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Independence of complex Cantor series and Cantor products

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Abstract

Two representations of a complex number via a Cantor series and a Cantor product are introduced. The problem of finding criteria for algebraic and linear independence based on these representations is discussed. Some criteria based on generalized Liouville's theorem about approximation by numbers from a fixed algebraic number field are mentioned.

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1 Introduction

Algebraic and/or linear independence criteria for real numbers represented by various representations have been of interest for quite some time, see e.g. [1], [2], [3], [7]. Here, a brief discussion about independence criteria for complex numbers represented via Cantor series and Cantor product expansions is given. Based on Cantor series expansions for real numbers, we extend them to complex numbers in the usual manner. The representation of real numbers as Cantor products is well-known ([8, Section 33]), while that of complex numbers has only recently been established in [6] and we make use of these results in our work here.

We now recall some well-known facts about real Cantor series and products.

Proposition 1. *I) ([8, Section 33], [4]) Let* $\mathcal{M} := \{m_k\}_{k\geq 1}$, $m_k \geq 2$, *be a sequence of natural numbers. A real number* $\alpha \in [0, 1)$ *is uniquely representable as a (real) Cantor (or* $\mathcal{M}$*-Cantor) series of the form*

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{m_1 m_2 \cdots m_k}, \tag{1.1}$$

where

$$c_k \in \{0,1,\dots,m_k-1\}\,,\quad c_k < m_k - 1 \text{ for infinitely many } k. \tag{1.2}$$

Moreover, assuming that each prime divides infinitely many of the m_k, then α is irrational if and only if both $c_k > 0$ and $c_k < m_k - 1$ hold infinitely often.

II) ([8, Section 35]) A real number $\alpha > 1$ is uniquely representable as a (real) Cantor product of the form

$$\alpha = \prod_{k=1}^{\infty}\left(1+\frac{1}{a_k}\right),$$

where $a_k \in \mathbb{N}$ are subject to

$$a_{k+1} \geq a_k^2, \text{ and there are infinitely many } k \text{ such that } a_k \geq 2. \tag{1.3}$$

Moreover, α is rational if and only if $a_{k+1} = a_k^2$ for all sufficiently large k.

III) ([5, Theorem 2.1]) A necessary and sufficient condition for α given by the convergent Cantor series (1.1) to be irrational is that, for every $B \in \mathbb{N}$, we can find an integer A and a subsequence $k_1, k_2, k_3, \dots$ such that

$$\frac{A}{B} < \alpha_{k_j} < \frac{A+1}{B} \quad (j \in \mathbb{N}),$$

where $\alpha = \alpha_1$ and for $j > 1$,

$$\alpha_j = \frac{c_j}{m_j} + \frac{c_{j+1}}{m_j m_{j+1}} + \frac{c_{j+2}}{m_j m_{j+1} m_{j+2}} + \cdots .$$

IV) ([5, Theorem 2.2]) Let $1 \leq k_1 \leq k_2 \leq \cdots$ be a sequence of positive integers and let

$$\frac{c_1}{m_1} + \frac{c_2}{m_1 m_2} + \cdots + \frac{c_{k_1}}{m_1 m_2 \cdots m_{k_1}} = \frac{C_1}{M_1}, \quad M_1 = m_1 m_2 \cdots m_{k_1}$$
$$\frac{c_{k_1+1}}{m_{k_1+1}} + \frac{c_{k_1+2}}{m_{k_1+1} m_{k_1+2}} + \cdots + \frac{c_{k_2}}{m_{k_1+1} m_{k_1+2} \cdots m_{k_2}} = \frac{C_2}{M_2},$$

where $M_2 = m_{k_1+1} m_{k_1+2} \cdots m_{k_2}$, and so on. Then (1.1) thus reduces to

$$\alpha = \frac{C_1}{M_1} + \frac{C_2}{M_1 M_2} + \cdots + \frac{C_k}{M_1 M_2 \cdots M_k} + \cdots$$

with $M_k \geq 2$ and $0 \leq C_k \leq M_k - 1$ for each k. That is, from (1.1) another Cantor series expansion with respect to the new sequence $M_1, M_2, \dots$ is introduced; such a procedure is referred to as a condensation. A necessary and sufficient condition for the series (1.1), under (1.2), to be rational is that there exist coprime integers $0 \leq a \leq b$, a condensation and an integer N such that, for all $k \geq N$,

$$C_k = \frac{a}{b}\,(M_k - 1)\,.$$

2 Complex Cantor series and products

Since each $\beta \in \mathbb{C}$ can be uniquely written as

$$\beta = \alpha_x + i\alpha_y \qquad (i := \sqrt{-1}),$$

where α_x and α_y are real numbers.

Throughout the rest of the paper, we restrict our attention to the case where both α_x *and* α_y *lie in the open interval* $(0,1)$.

From (1.1), both α_x and α_y can uniquely be represented as

$$\alpha_x = \sum_{k=1}^{\infty} \frac{c_k}{m_1 m_2 \cdots m_k}, \quad \alpha_y = \sum_{k=1}^{\infty} \frac{d_k}{m_1 m_2 \cdots m_k},$$

where

$$c_k \text{ and } d_k \in \{0, \ldots, m_k - 1\} \text{ and both are } < m_k - 1 \text{ for infinitely many } k. \tag{2.1}$$

We define the complex Cantor or complex $\mathcal{M}$-Cantor series expansion for β as

$$\beta = \sum_{k=1}^{\infty} \frac{c_k + i\, d_k}{m_1 m_2 \cdots m_k}$$

where c_k, d_k are subject to the conditions in (2.1).

We next define complex Cantor products. For $D \in \mathbb{N}$, let

$$\theta_D = \begin{cases} \frac{1}{2}\left(1 + \sqrt{-D}\right) & \text{if } \; -D \equiv 1 \mod 4 \\ \sqrt{-D} & \text{if } \; -D \not\equiv 1 \mod 4 \end{cases}$$

so that the ring of integers of $\mathbb{Q}(\sqrt{-D})$ is $\mathbb{Z}[\theta_D] = \{u + v\theta_D \; ; \; u, v \in \mathbb{Z}\}$. In 1989, A. Knopfmacher ([6]) introduced a new expansion for a complex number as a product of algebraic integers in a quadratic field $\mathbb{Q}(\sqrt{-D})$ which is analogous to the Cantor product expansion of a real number. His result reads: each $\beta \in \mathbb{C}$ has a representation

$$\beta = \prod_{k=1}^{\infty} \left(1 + \frac{1}{a_k}\right),$$

where

$$a_k \in \mathbb{Z}[\theta_D] \setminus \{0\} \qquad (k \geq 1),$$

and for k sufficiently large

$$|a_{k+1}| \geq \begin{cases} \mu^{-1}|a_k|^2 - (\mu^{-1} + 1)|a_k| - (1 + \mu), \; \mu = \frac{D+1}{4\sqrt{D}} \text{ if } \; -D \equiv 1 \mod 4 \\ \gamma^{-1}|a_k|^2 - (\gamma^{-1} + 1)|a_k| - (1 + \gamma), \; \gamma = \frac{\sqrt{D+1}}{2} \text{ if } \; -D \not\equiv 1 \mod 4. \end{cases}$$

Furthermore, for $D = 1$ or 2, the product terminates if and only if $\beta \in \mathbb{Q}(\sqrt{-D})$.

3 Independence criteria

Our algebraic and/or linear independence criteria were originated and closely related to the works in [1], [2], [3] and [7]. As seen in [2], a possible proof of algebraic independence is done via Liouville type arguments. Since the usual proof of Liouville's theorem is based on approximating algebraic numbers by rationals, extending this approach, one obtains two versions of generalized Liouville's theorem dealing with approximation by elements from a fixed algebraic number field.

Proposition 2. *I) Let β be an algebraic number of degree $m \geq 1$, all of whose conjugates (over $\mathbb{Q}$) are $\beta = \beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \ldots, \beta^{(m)}$. Let $a_0, a_1, \ldots, a_n (\neq 0)$ be given algebraic integers, let K be a Galois extension of $\mathbb{Q}$ containing all the a_i $(0 \leq i \leq n)$ and β, with $[K : \mathbb{Q}] = d$, and let $G := \mathrm{Gal}\,(K/\mathbb{Q})$ be the Galois group of K over $\mathbb{Q}$. If the polynomial $a(x) = a_0 + a_1 x + \cdots + a_n x^n$ satisfies $a\left(\beta^{(i)}\right) \neq 0$ for all $1 \leq i \leq m$, then there exists a positive constant $c = c(m, n, d, \beta)$, independent of $a_0, a_1, \ldots, a_n$, such that*

$$|a(\beta)| \geq \frac{c}{\mathcal{H}^{md-1}},$$

where $\mathcal{H} = \max_{0 \leq i \leq n} \{\mathcal{H}(a_i)\}$, and $\mathcal{H}(\gamma) = \max_{\sigma \in G}\{|\sigma(\gamma)|\}$ $(\gamma \in K)$ (called the house of γ).

II) Let

$$f(x_1, \ldots, x_r) = \sum_{i_1, \ldots, i_r} a_{i_1, \ldots, i_r}\, x_1^{i_1} \cdots x_r^{i_r} \in \mathbb{Z}[x_1, \ldots, x_r] \setminus \{0\} \tag{3.1}$$

be of degree $\deg_{x_j} f = d_j$ with respect to the variable x_j $(j = 1, \ldots, r)$. Let $P_1, \ldots, P_r, Q_1, \ldots, Q_r$ be nonzero algebraic integers, let K $(\supseteq \mathbb{Q})$ be an algebraic number field of degree d containing all the P_j, Q_j $(1 \leq j \leq r)$, and let $G := \mathrm{Gal}(K/\mathbb{Q})$ be its Galois group. If $f\left(P_1/Q_1, \ldots, P_r/Q_r\right) \neq 0$, then

$$\left| f\left(\frac{P_1}{Q_1}, \ldots, \frac{P_r}{Q_r}\right) \right| \geq \frac{1}{M_1^{dd_1} \cdots M_r^{dd_r} \left(A\, n_f\right)^{d-1}}$$

where $A := \max_{i_1, \ldots, i_r} |a_{i_1, \ldots, i_r}|$, n_f the number of terms in the sum (3.1), and $M_j = \max\{\mathcal{H}(P_j), \mathcal{H}(Q_j)\}$ $(1 \leq j \leq r)$, $\mathcal{H}$ denoting houses.

Sketch of proof. Let $f(x)$ be the minimal polynomial (over $\mathbb{Z}$) of β. Using the fact that the resultant of $f(x)$ and $a(x)$ is $R \neq 0$, we have $\left|\prod_{\sigma \in G} \sigma(R)\right| \geq 1$. On the other hand, from $|\sigma(R)| \leq (m+n)!\, F^n\, \mathcal{H}^m$, where $F := \max_{0 \leq j \leq m} \{|f_j|\}$, we get

$$|R| = \frac{\prod_{\sigma \in G} |\sigma(R)|}{\prod_{\sigma \in G,\, \sigma \neq \text{identity}} |\sigma(R)|} \geq \frac{1}{\{(m+n)!\, F^n\, \mathcal{H}^m\}^{d-1}}.$$

Since $\left|a(\beta^{(i)})\right| \leq L(a) \max\left(1, |\beta^{(i)}|^n\right)$ $(1 \leq i \leq m)$, where $L(a)$ is the length of $a(x)$, we get $|R| \leq |f_m|^n\, |a\,(\beta)|\, \{L(a) \max\left(1, \mathcal{H}(\beta)^n\right)\}^{m-1}$.

Taking $a(x)$ to be a linear polynomial in Proposition 2 I), the following generalized version of the classical Liouvill's theorem is recovered.

Corollary 1. *Let β be an algebraic number (over $\mathbb{Q}$) of degree $m \geq 1$ and let p, q $(\neq 0)$ be given algebraic integers. Let K be a Galois extension of $\mathbb{Q}$ containing p, q, β of degree $[K : \mathbb{Q}] = d$ and let $G := \mathrm{Gal}(K/\mathbb{Q})$ be its Galois group. If $p - q\sigma(\beta) \neq 0$ for all $\sigma \in G$, then there exists a positive constant $c = c(m, d, \beta)$, independent of p and q, such that*

$$|p - q\beta| \geq \frac{c}{\max\{\mathcal{H}(p), \mathcal{H}(q)\}^{md-1}}$$

where $\mathcal{H}(\gamma)$ denotes the house of $\gamma \in K$.

Using Proposition 2, one can deduce the following independence criteria.

Theorem 1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\setminus\{0\}$. Assume that there are infinitely many algebraic integers $P_{N,j}$, $Q_{N,j}$ $(\neq 0)$ $(1 \leq j \leq n,\ N \in \mathbb{N})$ all belonging to a finite Galois extension K (over $\mathbb{Q}$) with $[K : \mathbb{Q}] = d$, and $G = \mathrm{Gal}(K/\mathbb{Q})$ its Galois group, such that*

(i) $P_{N,j} - Q_{N,j}\sigma(\alpha_j) \neq 0 \quad (1 \leq j \leq n,\ \sigma \in G)$*;*

(ii) $M_{N,j} := \max\{\mathcal{H}(P_{N,j}), \mathcal{H}(Q_{N,j})\} \to \infty \quad (N \to \infty,\ 1 \leq j \leq n)$*,*
where $\mathcal{H}(\gamma)$ denotes the house of $\gamma \in K$;

(iii)

$$\lim_{N\to\infty} \frac{|\alpha_{j-1} - P_{N,j-1}/Q_{N,j-1}|}{|\alpha_j - P_{N,j}/Q_{N,j}|} = 0 \qquad (2 \leq j \leq n),$$

provided $n \geq 2$, and

(iv) for each positive number E, there exists $N_0 = N_0(E) \in \mathbb{N}$ such that

$$\left|\alpha_j - \frac{P_{N,j}}{Q_{N,j}}\right| \leq \frac{1}{(M_{N,1}M_{N,2}\cdots M_{N,j})^E} \quad (N \geq N_0,\ 1 \leq j \leq n).$$

Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are $\mathbb{Q}$-algebraically independent.

Moreover, if the condition (iv) is replaced by
(iv') there is a positive-valued function g of natural argument with $g(N) \to \infty$ $(N \to \infty)$ and there is an $N_0 = N_0(g) \in \mathbb{N}$ such that

$$\left|\alpha_j - \frac{P_{N,j}}{Q_{N,j}}\right| \leq \frac{1}{M_{N,1}M_{N,2}\cdots M_{N,j}\, g(N)} \quad (N \geq N_0,\ 1 \leq j \leq n),$$

then $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent.

4 Criteria for Cantor series

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be nonzero real numbers in the interval $(0,1)$ whose *infinite* $\mathcal{M}$-Cantor series are

$$\alpha_j := \sum_{k=1}^{\infty} \frac{c_{k,j}}{m_1 m_2 \cdots m_k} \qquad (1 \le j \le n)$$

subject to the digit conditions (1.2). For $N \in \mathbb{N}$, let

$$\frac{P_{N,j}}{Q_N} = \sum_{k=1}^{N} \frac{c_{k,j}}{m_1 m_2 \cdots m_k} \quad (1 \le j \le n), \quad Q_N := m_1 m_2 \cdots m_N.$$

Note that

$$M_{N,j} = \max\{\mathcal{H}(P_{N,j}), \mathcal{H}(Q_N)\} = \max\{|P_{N,j}|, |Q_N|\} = |Q_N| \to \infty \quad (N \to \infty).$$

As each Cantor series is infinite, we see that $P_{N,j} - Q_N\, \alpha_j \ne 0 \quad (1 \le j \le n)$.

Theorem 2. *Using the above notation and restrictions, suppose there exists an increasing function* $f : \mathbb{N} \to \mathbb{Z}$ *such that*

$$c_{k,j} = f(k)\, c_{k,j-1} \qquad (2 \le j \le n, \ k \in \mathbb{N}).$$

(i) If there is a function $g : \mathbb{N} \to \mathbb{Z}$ *with* $\min\{f(k), g(k)\} \to \infty \quad (k \to \infty)$ *and there exists a subsequence* $\{c_{k_\ell,j}\}_{\ell \ge 1}$ *of nonzero digits satisfying*

$$\frac{Q_{k_\ell+1}}{1 + c_{k_\ell+1,j}} \ge (M_{k_\ell,1} M_{k_\ell,2} \cdots M_{k_\ell,j})^{g(k_\ell)} \qquad (1 \le j \le n),$$

then $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are* $\mathbb{Q}$*-algebraically independent.*
(ii) If there is a function $g : \mathbb{N} \to \mathbb{Z}$ *with* $\min\{f(k), g(k)\} \to \infty \quad (k \to \infty)$ *and there exists a subsequence* $\{c_{k_\ell,j}\}_{\ell \ge 1}$ *of nonzero digits satisfying*

$$\frac{Q_{k_\ell+1}}{1 + c_{k_\ell+1,j}} \ge g(k_\ell)\, M_{k_\ell,1} M_{k_\ell,2} \cdots M_{k_\ell,j} \quad (1 \le j \le n),$$

then $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ *are* $\mathbb{Q}$*-linearly independent.*

Remarks. In order to compare Theorem 2 with classical results, consider the case $n = 1$.
I. Theorem 2 (ii) gives a criteria for the irrationality of $\alpha \in (0,1)$ that there exists a subsequence $\{c_{k_\ell}\}_{\ell \ge 1}$ of nonzero digits and a function $g : \mathbb{N} \to \mathbb{Z}$ with $g(k) \to \infty \quad (k \to \infty)$ such that

$$\frac{m_{k_\ell+1}}{1 + c_{k_\ell+1}} \ge g(k_\ell).$$

This criteria is not equivalent with the one given in Proposition 1 I) as witnessed via the following examples.

(a) The series $\frac{1}{10} + \frac{1}{10\cdot 10^2} + \frac{1}{10\cdot 10^2\cdot 10^3} + \cdots$ satisfies the criteria in Theorem 2 (ii), and so represents an irrational number. However, it does not satisfy the conditions in Proposition 1 I).
(b) The series $e - 2 = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots$ satisfies the conditions in both Theorem 2 (ii) and Proposition 1 I), and so represents an irrational number.
II. A criteria for the transcendence of $\alpha \in (0,1)$ given by Theorem 2 (i) is that there exists a subsequence $\{c_{k_\ell}\}_{\ell\geq 1}$ of nonzero digits and a function $g : \mathbb{N} \to \mathbb{Z}$ with $g(k) \to \infty \ \ (k \to \infty)$ such that

$$\frac{Q_{k_\ell+1}}{1 + c_{k_\ell+1}} \geq (Q_{k_\ell})^{g(k_\ell)}. \tag{4.1}$$

For a fixed $w > 0$, there is $N_0 \in \mathbb{N}$ such that for all $N_\ell \geq N_0$, we have $g(N_\ell) > w$. For $N_\ell \geq N_0$, if $\alpha \in (0,1)$ satisfies (4.1), then

$$\left|\alpha - \frac{P_{N_\ell}}{Q_{N_\ell}}\right| \leq \frac{c_{N_\ell+1}+1}{Q_{N_\ell+1}} \leq \frac{1}{(Q_{N_\ell})^{g(N_\ell)}} < \frac{1}{(Q_{N_\ell})^{w}},$$

showing that α is a Liouville number.

We turn next to complex numbers. Let

$$\beta_j := \alpha_x^{(j)} + i\,\alpha_y^{(j)} \qquad (j = 1, 2, \ldots, n)$$

be n nonzero complex numbers whose Cantor series representations are

$$\beta_j = \sum_{k=1}^{\infty} \frac{c_{k,j} + i\,d_{k,j}}{m_1 m_2 \cdots m_k} \qquad (j = 1, \ldots, n)$$

where $c_{k,j}, d_{k,j}$ are subject to the digit conditions (2.1). Assume that both $\{c_{k,j}\}_{k\geq 1}$, $\{d_{k,j}\}_{k\geq 1}$ are infinite sequences for each $j \in \{1, 2, \ldots, n\}$ and call its coresponding Cantor series doubly infinite. For convenience, we relabel the digits (in any fixed order) as

$$\{c_{k,1}, c_{k,2}, \ldots, c_{k,n}, d_{k,1}, d_{k,2}, \ldots, d_{k,n}\} =: \{e_{k,1}, e_{k,2}, \ldots, e_{k,2n}\} \quad (k \in \mathbb{N}).$$

For $N \in \mathbb{N}$, let

$$\frac{P_{N,j}}{Q_N} = \sum_{k=1}^{N} \frac{e_{k,j}}{m_1 m_2 \cdots m_k} \quad (j = 1, 2, \ldots, 2n), \quad Q_N := m_1 m_2 \cdots m_N.$$

Clearly,

$$M_{N,j} = \max\{\mathcal{H}(P_{N,j}), \mathcal{H}(Q_N)\} \to \infty \ \ (N \to \infty).$$

As each Cantor series is doubly infinite, we see that $P_{N,j} - Q_N\,\beta_j \neq 0 \ \ (1 \leq j \leq n)$.

Theorem 3. *Using the preceding notation, let $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{N} \to \mathbb{Z}$ with $f(k)$ being increasing and* $\min\{f(k), g(k)\} \to \infty \ \ (k \to \infty)$. *Since the Cantor series is doubly infinite, then for each $1 \le j \le 2n$ we assume that there exists a subsequence $\{e_{k_\ell,j}\}_{\ell \ge 1}$ of nonzero digits satisfying*

$$e_{k,j} = f(k)\, e_{k,j-1} \qquad (2 \le j \le 2n,\ k \in \mathbb{N}),$$

and

$$\frac{Q_{k_\ell+1}}{1 + e_{k_\ell+1,j}} \ge (M_{k_\ell,1} M_{k_\ell,2} \cdots M_{k_\ell,j})^{g(k_\ell)} \qquad (1 \le j \le 2n),$$

then $\beta_1, \beta_2, \dots, \beta_n$ are $\mathbb{Q}$-algebraically independent.

As for linear independence, we have:

Theorem 4. *Using the preceding notation, let $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{N} \to \mathbb{Z}$ with $f(k)$ being increasing and* $\min\{f(k), g(k)\} \to \infty \ \ (k \to \infty)$. *Since the Cantor series is doubly infinite, then for each $1 \le j \le n$ we assume that there exists subsequences $\{c_{k_\ell,j}\}_{\ell \ge 1}$, $\{d_{k_\ell,j}\}_{\ell \ge 1}$ of nonzero digits satisfying*

$$c_{k,j} = f(k)\, c_{k,j-1} \ \ (2 \le j \le n), \ \ \frac{Q_{k_\ell+1}}{1 + c_{k_\ell+1,j}} \ge g(k_\ell)\, M_{k_\ell,1} M_{k_\ell,2} \cdots M_{k_\ell,j} \ \ (1 \le j \le n)$$

or

$$d_{k,j} = f(k)\, d_{k,j-1} \ \ (2 \le j \le n), \ \ \frac{Q_{k_\ell+1}}{1 + d_{k_\ell+1,j}} \ge g(k_\ell)\, M_{k_\ell,1} M_{k_\ell,2} \cdots M_{k_\ell,j} \ \ (1 \le j \le n),$$

then $1, \beta_1, \beta_2, \dots, \beta_n$ are $\mathbb{Q}$-linearly independent.

5 Criteria for Cantor products

Let $\alpha_j > 1 \ \ (j = 1, \dots, n)$ be real numbers whose *infinite* Cantor product representations are

$$\alpha_j := \prod_{k=1}^{\infty} \left(1 + \frac{1}{a_{k,j}}\right) \qquad (1 \le j \le n),$$

where for each j, the positive integers $a_{k,j}$ are subject to the restrictions (1.3). For $N \in \mathbb{N}$, $1 \le j \le n$, let

$$\frac{P_{N,j}}{Q_{N,j}} = \prod_{k=1}^{N} \left(1 + \frac{1}{a_{k,j}}\right), \ Q_{N,j} = a_{1,j} a_{2,j} \cdots a_{N,j},\ P_{N,j} = (a_{1,j}+1)(a_{2,j}+1) \cdots (a_{N,j}+1).$$

Observe that

$$M_{N,j} = \max\{\mathcal{H}(P_{N,j}), \mathcal{H}(Q_{N,j})\} = \max\{|P_{N,j}|, |Q_{N,j}|\} = |P_{N,j}| \to \infty \ \ (N \to \infty).$$

Since each product is infinite, we infer that $P_{N,j} - Q_{N,j}\alpha_j \ne 0 \ \ (1 \le j \le n)$.

Theorem 5. *Using the preceding notation and restrictions, assume that there is a subsequence* $\{k_\ell\}_{\ell\geq 1}$ *of* $\mathbb{N}$ *such that*

$$\frac{a_{k_\ell,j}}{a_{k_\ell,j-1}-1}\to 0 \qquad (2\leq j\leq n,\ \ell\to\infty).$$

(i) If there exists $g:\mathbb{N}\to\mathbb{Z}$ *with* $g(k)\to\infty\ \ (k\to\infty)$ *such that*

$$(a_{k_\ell+1,j}-1)\frac{Q_{k_\ell,j}}{P_{k_\ell,j}}\geq (M_{k_\ell,1}\,M_{k_\ell,2}\cdots M_{k_\ell,j})^{g(k_\ell)} \quad (1\leq j\leq n),$$

then $\alpha_1,\alpha_2,\ldots,\alpha_n$ *are* $\mathbb{Q}$*-algebraically independent.*
(ii) If there exists $g:\mathbb{N}\to\mathbb{Z}$ *with* $g(k)\to\infty\ \ (k\to\infty)$ *such that*

$$(a_{k_\ell+1,j}-1)\frac{Q_{k_\ell,j}}{P_{k_\ell,j}}\geq g(k_\ell)\,M_{k_\ell,1}\,M_{k_\ell,2}\cdots M_{k_\ell,j} \qquad (1\leq j\leq n),$$

then $1,\alpha_1,\alpha_2,\ldots,\alpha_n$ *are* $\mathbb{Q}$*-linearly independent*

Remarks. Let us compare Theorem 5 with classical results by looking at the case $n=1$.
A. A criteria for irrationality of $\alpha>1$ given by Theorem 5 (ii) is that there exists a subsequence $\{k_\ell\}_{\ell\geq 1}$ of $\mathbb{N}$ and a function $g:\mathbb{N}\to\mathbb{Z}$ with $g(k)\to\infty\ \ (k\to\infty)$ such that

$$a_{k_\ell+1}\geq g(k_\ell)\frac{((a_1+1)(a_2+1)\cdots(a_{k_\ell}+1))^2}{a_1a_2\cdots a_{k_\ell}}+1. \tag{5.1}$$

This is not equivalent to the corresponding criteria in Proposition 1 II) (i.e., $a_{k+1}>a_k^2$ for sufficiently large k) as seen by taking the subsequence satisfying (5.1) to be that of even subscripts $k_\ell=2\ell$, while the elements of the remaining subscripts are taken suitably yet arbitrarily.
B. Theorem 5 (i) yields the conditions for the transcendence of $\alpha>1$ that there exists a subsequence $\{k_\ell\}_{\ell\geq 1}$ of $\mathbb{N}$ and a function $g:\mathbb{N}\to\mathbb{Z}$ with $g(k)\to\infty\ \ (k\to\infty)$ such that

$$(a_{k_\ell+1}-1)\,\frac{Q_{k_\ell}}{P_{k_\ell}}\geq (P_{k_\ell})^{g(k_\ell)}\,. \tag{5.2}$$

For a fixed $w>0$, there is $N_0\in\mathbb{N}$ such that for all $N_\ell\geq N_0$, we have $g(N_\ell)>w$. For $N_\ell\geq N_0$, if α satisfies (5.2), then

$$\left|\alpha-\frac{P_{N_\ell}}{Q_{N_\ell}}\right|\leq\frac{P_{N_\ell}}{Q_{N_\ell}\,(a_{N_\ell+1}-1)}\leq\frac{1}{(P_{N_\ell})^{g(N_\ell)}}<\frac{1}{(Q_{N_\ell})^{g(N_\ell)}}<\frac{1}{(Q_{N_\ell})^{w}},$$

showing that α is a Liouville number.

Regarding complex numbers, consider the case $D=1$ of Knopfmacher's result mentioned in Section 2. Let $\beta_1,\ldots,\beta_n$ be n nonzero complex numbers having *infinite* Cantor product representations of the form

$$\beta_j=\prod_{k=1}^{\infty}\left(1+\frac{1}{a_{k,j}}\right) \qquad (j=1,\ldots,n),$$

where $a_{k,j}$ are nonzero Gaussian integers subject to the condition that, for k sufficiently large,

$$|a_{k+1}| \geq \sqrt{2}\,|a_k|^2 - \left(\sqrt{2}+1\right)|a_k| - \left(1+1/\sqrt{2}\right).$$

For $N \in \mathbb{N},\ 1 \leq j \leq n$, let

$$\frac{P_{N,j}}{Q_{N,j}} = \prod_{k=1}^{N}\left(1+\frac{1}{a_{k,j}}\right),\quad Q_{N,j} = a_{1,j}a_{2,j}\cdots a_{N,j},\quad P_{N,j} = (1+a_{1,j})(1+a_{2,j})\cdots(1+a_{N,j}).$$

Since each product representation is infinite, we clearly have $P_{N,j} - Q_{N,j}\beta_j \neq 0\ (1 \leq j \leq n)$ From the product convergence, as $N \to \infty$, we have $|a_{N,j}| \to \infty$ and so $|Q_{N,j}| \to \infty$, yielding

$$M_{N,j} = \max\{\mathcal{H}(P_{N,j}), \mathcal{H}(Q_{N,j})\} \to \infty \quad (j = 1, \ldots, n).$$

Theorem 6. *Adopting the preceding notation and restrictions, assume further that there exists a subsequence $\{k_\ell\}_{\ell \geq 1}$ of $\mathbb{N}$ and a large integer K_0 such that*

$$|a_{k_\ell,j}| \geq 3 \qquad (k_\ell \geq K_0, 1 \leq j \leq n)$$

$$\frac{1}{|a_{k_\ell,j-1}|-1}\cdot\frac{|a_{k_\ell,j}|(|a_{k_\ell,j}|-1)}{|a_{k_\ell,j}|-2} \to 0 \qquad (2 \leq j \leq n,\ \ell \to \infty).$$

(i) If there exists $g : \mathbb{N} \to \mathbb{Z}$ with $g(k) \to \infty\ \ (k \to \infty)$ such that

$$\frac{|Q_{k_\ell,j}|(|a_{k_\ell+1,j}|-1)}{|P_{k_\ell,j}|} \geq (M_{k_\ell,1}M_{k_\ell,2}\cdots M_{k_\ell,j})^{g(k_\ell)} \qquad (1 \leq j \leq n),$$

then $\beta_1, \beta_2, \ldots, \beta_n$ are $\mathbb{Q}$-algebraically independent.
(ii) If there exists $g : \mathbb{N} \to \mathbb{Z}$ with $g(k) \to \infty\ \ (k \to \infty)$ such that

$$\frac{|Q_{k_\ell,j}|(|a_{k_\ell+1,j}|-1)}{|P_{k_\ell,j}|} \geq g(k_\ell)\, M_{k_\ell,1}M_{k_\ell,2}\cdots M_{k_\ell,j} \qquad (1 \leq j \leq n),$$

then $1, \beta_1, \beta_2, \ldots, \beta_n$ are $\mathbb{Q}$-linearly independent.

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