INTERPOLATING CARLITZ ZETA VALUES

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This note is a survey of some results obtained in collaboration with B. Anglès and F. Tavares Ribeiro [5] on a new class of $L$-series arising in the theory of function fields of positive characteristic recently introduced in [14]. Complete proofs and wider investigations can be found in [4, 5].

1. Preliminaries.

We set $A = \mathbb{F}_q[\theta]$, $K = \mathbb{F}_q(\theta)$, $K_\infty = \mathbb{F}_q((\theta^{-1}))$ and we denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of $K_\infty$.

The Carlitz zeta values are the series
\[
\zeta_C(n) := \sum_{a \in A^+} a^{-n} \in K_\infty, \quad n \geq 1,
\]
where the sum runs over the set $A^+$ of monic polynomials. In analogy with the classical zeta values
\[
\zeta(n) = \sum_{i \geq 1} i^{-n}
\]
with $n$ integer (convergence occurs only if $n \geq 2$).

It was proved by Carlitz [8] that, if $n \equiv 0 \pmod{q - 1}$,
\[
\zeta_C(n) \in K_\times \tilde{\pi}^n,
\]
where $\tilde{\pi}$ is the value in $\mathbb{C}_\infty$ of an infinite product
\[
\tilde{\pi} := -(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty,
\]
uniquely defined up to the multiplication by an element of $\mathbb{F}_q^\times = \mathbb{F}_q \setminus \{0\}$ (corresponding to the choice of a root $(-\theta)^{\frac{1}{q-1}}$). We notice that $v_\infty(\tilde{\pi}) = -\frac{q-1}{q-1}$, where $v_\infty$ is the valuation of $\mathbb{C}_\infty$ (so that $v_\infty(\theta) = -1$).

The element $\tilde{\pi}$ is a fundamental period of the Carlitz exponential $\exp_C$ (Goss, [11, §3.2]), that is, the unique surjective, entire, $\mathbb{F}_q$-linear function
\[
\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty
\]
of kernel $\tilde{\pi}\mathbb{F}_q[\theta]$ such that its first derivative satisfies $\exp'_C = 1$.

We have the following arithmetical analogy between the Carlitz zeta values $\zeta_C(n) \in K_\times^ (n \geq 1)$ and the special values $\zeta(n) (n \geq 2)$ of Riemann's zeta function, which was pointed out by Lenny Taelman.

For $n \geq 1$, we have the functor of Quillen $K$-theory $K_{2n-1}$, which, evaluated at $\mathbb{F}_p$, gives the finite group $K_{2n-1}(\mathbb{F}_p)$, cyclic of cardinality $p^n - 1$ (Quillen's theorem). Moreover, the
evaluation \( \text{Lie}(K_{2n-1})(\mathbb{F}_p) \) of the functor \( \text{Lie}(K_{2n-1}) \) (\(^1\)) has cardinality \(|\text{Lie}(K_{2n-1})(\mathbb{F}_p)| = p^n\) (this can be deduced, for example, from the paper of Hesselholt and Madsen [12, Theorem E]). Now, this yields the Eulerian product

\[
\zeta(n) = \prod_p \left( \frac{|\text{Lie}(K_{2n-1})(\mathbb{F}_p)|}{|K_{2n-1}(\mathbb{F}_p)|} \right)
\]

which diverges of course for \( n = 1 \). We note that the cardinalities above can also be viewed as positive generators of Fitting ideals of finite \( \mathbb{Z} \)-modules.

We set \( A = \mathbb{F}_q[\theta] \). The Carlitz module \( C \) is the functor from \( A \)-algebras to \( A \)-modules which sends an \( A \)-algebra \( A \) to the unique \( A \)-module which has \( A \) as underlying abelian group, and such that the (left) multiplication by \( \theta \) of an element \( x \) of \( A \) is \( C_\theta(x) = \theta x + x^q \).

Let \( P \) be a prime of \( A \) (that is, a monic irreducible polynomial of \( A \)). To the \( A \)-algebra \( A/PA \), we can associate the \( A \)-module \( C(A/PA) \), which is a finite \( A \)-module, to which we can associate the unique monic generator of its Fitting ideal \([C(A/PA)]_A\). In virtue of Goss, [11, Theorem 3.6.3], we have

\[
[C(A/PA)]_A = P - 1.
\]

More generally, Anderson and Thakur have introduced in [3], for \( n \geq 1 \), a \( t \)-module called the \( n \)-th tensor power of the Carlitz module \( C \), denoted by \( C^\otimes n \), which allows to extend the above formula for \( \zeta(n) \) in our framework. Indeed, for all \( n \geq 1 \) and \( P \) a prime of \( A \), the \( A \)-module \( C^\otimes n(A/PA) \) is finite and the monic generator of its Fitting ideal is \( P^n - 1 \) [3, Proposition 1.10.3]. Furthermore, \([\text{Lie}(C^\otimes n)(A/PA)]_A = P^n \) and

\[
\zeta_C(n) = \prod_p \left( \frac{[\text{Lie}(C^\otimes n)(A/PA)]_A}{[C^\otimes n(A/PA)]_A} \right),
\]

convergence being ensured even with \( n \geq 1 \).

The value \( \zeta_C(1) \) is somewhat distinguished also because its classical counterpart \( \zeta(1) \) is a divergent series. If \( q = 2 \), then Carlitz result (2) implies that \( \zeta_C(1) \in K^\times \tilde{\pi} \) so that \( \text{exp}_C(\zeta_C(1)) \) is a torsion point for \( C \) in this case. A little computation shows that \( \zeta_C(1) = \frac{\tilde{\pi}}{\theta(\theta+1)} \) so that \( \text{exp}_C(\zeta_C(1)) \) is a point of \( \theta(\theta+1) \)-torsion and in fact, we find \( \text{exp}_C(\zeta_C(1)) = 1 \) (note that if \( q = 2 \), \( C_{\theta(\theta+1)}(1) = (\theta^2 + \theta + 1 + (\theta^2 + \theta)\tau + \tau^2)(1) = 0 \); if \( q > 2 \), \( 1 \) is always a point of infinite order).

In [8], Carlitz proves that

\[
\text{exp}_C(\zeta_C(1)) = 1
\]

for all \( q \); this is a completely different relation, if compared with (2).

Taelman [17] recently exhibited an appropriate setting to interpret the above formula as an instance of the class number formula. He worked more generally in the framework of Drinfeld modules defined over the ring of integers \( R \) of a finite extension \( L \) of \( K \).

Taelman associated to each such Drinfeld module \( \phi \) a finite \( A \)-module \( H(\phi/R) \) called the class module and a finitely generated \( A \)-module \( U(\phi/R) \) called the unit module. Taelman also introduced, for each such Drinfeld module \( \phi/R \), an \( L \)-series value

\[
L(\phi/R) = \prod_m \left( \frac{[\text{Lie}(\phi)(R/mR)]_A}{[\phi(R/mR)]_A} \right),
\]

\(^1\)We recall that if \( F : (\text{Rings}) \to (\text{Ab. groups}) \) is a functor, \( \text{Lie}(F) \) denotes the functor \( \text{Ker}(F(A[e]) \to F(A)) \), \( e \mapsto 0 \), where \( A[e] \) denotes the ring of dual numbers.
where the product runs over the maximal ideals of \( R \) (the convergence can be checked easily). Taelman fundamental Theorem [17, Theorem 1] states that

\[
L(\phi/R) = [H(\phi/R)]_A \text{Reg}(U(\phi/R)),
\]

where \( \text{Reg}(U(\phi/R)) \) denotes a regulator of the unit module defined by Taelman. It is easy to see that \( L(\phi/R) \) becomes \( \zeta_C(1) \) in the case of \( \phi = C \) and \( R = A \). In particular, since \( \exp_C \) induces an isometry of the disk \( \{ z \in \mathbb{C}_\infty; v_\infty(z) > -\frac{2}{q-1} \} \), the class \( A \)-module

\[
H(C/A) = \frac{C(K_\infty)}{\exp_C(K_\infty) + C(A)}
\]
is trivial. For similar reasons, the unit \( A \)-module

\[
U(C/A) = \{ f \in K_\infty; \exp_C(f) \in C(A) \}
\]
is the free \( A \)-submodule of \( K_\infty \) generated by \( \log_C(1) \), the Carlitz logarithm evaluated at one (this is the local composition inverse of \( \exp_C \) at 0 and converges at one). From this, Carlitz formula (4) follows.

A generalization of Taelman’s Theorem was recently considered by Jiangxue Fang [10] to certain \( L \)-series values associated to Anderson’s \( t \)-modules. If \( E \) is a \( t \)-module defined over \( R \) (the ring of integers of \( L \) a finite extension of \( K \)), the definition of \( L(E/R) \) is formally the same as Taelman’s for Drinfeld modules, and we have \( L(C^{\otimes n}/A) = \zeta_C(n) \). Fang’s Theorem [10, Theorem 1.7] states a generalization of Taelman’s class number formula in this setting. His results makes a fundamental use of the machinery of shtukas as in Lafforgue’s paper [13].

2. Results.

In the preprint [5], we have generalized the formulas (2) and (4) in a different direction. For the sake of simplicity, we are now going to present a particular case of our results. For this purpose, we are going to introduce a generalization of the Carlitz module functor.

2.1. The Carlitz functor revisited. Let \( t_1, \ldots, t_s \) be indeterminates, let us denote by \( A_s \) the polynomial algebra \( A[t_1, \ldots, t_s] \). Let \( T_s \) be the standard Tate algebra of dimension \( s \), that is, the completion of the polynomial algebra \( \mathbb{C}_\infty[t_1, \ldots, t_s] \) for the Gauss norm \( \| \cdot \| \) associated to the absolute value \( | \cdot | \) of \( \mathbb{C}_\infty \) uniquely normalized by setting \( | \theta | = q^{-v_\infty(\theta)} = q \). We fix once and for all the embedding \( A[t_1, \ldots, t_s] \subset T_s \) determined by the embedding \( A \subset \mathbb{C}_\infty \).

The Carlitz module \( C(\mathbb{C}_\infty) \) over \( \mathbb{C}_\infty \) extends in an unique way to an \( A_s \)-module \( C(T_s) \) (we allow a slight abuse of notation; \( C(\mathbb{C}_\infty) \) is an \( A \)-module while \( C(T_s) \) is an \( A_s \)-module, but this will not lead to confusion). Explicitly, \( C(T_s) \) is the unique \( A_s \)-module having \( T_s \) with the usual multiplication as the underlying \( \mathbb{F}_q[t_1, \ldots, t_s] \)-module, and such that the (left) multiplication of an element \( x \in T_s \) by \( \theta \), denoted by \( C_\theta(x) \), is \( \theta x + \tau(x) \), where

\[
\tau : T_s \rightarrow T_s
\]
represents the \( \mathbb{F}_q[t_1, \ldots, t_s] \)-linear extension of \( \tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \).

To give a concrete example, let us consider \( f = t_1 - \theta \), which belongs to \( A_1 \) hence to \( T_1 \). Then, \( \tau(f) = t_1 - \theta^q \) and \( C_\theta(f) = t_1(\theta + 1) - (\theta^2 + \theta^q) \). In the case of \( s = 1 \), we also prefer to write \( T = T_1 \) and \( t = t_1 \).

Since \( \tau \) induces a continuous automorphism of \( T_s \) for all \( s \), there is an unique \( \mathbb{F}_q[t_1, \ldots, t_s] \)-linear extension

\[
\exp_C : T_s \rightarrow T_s
\]
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of \( \exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty \) which is a continuous, open \( \mathbb{F}_q[t_1, \ldots, t_s] \)-linear endomorphism of \( \mathbb{T}_s \).

We further have the following exact sequence of \( A_s \)-modules:

\[
0 \to \tilde{\pi} A_s \to \mathbb{T}_s \to C(\mathbb{T}_s) \to 0.
\]

Here, the third arrow is \( \exp_C \), and it is understood that \( C_{a}(\exp_C(f)) = \exp_C(af) \) for all \( a \in A_s \).

2.2. Torsion. The above function \( \exp_C \) has quite a rich torsion structure. If \( f \in A_s \) is such that \( f^{-1} \in \mathbb{T}_s \) (this means that \( f \) is a polynomial which has leading coefficient in \( \mathbb{F}_q^\times \) as a polynomial in \( \theta \), or in other words, \( f \in \mathbb{T}^\times \), group of units of \( \mathbb{T} \)), then

\[
C_f \left( \exp_C \left( \frac{\tilde{\pi} \theta^j}{f} \right) \right) = 0, \quad j = 0, \ldots, \deg_\theta(f) - 1.
\]

It is easily seen, under the hypothesis that \( f \) is a unit of \( \mathbb{T}_s \), that the functions \( \exp_C \left( \frac{\tilde{\pi} \theta^j}{f} \right) \) constitute an \( \mathbb{F}_q[t_1, \ldots, t_s] \)-basis of the submodule \( \text{Ker}(C_f) \subset C(\mathbb{T}_s) \), free of rank \( d \).

These functions can be used to construct Galois representations

\[
\text{Gal}(K^{\text{sep}}/K) \to \text{GL}_d(\mathbb{F}_q[[t_1, \ldots, t_s]])
\]

(here, \( K^{\text{sep}} \) denotes the separable closure of \( K \) in \( \mathbb{C}_\infty \)). More generally, we can attach similar Galois representations to the torsion modules of uniformizable Drinfeld modules of rank one defined over \( A_s \) introduced in [5]. The simplest case is given by the Anderson-Thakur function, first introduced by Anderson and Thakur in [3]:

\[
\omega = \exp_C \left( \frac{\tilde{\pi}}{\theta - t} \right) \in \mathbb{T}^\times,
\]

which is, by the above discussion, the generator of the \( \mathbb{F}_q[t] \)-module \( \text{Ker}(C_{\theta-t}) \subset \mathbb{T} \), free of rank one. Here, it is well known that the associated Galois representation

\[
\text{Gal}(K^{\text{sep}}/K) \to \text{GL}_1(\mathbb{F}_q[[t]])
\]

is surjective (use, for example [16, Theorem 0.2]). Since \( \tau(\omega) = (t-\theta)\omega \) (this is equivalent to saying that \( \omega \in \text{Ker}(C_{\theta-t}) \)), we also deduce:

**Proposition 1.** The following properties hold:

1. We have the product expansion

\[
\omega = (-\theta)^{-1} \prod_{i \geq 0} \left( 1 - \frac{t}{\theta^{q^i}} \right)^{-1},
\]

convergent in \( \mathbb{T} \).

2. \( \omega \), as an element of \( \mathbb{T} \), extends to a meromorphic function over \( \mathbb{C}_\infty \) and has, as unique singularities, simple poles at the points \( t = \theta, \theta^2, \theta^q, \ldots \). The residues can be explicitly computed. In particular, we have \( \text{Res}_{t=\theta}(\omega) = -\tilde{\pi} \).

3. The function \( 1/\omega \) extends to an entire function \( \mathbb{C}_\infty \to \mathbb{C}_\infty \) with unique zeros located at the poles of \( \omega \).
2.3. L-series values in $T_s$. We construct the Carlitz zeta values $\zeta_C(n; s) \in T_s$, $n > 0$. They are defined as follows for $n \geq 1$ an integer and $s \geq 0$:

$$\zeta_C(n; s) = \sum_{a \in A^+} a^{-n}a(t_1) \cdots a(t_s) \in T_s \cap K_\infty[[t_1, \ldots, t_s]].$$

It is easy to show that $\zeta_C(n, s) \in T_s^*$ and that $\|\zeta_C(n, s)\| = 1$. Carlitz zeta values are a special case of our construction with $s = 0$. In [4] it is proved that, in terms of the variables $t_1, \ldots, t_s$, these series define entire functions $C'_\infty \rightarrow C_\infty$. Therefore, evaluation at $t_i = \theta^{q^k}$, $i = 1, \ldots, s$ and $k_i \in \mathbb{Z}$ makes sense and, for $n > 0$,

$$\zeta_C(n) = \zeta_C(n; 0) = \zeta_C(n + q^{k_1} + \cdots + q^{k_s}; s)|_{t_i = \theta^{q^k}}.$$

In this respect, we can view these functions as interpolations of Carlitz zeta values.

In [5], we prove:

**Theorem 2.** For $s \geq 0$, we have

$$\exp_C(\zeta_C(1; s)\omega(t_1) \cdots \omega(t_s)) = P_s\omega(t_1) \cdots \omega(t_s),$$

where $P_s \in A_s$. Moreover, for $s > 1$, we have $P_s = 0$ if and only if $s \equiv 1 \pmod{q - 1}$. In this case, we have

$$\zeta_C(1; s) = \frac{\bar{\pi}B_s}{\omega(t_1) \cdots \omega(t_s)},$$

with $B_s \in A_s$.

For $n = 0$, we re-obtain Carlitz Theorem (4). The vanishing of $P_s$ is equivalent to (5), since this means that $\zeta_C(1; s)\omega(t_1) \cdots \omega(t_s)$ is in the kernel of $\exp_C$. Of course, formula (5) can be viewed as a generalization of Carlitz Theorem (2) in the case $n = 1$, but for various values of $s \equiv 1 \pmod{q - 1}$.

One of the ingredients of the proof of Theorem 2 is a variant of Taelman’s class number formula for Drinfeld modules defined over integral closures of $A$ in finite extensions of $K$ [17, Theorem 1]. This was obtained by F. Demeslay and a particular case of his result (corresponding to what we need to prove Theorem 2) appears in the appendix of [5].

Demeslay’s method is inspired by Taelman’s proof in [17] and uses a generalization of the notion of Drinfeld module introduced in [5] (the extension to $T_s$ of the Carlitz module is an example of this, but there exist many non-isomorphic Drinfeld modules of rank one over $T_s$ as soon as $s \geq 1$).

In [5] we prove a generalization of Theorem 2 which holds for more general Drinfeld modules of rank one over $T_s$, provided that they are defined over $A_s$. We point out that Demeslay is currently working on a generalisation of his class number formula which may well handle at once $t$-modules and Drinfeld $A_s$-modules (it would then encompass Fang’s and Taelman’s class number formulas).

Comparing (5) and (2) we are led to the following:

**Question 3.** Is it true that

$$\bar{\pi}^{-n}\zeta_C(n; s)\omega(t_1) \cdots \omega(t_s) \in K(t_1, \ldots, t_s)$$

if and only if $n \equiv s \pmod{q - 1}$?

This question is also suggested by the results in [4], in which we prove that $s > 1$ and $n \equiv s \pmod{q - 1}$ imply (6). For example, in [14] it is proved that

$$\zeta_C(1; 1) = \frac{\bar{\pi}}{(\theta - t)\omega(t)}.$$
Proposition 1, which provides analogies between Euler’s gamma function and the function \( \omega \) of Anderson and Thakur, also provides us (thanks to (7)) with the entire continuation \( \mathbb{C}_\infty \to \mathbb{C}_\infty \) of \( \zeta_C(1;1) \), and the whole phenomenology of the trivial zeros and the special values of \( \zeta_C(1;1) \) (as in (2)).

This gives to the functional identity (7) a role similar to that of the functional equation of Riemann’s zeta function and the second part of Theorem 2 gives a partial generalization of this. For further information, read [5].

2.3.1. A transcendence question. Let \( (\mathcal{A}, \nu) \) be an integral difference ring, that is, a domain \( \mathcal{A} \) together with an endomorphism \( \nu : \mathcal{A} \to \mathcal{A} \). A \( \nu \)-polynomial in \( X_1, \ldots, X_s \) over \( \mathcal{A} \) is a polynomial of

\[
\mathcal{A}[X_1, \ldots, X_s, \nu(X_1), \ldots, \nu(X_s), \nu^2(X_1), \ldots, \nu^2(X_s), \ldots]
\]

(in infinitely many indeterminates \( \nu^k(X_i), k \geq 0, 1 \leq i \leq s \)). Let \( \mathcal{B}/\mathcal{A} \) be an integral difference ring extension. We say that elements \( x_1, \ldots, x_n \in \mathcal{B} \) are \( \nu \)-independent over \( \mathcal{A} \) if the only \( \nu \)-polynomial in \( X_1, \ldots, X_n \) over \( \mathcal{A} \) vanishing at \( (x_1, \ldots, x_n) \) is the zero polynomial.

We can give the question 3 a transcendental flavor by choosing \( \mathcal{A} = A_{\infty} = A[t_1, t_2, \ldots] = \bigcup_s A_s \) with \( \nu = \tau_p \), the unique \( \mathbb{F}_p[t_1, t_2, \ldots] \)-linear endomorphism such that \( \tau_p(\theta) = \theta^p \) (here, \( p \) is the prime dividing \( q \)). We recall that, in [9], Chang and Yu have proved that the elements \( \pi, \zeta_C(n) \) of \( \mathbb{C}_\infty, n \geq 1, q - 1 \nmid n, p \nmid n \) are algebraically independent over \( K \).

The conditions on \( n \) allow us to avoid the Bernoulli-Carlitz relations (2) and the trivial relations \( \zeta(pn) = \zeta(n)^p \).

We interpret the elements \( \zeta_C(n; s) \) of \( \mathbb{T}_s \) as a generalization of Carlitz’ zeta values. This seems to legitimate the next question:

**Question 4.** Is it true that \( \pi \) and the series \( \zeta_C(n; s) \) with \( i \geq 1, s \geq 0, n \neq s \) (mod \( q - 1 \)) and \( p \nmid n \) are \( \tau_p \)-independent over \( A_{\infty} \)?

The conditions on \( n, s \) are required to avoid that the question has negative answer trivially. Indeed, if \( n \equiv s \) (mod \( q - 1 \)), it is proved in [4] that (6) holds and we know that \( x_i = \omega(t_i) \) is a solution of \( \tau^e_p(X_i) = (t_i - \theta)X_i \) where \( e \) is such that \( p^e = q \) so that \( \pi \) and \( \zeta_C(n; s) \) are in this case \( \tau \)-dependent (that is, not \( \tau \)-independent). On the other hand, there is the trivial relation \( \tau_p(\zeta_C(n; s)) = \zeta_C(pm; s) \) that we want to equally avoid.

2.3.2. Anderson log-algebraic Theorem revisited. Theorem 2 can be applied to deduce an operator theoretic version of Anderson’s log-algebraic Theorem (see [2]). We come back to the \( \tau \)-polynomials of §2.3.1. The ring

\[
\mathcal{A}_\tau = K[X_1, \ldots, X_s, \tau(X_1), \ldots, \tau(X_s), \tau^2(X_1), \ldots, \tau^2(X_s), \ldots]
\]

is endowed with a structure of difference ring with the operator \( \tau \) which sends \( c \in \mathbb{C}_\infty \) to \( c^\tau \) and \( \tau^k(X_i) \) to \( \tau^{k+1}(X_i) \). In particular, we have, for all \( d \in \mathbb{Z} \), the polynomials

\[
w_d := \sum_{a \in \mathcal{A}^+} a^{-1} C_a(X_1) \cdots C_a(X_s) \in \mathcal{A}_\tau.
\]

Let \( Z \) be a variable. We can define the formal series

\[
\mathcal{L}_\tau := \sum_{d \geq 0} 2^{q^d} w_d \in \mathcal{A}_\tau[[Z]].
\]

We prove, in [5]:

**Theorem 5.** \( \exp_C(\mathcal{L}_\tau) \in \mathcal{A}_\tau[Z] \).
The interest of this result relies on the fact that we can evaluate at $X_i$ elements of $\mathbb{T}_s$, not just elements of $\mathbb{C}_\infty$ as in Anderson’s original result.

3. Global $L$-series for $\varphi$-sheaves

3.1. Settings. We recall here the definition of the global $L$-function associated to a $\varphi$-sheaf. Our references are [6, 7, 11, 18].

We fix an absolutely irreducible smooth affine scheme $Y$ over $\mathbb{F}_q$ (we call it the coefficient scheme). We denote by $A$ the ring $H^0(Y, \mathcal{O}_Y)$. For any $\mathbb{F}_q$-scheme of finite type $X$ (called the base scheme), we write $X_Y := X \times_{\mathbb{F}_q} Y$.

If $X = \text{Spec}(A)$ for some finitely presented $\mathbb{F}_q$-algebra as above, then $X_Y$ is just $\text{Spec}(A \otimes_{\mathbb{F}_q} A)$. We denote by $\sigma : X \to X$

the map induced by the Frobenius morphism defined by $x \mapsto \sigma x = x^q$ on the sheaf $\mathcal{O}_X$.

We endow $X_Y$ with the scheme endomorphism $\varphi = \sigma \times \text{id}.$

**Definition 6** (Drinfeld). A $\varphi$-sheaf $\mathcal{F}$ of rank $r$ on $X$ over $A$ is a locally free $\mathcal{O}_{X_Y}$-module $\mathcal{F}$ of finite rank $r$, endowed with an injective morphism $\varphi : \sigma^* \mathcal{F} \to \mathcal{F}$.

A morphism of $\varphi$-sheaves is an $\mathcal{O}_{X_Y}$-linear morphism with respect to the action of $\varphi$.

Compare with Definition 3.2.1 of Böckle and Pink book [7] where more general sheaves are considered. In the present note we always suppose that the underlying sheaf $\mathcal{F}$ is locally free.

3.1.1. Example. We choose $X = A^1$ and $Y = A^s$ (case in which $A = \mathbb{F}_q[\theta]$ and $A = \mathbb{F}_q[t_1, \ldots, t_s]$). Then, $X_Y = \text{Spec}(A[t_1, \ldots, t_s])$. For $\mathcal{F}$, we choose the structure sheaf of $X_Y$. Then, $\sigma$ induces the map $P(\theta, t_1, \ldots, t_s) \in A[t_1, \ldots, t_s] \mapsto P(\theta^q, t_1, \ldots, t_s) \in A[t_1, \ldots, t_s]$.

Let $\mathcal{F}$ be a $\varphi$-sheaf on $X$ over $A$. Then, $\mathcal{F}$ can be identified with a projective $A[t_1, \ldots, t_s]$-module of finite rank $r$ which can be injected in its free extension by zero, which is a free $A[t_1, \ldots, t_s]$-module of finite rank $r'$ endowed with a $\sigma$-semilinear injective morphism acting as the Frobenius over $A$, and trivially on the variables $t_i$.

3.1.2. Example. Another important particular case is that of the base scheme $X = x = \text{Spec}(k_x)$ where $k_x$ is a finite extension of $\mathbb{F}_q$ of degree $d_x$, and coefficient scheme $Y$ absolutely irreducible smooth affine scheme over $\mathbb{F}_q$. Let $\mathcal{F}$ be a $\varphi$-sheaf on $x$ over $A$. Then, the dual characteristic polynomial (see Lemma-Definition 8.1.1 of [7])

$$\det_A (\text{id} - T \varphi | \mathcal{F}) \in 1 + TA[T]$$

is well defined ($T$ denotes an indeterminate). In fact, it belongs to $1 + T^{d_x} A[t^{d_x}]$ [7, Lemma 8.1.4]. The naive $L$-series of $\mathcal{F}_x$ is defined by

$$L(x, \mathcal{F}, T) = \det_A (\text{id} - T \varphi | \mathcal{F})^{-1} \in 1 + T^{d_x} A[[T^{d_x}]]$$

(see [7, Definition 8.1.6]).
3.2. Global $L$-functions. Now, let us consider, more generally, a scheme of finite type $X$ over $\mathbb{F}_q$ and a $\wp$-sheaf $\mathcal{F}$ on $X$ over $A$. We denote by $|X|$ the set of closed points of $X$. The choice of $x \in |X|$ determines a morphism $i : \text{Spec}(k_x) \to X$ and we can construct a pull-back (stalk at $x$) $i_*^x \mathcal{F}$ of $\mathcal{F}$: a $\wp$-sheaf on $\text{Spec}(k_x)$ over $A$ whose underlying sheaf is again locally free. We define, following [18, §2] or [7, Definition 8.1.8], the naive-$L$-function of $\mathcal{F}$ on $X$ over $A$ as the product:

$$L(X, \mathcal{F}, T) = \prod_{x \in |X|} L(x, i_*^x \mathcal{F}, T) \in 1 + TA[[T]].$$

The attribute of naive comes from the theory of crystals over function fields developed by Böckle and Pink [7]. They associate crystalline $L$-series canonically to $A$-crystals. In the case in which the underlying sheaf is locally free and $A$ is reduced, the naive and the crystalline $L$-series coincide (cf. [7, Corollary 9.4.3]). We recall that, among other results, Taguchi and Wan [18, Theorem 4.1] proved that $(\mathcal{F}$ being locally free and $A$ the ring of integers of a finite extension of $\mathbb{F}_q(t))$ $L(X, \mathcal{F}, T)$ is rational in $T$.

To define a $\textit{global} L$-function (following Goss, Taguchi and Wan [18, §8] and Böckle [6, Definition 2.8]) we have to make an assumption. We require that there exists a morphism

(9) \quad $f : X \to Y = \text{Spec}(A)$.

If $x \in |X|$, we set $p_x = f(x)$ and we have that the residue degree $d_{p_x}$ divides $d_x$.

3.2.1. Exponentiation. We now review Goss’ idea of exponentiation of an ideal. The exponentiation takes place in $A$ and for this reason, we need to suppose that $\text{dim}_{\mathbb{F}_q}(A) = 1$. Hence, we suppose, additionally, that $Y = \mathcal{C} = \overline{\mathbb{C}} \setminus \{\infty\}$, where $\overline{\mathbb{C}}$ is a smooth projective geometrically irreducible curve over $\mathbb{F}_q$, and $\infty$ is a point $\mathbb{F}_q$-rational on it (otherwise, the exponentiation of ideal becomes difficult to realize). In other words, $A = H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is a Dedekind ring.

Goss’ topological group of exponents is (for the $\infty$-adic theory)

$$S_\infty = \mathbb{C}_\infty \times \mathbb{Z}_p,$$

where $\mathbb{C}_\infty$ is the completion of an algebraic closure of $K_\infty$, the completion of the fraction field of $A$ at the chosen infinitesimal place.

Let $I$ be a fractional ideal of $A$ and $s = (z, n) \in S_\infty$. The exponentiation of $I$ by $s$ is, by definition, the element of $\mathbb{C}_\infty$:

$$I^s = z^{\deg I} \langle I \rangle^n$$

($(I)$ denotes the the one unit part of $I$, depending on a choice of uniformizer $\pi$ of $K_\infty$, see [11, §8.2] and $\langle I \rangle^n$ denotes the $\mathbb{Z}_p$-exponentiation by $n$). Then, the global $L$-series associated to the datum of $X, \mathcal{C}, f, \mathcal{F}, \pi$ etc. is:

$$L^{\text{glob}}(X, \mathcal{F}, s) = \prod_{x \in |X|} L(x, i_*^x \mathcal{F}, T)|_{T^{\mathfrak{p}_x} = \mathfrak{p}_x^{-s}}.$$
3.2.2. Example. If $X = \text{Spec}(A)$ with $A = \mathbb{F}_q[\theta]$ and $Y = C = \text{Spec}(A)$ with $A = \mathbb{F}_q[t]$ with the map $f$ in (9) corresponding to

$$\mathcal{A} \rightarrow A, \quad a(t) \mapsto a(\theta).$$

We set $\mathcal{E}$ to be the structure sheaf of $X_Y$ with $\varphi = (t - \theta)(\sigma \times \text{id})$. In this case, it is easy to see that, for all $x = (p)$ closed point of $X$ (ideal generated by a prime $p$, that is, a monic irreducible polynomial of $A$),

$$L(p, \mathcal{E}, T) = \frac{1}{1 - pT}$$

so that

$$L^\text{glob}(X, \mathcal{E}, s) = \prod_p \left(1 - \frac{p}{p^s}\right)^{-1}$$

is the Goss' zeta function evaluated at $s - s_1$ (where $s_1 = (\pi^{-1}, 1) \in S_\infty$).

3.3. Alternative construction for a global $L$-series. Here we suppose, additionally, that both $X, Y$ are affine schemes and $X = \text{Spec}(A)$ and $Y = \text{Spec}(A)$.

We want to propose a different construction taking into account certain hidden features. This will give the Carlitz zeta values of §2 as a special case.

We have more freedom on the choice of $Y$; so we do not restrict to the case of an affine curve. Similar hypotheses occur in the definition of $A$-premotives by Tamagawa (see [19, Definition, p. 155]).

Exponentiation of ideals is anyway still needed, and will be performed now in the ring $A$ so that we suppose $X$ to be an affine curve: $X = \mathcal{C}$ with $\mathcal{C}, \overline{\mathcal{C}}, \infty, K_\infty, C_\infty$ etc. as in §3.2.1 ($K_\infty$ is now the completion of the fraction field of $A$ at $\infty$). There is an injective homomorphism of groups

$$s : \mathbb{Z} \rightarrow S_\infty, \quad n \mapsto (\pi^{-n}, n),$$

determined by the choice of $\pi$. Instead of letting the variable $s$ varying in the group $S_\infty$, we can even reduce ourselves to choose $s = s_n$ in the image of $\mathbb{Z}_{>0}$.

We choose $A = \mathbb{F}_q[t_1, \ldots, t_n]/\mathcal{P}$, with $\mathcal{P}$ a prime ideal, or (0). We choose a norm $| \cdot |_\infty$ on $C_\infty$. Then, the ring $C_\infty \otimes_{\mathbb{F}_q} A$ is endowed with the norm induced by the Gauss norm on $C_\infty \otimes_{\mathbb{F}_q} \mathbb{F}_q[t_1, \ldots, t_n]$. We denote by $\mathcal{T}$ the affinoid Tate algebra $C_\infty \otimes_{\mathbb{F}_q} A$ obtained by completing $C_\infty \otimes_{\mathbb{F}_q} A$ for this Gauss norm. We denote by $\| \cdot \|_\mathcal{T}$ the standard norm of $\mathcal{T}$.

We construct local factors of our new global $L$-functions. Let $x$ be again a closed point of $X$ represented by a prime ideal $P$ and let us consider a $\varphi$-sheaf $\mathcal{E}$ on $X$ over $A$.

We do a different substitution in the local factor (8). We define the local factor at $x$ of our global $L$-series by setting:

$$\mathcal{L}(x, i_x^* \mathcal{E}, n)^{-1} = \det(id - P^{-s_n} \varphi | i_x^* \mathcal{E})^{-1} \in 1 + P^{-s_n} A[P^{-s_n}] \subset C_\infty \otimes A.$$

Since

$$\|\mathcal{L}(x, i_x^* \mathcal{E}, n)\|_\mathcal{T} = 1, \quad \|\mathcal{L}(x, i_x^* \mathcal{E}, n) - 1\|_\mathcal{T} \rightarrow 0$$

with the limit taken for the cofinite filter on $|X|$, the product

$$\mathcal{L}^\text{glob}(X, \mathcal{E}, n) := \prod_{x \in |X|} \mathcal{L}(x, i_x^* \mathcal{E}, n)$$

converges in $\mathcal{T}$ to our new global $L$-function. This can be viewed as a function in virtue of the fact that the elements of $\mathcal{T}$ themselves can be viewed as functions.
Example. When $X = \mathbb{A}^1$ and $Y = \mathbb{A}^s$ (case in which $A = \mathbb{F}_q[t_1, \ldots, t_s]$ so that the ring $A_s$ that we have used in §2 is $A_s = \mathbb{F}_q[t_1, \ldots, t_s] = \text{Spec}(X \times \mathbb{F}_q \mathbb{F}_q(Y))$, we choose $\mathcal{E}_s$ to be the structure sheaf of $X_Y$ with $\varphi$ defined by $(t_1 - \theta) \cdots (t_s - \theta)(\sigma \times \text{id})$. In this case, for all $n > 0$, it is easy to check that

$$L^{\text{glob}}(X, \underline{\mathcal{E}}_s, n) = \zeta_C(n; s).$$

Now, we can vary $t_1, \ldots, t_s$ in the polydisk $\{(t_1, \ldots, t_s) \in \mathbb{C}_\infty; |t_i| \leq 1\}$; this provides us with a different type of global $L$-function.

More generally, it is not difficult to show that, for $\phi$ a Drinfeld module of rank one with parameter $\alpha \in A_s$ as defined in [5], the $L$-series value $L(n, \phi)$ defined there is equal to $L^{\text{glob}}(X, \underline{\mathcal{E}}, n)$, where $\varphi$ now acts as $\alpha(\sigma \times \text{id})$. Explicitly,

$$L^{\text{glob}}(X, \underline{\mathcal{E}}_s, n) = L(n, \phi),$$

in the notation of [5].

4. Final remarks.

It is likely that the functions of §3.3 can be further generalized in yet another arithmetically interesting direction. To simplify, we focus on the function

$$L(\chi_t, 1) = \sum_{a \in A^+} \frac{a(t)}{a} \in \mathbb{T},$$

which corresponds to $X = Y = \mathbb{A}^1$ and $\mathcal{E} = \mathcal{O}_{X_Y}$ with $\varphi = (t - \theta)(\sigma \times \text{id})$. From now on, $A, K, K_\infty, \mathbb{C}_\infty$ are as in §1.

Let $u, v$ be variables of $\mathbb{C}_\infty$ such that $|u| \leq 1 < |v|$. We can define the series of functions of two variables:

$$L^\#(u, v) = \sum_{a \in A^+} \frac{a(u)}{a(v)}.$$

We have $L^\#(t, \theta) = \zeta_C(1; 1)$. By (7), and the first part of Proposition 1, we observe that

$$L^\#(u, v) = \prod_{i > 0} \frac{1 - \frac{u}{v^{q^i}}}{1 - \frac{v}{v^{q^i}}}.$$

In particular, if $x, y, z$ are variables of $\mathbb{C}_\infty$ such that $|z| \leq 1$ and $|x|, |y| < 1$, we have

$$\frac{L^\#(z, 1/y)}{L^\#(z, 1/x)} = \prod_{i > 0} \frac{1 - x^{q^i - 1}}{1 - y^{q^i - 1}} \prod_{i > 0} \frac{1 - y^{q^i}z}{1 - x^{q^i}z}.$$

We now follow Anderson, in [1]. In his definition of the function $\tau$ of an $A$-lattice, he introduces, in loc. cit. §2.3, the auxiliary formal series in four variables

$$h(t, x, y, z) = (1 - tz) \prod_{i \geq 0} \frac{1 - y^{q^i}z}{1 - x^{q^i}z} = \sum_{k \geq 0} h_k(t, x, y) z^k \in \mathbb{F}_q[[t, x, y, z]].$$

This can be viewed as a generating series for solitons, and by the above observation, $h(t, x, y, z)$ is related to the ratio of two of our series $L^\#$ by:

$$h(t, x, y, z) = \frac{(1 - tz)(1 - xz)}{1 - yz} \frac{L^\#(z, 1/y)}{L^\#(z, 1/x)} \prod_{i > 0} \frac{1 - y^{q^i - 1}}{1 - x^{q^i - 1}}.$$
INTERPOLATING CARLITZ ZETA VALUES

Another reason to investigate these functions is suggested in [15]; looking at [15, Proposition 28], we need to introduce the functions of two variables $|u| \leq 1 < |v|

$$\omega^\tau_t(u, v) = v_0 \prod_{i \geq 0} \tau^i \left(1 - \frac{u}{v}\right)^{-1}$$

(where $v_0$ is solution of $\tau(v_0) + v v_0 = 0$) to make visible analogues of the multiplication relations and the cyclotomic relations:

$$\omega^\tau_t(u, v) = \prod_{i=0}^{n-1} \tau^i(\omega^\tau_{\tau^n}(u, v)),$$

$$\omega^\tau_t(u, v^n) = \prod_{i=0}^{n-1} \tau^i(\omega^\tau(u, \zeta^i v)).$$

In the formulas above, $n > 0$, $\zeta$ is a solution of $X^n = 1$ in $\mathbb{C}_\infty$ such that $\zeta^k \neq 1$ for all $1 < k < n$. This suggests the study of the analytic and the arithmetic properties of the two variable series

$$L_t^\tau(u, v) = \sum_{a \in (\mathbb{F}_q^{ac})^\tau} \frac{a(u)}{a(v)}.$$

The sum takes place in the subring of $\mathbb{F}_q^{ac}[\theta]$ whose elements are the monic polynomials with coefficients fixed by $\tau$. Of course, $\mathbb{F}_q^{ac}$ denotes the algebraic closure of $\mathbb{F}_q$ in $\mathbb{C}_\infty$.

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