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MODULI OF FANO VARIETIES VIA KÄHLER-EINSTEIN METRICS

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ABSTRACT. In this short notes, we give an introduction to our recent approach to moduli of Fano manifolds via Kähler-Einstein geometry.

CONTENTS

1. INTRODUCTION

The purpose of this survey is to explain recent progresses on moduli of Fano varieties on special focus on the relation with Kähler-Einstein geometry and stability. It benefits by the recent equivalence theorem of K-stability and Kähler-Einstein metric existence [CDS1], [Tia2]. Personally speaking, the study was originally motivated by [Od1], [Od3] on the study of general type case and differential geometric background such as [Fuj], [FS], [Don1], [Don3] for smooth case and the result on Gromov-Hausdorff limit [DS].

Regarding general theory of constructing moduli of varieties, after the ground-breaking work of Mumford [GIT] resolving the case of smooth projective curves, as well as his introduction of the abstract method to attack those construction problems "Geometric Invariant Theory", there has been a tons of great efforts poured to get extension to higher dimensional case as well as singular cases, especially to get compactification of the moduli.

In the next section, we detail a little more those historical development of moduli of varieties from algebro-geometric perspective, with possible biases due to my personal perspective.

2. Algebro-geometric history of moduli of varieties

After [GIT] constructed the moduli of *smooth* projective curves, the famous [DM] established its geometric compactification as a moduli of *nodal* curves with canonical ample class - nowadays called *(Deligne-Mumford) stable curves.* Note that paper's main construction was as an algebraic stack which is not technically hard from modern perspective. (Indeed their main result is not the construction but the algebraic proof of irreducibility of the moduli as well as its properness, and the notion of algebraic stack they introduced is of course of great importance.) After a while, the moduli stack $\bar{\mathcal{M}}_g$ turned out to have a coarse moduli variety $\bar{M}_g$ which is projective [KnM], [Mum2], [Gie2]. The proofs of projectivity were via Geometric Invariant Theory again. (The more direct proof was recently provided by Li-Wang [LW].)

Higher dimensional generalizations of the above story has been facing many difficulties and not fully settled yet. The GIT stability of canonical models in 2-dimensional case was proved by Gieseker [Gie1]. However those surfaces do not have compact moduli so to get compactification one needs to discuss what was the right degenerations to put on the boundaries. Shepherd-Barron observed that we can put some suitable degenerations on the boundary via birational geometric methods while they are *not* necessarily asymptotically GIT stable. This birational geometric idea was systematically studied and put in more general context by Kollár-Shepherd-Barron [KSB] and later by Alexeev in higher dimensions [Al1]. Nowadays this construction of moduli of "general type" (more precisely of semi-log-canonical varieties with canonical ample classes) is often being called KSBA (Kollár-Shepherd-Barron-Alexeev) construction. The construction benefits recent breakthrough [BCHM] of the Minimal Model Program and its some continuations.

Unfortunately the idea of KSBA works only for varieties whose *canonical class is ample* (which is in particular of "general type" if the variety is smooth). On some class of variety X we have natural boundary divisor D (on X) and putting that boundary to form a pair (X, D), we regard the pair as having ample (log!)-canonical class $K_X + D$ so that we can apply KSBA's idea. The works of Alexeev

[Al2], Hacking-Keel-Tevelev [HKT], Laza [Laz] follow this line. However, in general, those boundaries D are additional informations (i.e. there is no canonical choice of such boundaries) so that it does not give a desired compactification, as even raising the dimension of the original moduli and changing the problem.

So it would be wonderful if we get some generalization for other class of varieties (*without* putting boundaries as additional information), which is the main point of my idea of using stability notions coming from differential geometric quest of canonical Kähler metrics such as Kähler-Einstein metrics or constant scalar curvature Kähler metrics. Indeed, the following theorem holds.

Theorem 2.1 ([Od1],[Od3]). *For projective variety X with $\mathbb{Q}$-Cartier K_X, the followings are equivalent.*

- *X is KSBA stable i.e. X has only semi-log-canonical singularities.*
- *(X, K_X) is K-stable.*
- *(X, K_X) is K-semistable.*

Recall that the K-stability [Tia1],[Don2] is a closely related version of the classical GIT stability notion for projective varieties [GIT], [Mum2]. Their motivation was to formulate the following conjecture:

Conjecture 2.2 (Yau-Tian-Donaldson conjecture). *For a polarized projective manifold (X, L), the existence of constant scalar curvature Kähler metric in the Kähler class $c_1(L)$ is equivalent to the K-(poly)stability of (X, L).*

About the theorem 2.1, note the original stability works for one side of the claim "semistable implies semi-log-canonicity" [Od1]. The main idea of the proofs of Theorem 2.1 is to use the MMP or basic discrepancy arguments to test configurations (which encodes one parameter subgroups) and see the behaviour of Donaldson-Futaki invariants after (equivariant) Riemann-Roch type formula for the invariants. The above theorem 2.1 is recently linked back to differential geomery by Berman-Guenancia [BG] as follows.

Theorem 2.3 (Berman-Guenancia [BG]). *In the same setting as in Theorem 2.1, they are also equivalent to the following metric existence condition.*

($*$) *There is a Kähler-Einstein metric on X^{reg} which extends as a current to X and volume $(K_X)^n$.*

This can be seen as another case of the Yau-Tian-Donaldson correspondence. More details on the general conjectural pictures and background related to moduli of K-stable varieties can be found in [Od4].

3. Some metric geometry

This section is some crash course on metric geometry to help some algebraic readers to understand the rest of the article better, and obviously most of the contents are just classical and basically copied from standard references.

The point of bringing metric perspective into moduli of varieties is, giving that additional structures, we naturally get Hausdorff property of moduli as well as the *canonical* limits as Gromov-Hausdorff limits or its versions.

Note that the word "*metric*" can mean two things - either Riemannian metrics on manifolds or the distance structure on general sets or topological spaces. We refer the details to the textbook [BBI] on which my understanding also heavily rely.

3.1. **Hausdorff distance.** Given an ambient metric space M and two subsets (with induced metrics) $X, Y \subset M$, we define the *Hausdorff distance* $d_H(X, Y; M)$ as

$$\inf\{r > 0 \mid \forall x \in X, \exists y \in Y, d(x,y) < r, \forall y \in Y, \exists x \in X, d(x,y) < r\}.$$

If M is obvious from the context, we omit it. This gives a pseudo distance between subsets of M.

Example 3.1. Consider a subset $Z \in M$ which satisfies $d_H(Z, M; M) < \epsilon$. Then Z is called *ϵ-net.* If M is a Riemannian manifolds, we can see that for any $\epsilon > 0$, we can take ϵ-net Z_ϵ as a discrete subset.

Example 3.2. $d_H(S, \bar{S}) = 0$ for any $S \subset M$, where $\bar{S}$ denotes the closure of S in M.

Set $\mathcal{C}(X) :=$ { closed subsets of X} with Hausdorff distance d_H. Then the followings are known.

Theorem 3.3. *(i) $\mathcal{C}(X)$ is complete if so is X.*
(ii) $\mathcal{C}(X)$ is compact if so is X (Blaschke).

3.2. **Gromov-Hausdorff distance.** We now pass the distance structure among subsets of given metric space to abstract metric spaces after the idea of M. Gromov.

Definition 3.4. Let X, Y are both metric spaces. The *Gromov-Hausdorff distance* of those two are defined as:

$d_{GH}(X, Y) := \inf\{d_H(X, Y; M) \mid X, Y \subset M \text{with distances preserved.}\}$.

Using this notion, we can see the moduli of compact Riemannian manifolds are, in particular, Hausdorff (see also [Ebi]).

About the (pre)compactness of set of metric spaces, there is a famous useful criterion due to Gromov. Let us prepare the following basic definition.

Definition 3.5. A class of compact metric spaces $\mathcal{X}$ is said to be *uniformly totally bounded* if it satisfies the following two conditions.

(i) $\exists D > 0$ such that $diam(X) \leq D$ for any $X \in \mathcal{X}$.

(ii) $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{Z}_{>0}$ such that $X \in \mathcal{X}$ has ϵ-net of cardinality less than $N(\epsilon)$.

Then the Gromov precompactness theorem is then this.

Theorem 3.6 (Gromov). *Any uniformly totally bounded class of compact metric spaces form precompact moduli via Gromov-Hausdorff distance.*

The core idea of the known proof is, given a sequence $\{X_i\}_i$ in $\mathcal{X}$, to regard each $X_i \in \mathcal{X}$ as a limit of ϵ-net where ϵ tends to zero. Then we cook up the limit as some quotient of a set of certain sequences of X_i.

A famous example is the set of compact Riemannian n-dimensional manifolds whose Ricci curvature is at least $(n-1)k$ and diameter at most D, for some fixed $k, D > 0$. It is fairly nontrivial to confirm this does work as an example but it can be proved by constructing maps from each metric space to the space form of curvature k. We apply this in combination with Myers theorem for Kahler-Einstein Fanos, which asserts the boundedness of their diameters later.

3.3. **Pointed Gromov-Hausdorff limits.** For non-compact spaces, we need some modification of the notion because there is certain examples of sequences of compact metric spaces whose diameter diverges even though it "converges" to a metric space with infinite diameter at least intuitively. Here are some examples:

Example 3.7. Consider the intervals $[-a, a]$ with $a > 0$. If a goes to infinity, this should be regarded as converging to the whole line $\mathbb{R}$.

Example 3.8. Recall that (Deligne-Mumford) stable curve which is not smooth also has hyperbolic metric like the case of smooth hyperbolic curve. As it is algebro-geometrically a limit of smooth hyperbolic curve, it should be regarded as a "limit" of those smooth hyperbolic curves (with the hyperbolic metrics while preserving the curvature).

To justify these convergence, the following notion is useful.

Definition 3.9. A sequence of metric spaces with base points $X_i \ni p_i (i \in \mathbb{Z}_{>0})$ has $X_\infty \ni p_\infty$ as the *pointed Gromov-Hausdorff limit* if for all $r > 0$, the balls $B_r(p_i)$ converges to $B_r(p_\infty)$ in the usual Gromov-Hausdorff sense.

Then we expect the following, in relation with Kollár-Shepherd-Barron-Alexeev stable varieties.

Example 3.10. Recall that very recently Berman-Guenancia [BG] constructed singular Kähler-Einstein metrics on KSBA stable varieties extending the example 3.8.

Consider the $\mathbb{Q}$-Gorenstein deformation (flat family) of KSBA stable varieties $f: \mathcal{X} \to C$ over a curve with a section $s: C \to \mathcal{X}$ which does not intersect with singularities of fibers. Then we expect that the pointed Gromov-Hausdorff limit of the singular Kähler-Einstein metrics on $\mathcal{X}|_{p_i}$, while $p_i \to p \in C$ and preserving the volume, is just their singular Kähler-Einstein metric on $\mathcal{X}_p$ again.

3.4. **Collapse and non-collapse.** An interesting feature of Gromov-Hausdorff limit is that sometimes the dimension can jump down. Consider the following example.

Example 3.11. If we consider elliptic curve $\mathbb{C}/ < 1, ti >$ with usual Euclidean metric where $t \to 0$ converges to a circle $\mathbb{R}/\mathbb{Z}$. Instead, if we take $\mathbb{C}/ < t, t^{-1}i >$ with $t \to 0$, then it converges to a line $\mathbb{R}$ as a pointed Gromov-Hausdorff limit.

These convergence with dimensions jumping down is called *collapse.* Pointed Riemannian manifolds $M_i \ni p_i$ does not collapse if $vol(B_r(p_i)) > c \cdot r^n$ for uniform $c > 0$ where i and r ranges all over.

We end the section by seeing some Kähler-Einstein situations where collapse never occurs. For that we need to recall the following two (even more) classical results:

Theorem 3.12 (Myers). *If a compact n-dimensional Riemannian manifold M has Ricci curvature at least $(n-1)k$ with some uniform constant $k > 0$, then the diameter is at most $\pi/\sqrt{k}$.*

Theorem 3.13 (Bishop-Gromov inequality). *Suppose a pointed Riemannian manifolds $M \ni p$ has Ricci curvature at least $(n-1)k$ with some constant $k > 0$. And let $V_r(k) := vol(B_r)$ be the volume of radius r ball in the space form with Ricci curvature $(n-1)k$.*

Then the ratio $vol(B_r(p))/V_r(k)$ with does not increases when r increases.

Following these two theorems, we see easily that

Proposition 3.14. *Any sequence of Kähler-Einstein Fano n-manifolds with volume preserved has non-collapsed Gromov-Hausdorff limit.*

Indeed the diameter should be bounded above by Myers theorem so that we can apply the Gromov precompactness theorem to show we have a Gromov-Hausdorff limit. Furthermore, by the Bishop-Gromov inequality it cannot collapse.

We can show the same thing for a sequence of Ricci flat manifolds if the diameter converges to a positive finite number. It is the case when it is a sequence of smooth Calabi-Yau varieties degenerating to a log terminal Calabi-Yau variety (see Tosatti [Tos]).

4. Moduli of Fano varieties

Finishing the preparatory background in the previous section, now we can discuss the main issue. As we wrote in the end of the section 2, the KSBA projective moduli of semi-log-canonical varieties with ample canonical classes turned out to correspond to K-stability [Od1], [Od3]. Then a natural question is to extend the moduli of more general K-stable varieties as we first discussed in [Od2].

Meanwhile the following result was proved.

Theorem 4.1 (Donaldson-Sun [DS]). *Gromov-Hausdorff limits of Kähler-Einstein Fano n-dimensional manifolds are n-dimensional $\mathbb{Q}$-Fano varieties (i.e. singular Fano varieties with only log terminal singularities) with singular Kähler-Einstein metrics.*

The key idea is to show lower boundedness of (asymptotic) Bergman kernel - called "partial C^0 estimates" - to give identifying map between the Gromov-Hausdorff limit and a limit in Hilbert scheme. Originally the theorem was proved for the existence problem of Kähler-Einstein metrics - to construct a destabilizing test configuration of Fano manifolds without Kähler-Einstein metrics. Indeed the extension to conical singular situation is the key to [CDS1] and [Tia2]. Nevertheless, the theorem implied some possible application to moduli as well which matches to the author's algebraic results mentioned above.

Motivated by two of these aspects, the followings are proved.

Theorem 4.2 ([Od4]). *Kähler-Einstein Fano manifolds with discrete automorphism groups form Hausdorff moduli algebraic space. Furthermore, it is an orbifold i.e. has only quotient singularities.*

We expect that putting Gromov-Hausdorff limits on infinity, we get projective geometric compactification and this is the moduli of K-(semi)stable Fano varieties ("K-moduli"). In del Pezzo surfaces case, in [OSS], we have described the moduli and degenerations explicitly whose rough statement is as follows. Mabuchi-Mukai [MM] pioneered this direction.

Theorem 4.3 ([OSS]). *The Gromov-Hausdorff compactification of moduli of del Pezzo surfaces are some algebro-geometric (explicit, étale locally GIT) compactification which is at least compact algebraic space. (Other than degree* 1 *case, we confirmed projectivity as well.)*

We note that the Gromov-Hausdorff convergence we consider in the above theorem is in a slightly different sense since we also concern the continuity of *complex structures* as well, whose definition is naturally done in [DS] (also we recommend [Spo]). We consult [OSS] for more precise statement of Theorem 4.3 which includes formulation at the stacky level, as well as the explicit construction of the moduli in the del Pezzo case. Another reference we can refer to is [Od4] whose last two sections gives a review of general conjecture picture (not sticking to Fano case) as well as partial results on canonical limits.

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