

It follows from Lemma 3.1, the commutativity $p = \pi \circ q$ and (3.2) that

$$\pi_*(\theta^N) = \pi_* q_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}q^*\theta^N) = \pi_* q_*\Big(\prod_{i=0}^{d-1}\xi_i^{d-1-i}\Big(\sum_{i=0}^{d-1}\xi_i\Big)^N\Big)$$

$$(3.3) \qquad = p_*\Big(\prod_{i=0}^{d-1}\xi_i^{d-1-i}\Big(\sum_{i=0}^{d-1}\xi_i\Big)^N\Big)$$

$$= \mathtt{coeff}_{\overline{\xi_0},\ldots,\overline{\xi_{d-1}}}\Big(\prod_{i=0}^{d-1}\xi_i^{d-1-i}\Big(\sum_{i=0}^{d-1}\xi_i\Big)^N; r-1,\ldots,r-d\Big),$$

where $\mathtt{coeff}_{\overline{\xi_0},\ldots,\overline{\xi_{d-1}}}(\cdots; r-1,\ldots,r-d)$ denotes the coefficient of $\cdots$ in $\overline{\xi_0}^{r-1}\overline{\xi_1}^{r-2}\cdots\overline{\xi_{d-1}}^{r-d}$.

Now one can show that

Lemma 3.2.

$$\mathtt{coeff}_{\overline{\xi_i}}(\xi_i^{p_i}; r-i-1) = \mathtt{const}_{t_i}(t_i^{-p_i+r-i-1}s(\mathcal{E}_i, t_i)),$$

where $\mathtt{const}_{t_i}(\cdots)$ *the constant term in the Laurent expansion of* $\cdots$ *in* t_i.

Applying Lemma 3.2 repeatedly, one obtains

Lemma 3.3.

$$\mathtt{coeff}_{\overline{\xi_0},\ldots,\overline{\xi_{d-1}}}\big(\xi_0^{p_0}\cdots\xi_{d-1}^{p_{d-1}}; r-1,\ldots,r-d\big)$$

$$= \mathtt{const}_{\underline{t}}\Big(\Delta(t_0,\ldots,t_{d-1})\prod_{i=0}^{d-1}t_i^{-p_i+r-d}s(\mathcal{E}_0, t_i)\Big),$$

where $\underline{t} := (t_0, \ldots, t_{d-1})$, *and* $\Delta(t_0, \ldots, t_{d-1}) := \prod_{0\le i<j\le d-1}(t_i - t_j)$ *the Vandermonde polynomial of* $(t_0, \ldots, t_{d-1})$.

By virtue of (3.3) and Lemma 3.3, one can show

Proposition 3.4. *For a non-negative integer* N,

$$\pi_*\theta^N = \mathtt{const}_{\underline{t}}(P_N(\underline{t})),$$

where $\pi_* : A^*(\mathbb{G}_X(d,\mathcal{E})) \to A^{*-d(r-d)}(X)$ *is the push-forward by* π, $s(\mathcal{E}, t)$ *is the Segre series of* $\mathcal{E}$ *in* t, *and*

$$P_N(\underline{t}) := \Delta(\underline{t})\left(\sum_{i=0}^{d-1}\frac{1}{t_i}\right)^N\prod_{i=0}^{d-1}t_i^{-(d-1-i)+r-d}s(\mathcal{E}_0, t_i).$$

Now, to prove Theorem 2.1 (1), just expand the Laurent series $P_N(\underline{t})$ by the multinomial theorem with the following

Lemma 3.5 ([2, Example A.9.3]).

$$\det\left[\frac{1}{(x_i+j)!}\right]_{0\le i,j\le d-1}=\frac{\Delta(x_i)}{\{x_i+d-1\}!}.$$

For Theorem 2.1 (2), we have two proofs, where we use a consequence of Cauchy identity [6, Chapter I, (4.3)] and Jacobi-Trudi identity [2, Lemma A.9.3], as follows:

Lemma 3.6.

$$\prod_{i=0}^{d-1}s(\mathcal{E},t_i)=\sum_{\lambda\ge 0}\Delta_\lambda(s(\mathcal{E}))s_\lambda(\underline{t}).$$

One of our proofs is obtained just by expanding $P_N(\underline{t})$, similarly to the proof of Theorem 2.1 (1). For the other, we establish a formula of Kadell type for confluent Selberg integral, due to Terasoma, as follows (Cf. [3]):

Proposition 3.7. *Set*

$$W_{\exp}(x,\underline{t}):=\prod_{i=0}^{d-1}t_i^{x-1}\prod_{i=0}^{d-1}\exp(-t_i)\prod_{i<j}(t_i-t_j)^2,$$

$$I_{\mathrm{conf}}(\lambda,x):=\int_{[0,+\infty)^d}s_\lambda(\underline{t})W_{\exp}(x,\underline{t})d\underline{t}.$$

Then

$$I_{\mathrm{conf}}(\lambda,x)=d!\Delta(\lambda_i-i)\Gamma\{x+d-i+\lambda_i\},$$

for a real number $x>0$, where $\underline{t}:=(t_0,\dots,t_{d-1})$ and $d\underline{t}:=dt_0\cdots dt_{d-1}$.

Remark 3.8. Symmetrising the Laurent series $P_N(\underline{t})$ with respect to the variables $\underline{t}$, one sees that $\mathtt{const}_{\underline{t}}(P_N(\underline{t}))$ is equal to the constant term of the Laurent series,

$$P_N^s(\underline{t}):=\frac{(-1)^{\frac{d(d-1)}{2}}}{d!}\prod_{0\le i<j\le d-1}\left(\frac{1}{t_i}-\frac{1}{t_j}\right)^2\left(\sum_{i=0}^{d-1}\frac{1}{t_i}\right)^N\prod_{i=0}^{d-1}t_i^{r-1}s(\mathcal{E},t_i).$$

Roughly speaking, to obtain the constant term of $P_N(\underline{t})$, we calculate the residue of $P_N^s(t_0^{-1},\dots,t_{d-1}^{-1})(t_0\cdots t_{d-1})^{-1}$ by using Proposition 3.7 (see Remark 3.8): Indeed, we have $\mathtt{const}_{\underline{t}}(P_N(\underline{t}))=\mathtt{const}_{\underline{t}}(P_N^s(t_0^{-1},\dots,t_{d-1}^{-1}))$.

4. Example

Example 4.1. $\deg \mathbb{G}_{\mathbb{P}^4}(2, T_{\mathbb{P}^4}) = 5040$. This number is exactly equal to the factorial of 7 (pointed out by Agaoka): $5040 = 7!$. I guess this would be nothing but a coincidence without rationale (what do you think?).

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