数理解析研究所講究録
第 1898 巻 2014 年 6-15

Algebraic independence of values of Carlitz multiple polylogarithms

九州大学・数理学府 三柴 善範*
Yoshinori Mishiba
Graduate School of Mathematics,
Kyushu University

Abstract

This is a summary of my talk in the conference "Analytic Number Theory – Arithmetic Properties of Transcendental Functions and their Applications" at RIMS and my papers [M1] and [M2] from the viewpoint of multiple polylogarithms. We explain our results on the algebraic independence of values of Carlitz multiple polylogarithms which are function field analogues (in characteristic p) of the classical multiple polylogarithms.

1 Classical case

First, we recall the classical multiple polylogarithms. In characteristic zero, we have the following exact sequence:

$$0 \longrightarrow 2\pi\sqrt{-1}\cdot\mathbb{Z} \longrightarrow \operatorname{Lie}\mathbb{G}_m(\mathbb{C}) \xrightarrow{\exp} \mathbb{G}_m(\mathbb{C}) \longrightarrow 1.$$

The logarithmic function is a local section to the map exp around the unit element $1 \in \mathbb{C}^\times$. It is defined by

$$\mathbb{C}^\times \supset \{1-z | z \in \mathbb{C},\ |z|<1\} \ni 1-z \mapsto -\sum_{m=1}^{\infty}\frac{z^m}{m} =: \log(1-z).$$

Then for a positive integer $n \geq 1$, the *n-th polylogarithm* is the function defined by

$$\mathrm{Li}_{\mathrm{n}}^{\mathbb{C}}(z) := \sum_{m=1}^{\infty}\frac{z^m}{m^n},$$

which converges on $|z|<1$ (resp. $|z|\leq 1$) for $n=1$ (resp. $n \geq 2$). More generally, let $\underline{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ be a d-tuple of positive integers, and $\underline{z} = (z_1, \ldots, z_d)$ a d-tuple of variables. The *multiple polylogarithm* is the function defined by

$$\mathrm{Li}_{\underline{\mathrm{n}}}^{\mathbb{C}}(\underline{z}) := \sum_{m_1 > \cdots > m_d \geq 1}\frac{z_1^{m_1}\cdots z_d^{m_d}}{m_1^{n_1}\cdots m_d^{n_d}},$$

*Graduate School of Mathematics, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, JAPAN.
e-mail: y-mishiba@math.kyushu-u.ac.jp
Supported by the JSPS Research Fellowships for Young Scientists.

which converges if $|z_i| \le 1$ for each i (and $|z_1| < 1$ when $n_1 = 1$). Such $\underline{n}$ is called an *index* and $\sum_i n_i$ (resp. d) is called the weight (resp. depth) of $\mathrm{Li}_{\underline{n}}^{\mathbb{C}}$.

The values $\mathrm{Li}_{\underline{n}}^{\mathbb{C}}(1, \dots, 1)$ ($n_1 \ge 2$) are called the multiple zeta values. There are numerous studies on the relations among these values. More generally, we are interested in the algebraic independence of the values of multiple polylogarithms at given algebraic points. Namely, we want to determine when $\mathrm{Li}_{\underline{n}_1}^{\mathbb{C}}(\underline{\alpha}_1), \dots, \mathrm{Li}_{\underline{n}_r}^{\mathbb{C}}(\underline{\alpha}_r)$ are algebraically independent over $\overline{\mathbb{Q}}$ for given indices $\underline{n}_1, \dots, \underline{n}_r$ and algebraic points $\underline{\alpha}_1, \dots, \underline{\alpha}_r$ satisfying the respective convergence conditions. This problem seems very difficult. Showing the linear independence of them over $\mathbb{Q}$ also seems difficult. In the depth one case, there exist some results (see [Ha], [HO],. [N], [R]). Algebraic independence results are not known except that of Lindemann's ([L]), where he proved that $\mathrm{Li}_2^{\mathbb{C}}(1) = \pi^2/6 \notin \overline{\mathbb{Q}}$.

2 Characteristic p

Next, we explain the positive characteristic case. Let p be a prime number and q a power of p. We denote two independent variables by θ and t. Let $K := \mathbb{F}_q(\theta)$ be the rational function field over $\mathbb{F}_q$, $K_\infty := \mathbb{F}_q((\theta^{-1}))$ the ∞-adic completion of K, $\mathbb{C}_\infty$ the ∞-adic completion of a fixed algebraic closure of K_∞, and $\overline{K}$ the algebraic closure of K in $\mathbb{C}_\infty$. These are function field analogues of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\overline{\mathbb{Q}}$. We fix an ∞-adic multiplicative valuation $|-|_\infty$ on $\mathbb{C}_\infty$. Let C be the *Carlitz module* (over $\overline{K}$). Thus C is the additive group scheme $\mathbb{G}_a$ equipped with the $\mathbb{F}_q[t]$-action defined by

$$t.z := \theta z + z^q \text{ and } a.z := az \;\; (z \in \mathbb{C}_\infty, \; a \in \mathbb{F}_q).$$

The Carlitz module C is a function field analogue of the multiplicative group $\mathbb{G}_m$ in characteristic zero. Note that the Lie algebra of C is also the additive group $\mathbb{G}_a$, but the induced action of t is computed as $t.z = \theta z$ (for $z \in \operatorname{Lie} C(\mathbb{C}_\infty)$). As before, we have the following exact sequence of $\mathbb{F}_q[t]$-modules:

$$0 \longrightarrow \widetilde{\pi} \cdot \mathbb{F}_q[\theta] \longrightarrow \operatorname{Lie} C(\mathbb{C}_\infty) \xrightarrow{\exp_C} C(\mathbb{C}_\infty) \longrightarrow 0,$$

where

$$\exp_C(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q) \cdots (\theta^{q^i} - \theta^{q^{i-1}})}$$

and

$$\widetilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} \cdot K_\infty^\times.$$

The function $\exp_C \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called the *Carlitz exponential* and $\widetilde{\pi}$ is called the *Carlitz period*. These are analogous objects of the classical exponential function and its fundamental period $2\pi\sqrt{-1}$. As before, the map $\exp_C$ has a local section around the unit element $0 \in \mathbb{C}_\infty$. It is defined by

$$\log_C(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q) \cdots (\theta - \theta^{q^i})},$$

which converges on $|z|_\infty < |\theta|_\infty^{\frac{q}{q-1}}$. For a positive integer $n \geq 1$, the *n-th Carlitz polylogarithm* was introduced by Anderson and Thakur in [AT1]. It is defined by

$$\mathrm{Li}_n(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{((\theta-\theta^q)\cdots(\theta-\theta^{q^i}))^n},$$

which converges on $|z|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. They also showed that the n-th Carlitz polylogarithm appears as the last coordinate of the logarithmic function of the n-th tensor power of the Carlitz module C. For an index $\underline{n} = (n_1, \ldots, n_d)$, Chang ([C]) defined the *Carlitz multiple polylogarithm* as

$$\mathrm{Li}_{\underline{n}}(\underline{z}) := \sum_{i_1 > \cdots > i_d \geq 0} \frac{z_1^{q^{i_1}} \cdots z_d^{q^{i_d}}}{((\theta-\theta^q)\cdots(\theta-\theta^{q^{i_1}}))^{n_1} \cdots ((\theta-\theta^q)\cdots(\theta-\theta^{q^{i_d}}))^{n_d}}.$$

The function $\mathrm{Li}_{\underline{n}}(\underline{z})$ converges if $|z_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. We call $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{C}_\infty^d$ an algebraic point if $\alpha_i \in \overline{K}$ for each i, and non-trivial if $\alpha_i \neq 0$ for each i. We have the *harmonic product formulas* among values of Carlitz multiple polylogarithms (see [C]). For example,

$$\mathrm{Li}_{n_1}(\alpha_1)\,\mathrm{Li}_{n_2}(\alpha_2) = \mathrm{Li}_{n_1,n_2}(\alpha_1, \alpha_2) + \mathrm{Li}_{n_2,n_1}(\alpha_2, \alpha_1) + \mathrm{Li}_{n_1+n_2}(\alpha_1\alpha_2).$$

By using Anderson and Thakur's theory ([AT1], [AT2]), Chang also showed that the multizeta values at $\underline{n}$ in characteristic p is a K-linear combination of $\mathrm{Li}_{\underline{n}}$ at some points in $\mathbb{F}_q[\theta]^d$.

We are interested in the algebraic independence of $\mathrm{Li}_{\underline{n}}(\underline{\alpha})$'s over $\overline{K}$ for given indices $\underline{n}$ and non-trivial algebraic points $\underline{\alpha}$ which satisfy the respective convergence conditions. Papanikolas ([P], $n = 1$), Chang and Yu ([CY], $n \geq 1$) proved that for a positive integer $n \geq 1$ and $\alpha_1, \ldots, \alpha_r \in \overline{K}^\times$ with $|\alpha_j|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for each j, if $\widetilde{\pi}^n, \mathrm{Li}_n(\alpha_1), \ldots, \mathrm{Li}_n(\alpha_r)$ are linearly independent over K, then they are algebraically independent over $\overline{K}$. Moreover, in [CY], Chang and Yu proved the following theorem: Let $n_1, \ldots, n_d \geq 1$ be positive integers such that n_i/n_j is not an integral power of p for each $i \neq j$. For each i, take non-trivial algebraic points $\alpha_{i1}, \ldots, \alpha_{ir_i} \in \overline{K}^\times$ such that $|\alpha_{ij}|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$. If $\widetilde{\pi}^{n_i}, \mathrm{Li}_{n_i}(\alpha_{i1}), \ldots, \mathrm{Li}_{n_i}(\alpha_{ir_i})$ are linearly independent over K for each i, then

$$\mathrm{tr.deg}_{\overline{K}}\,\overline{K}(\widetilde{\pi}, \mathrm{Li}_{n_i}(\alpha_{ij}) | 1 \leq i \leq d,\ 1 \leq j \leq r_i) = 1 + \sum_{i=1}^{d} r_i.$$

Note that these results treat only depth one elements. We want to consider higher depth elements. Chang ([C]) showed that $\mathrm{Li}_{\underline{n}}(\underline{\alpha}) \neq 0$ for each non-trivial point $\underline{\alpha}$ (with the convergence condition), and the values of Carlitz multiple polylogarithms of different weights at non-trivial algebraic points are linearly independent over $\overline{K}$. However, note that his results do not treat the algebraic independence of given elements. Our results in [M1] and [M2] are about the algebraic independence of values of Carlitz multiple polylogarithms which may have higher depths. In particular, we treat the elements of the set

$$S(\underline{n}, \underline{\alpha}) := \{\widetilde{\pi}\} \cup \{\mathrm{Li}_{n_j, n_{j+1}, \ldots, n_i}(\alpha_j, \alpha_{j+1}, \ldots, \alpha_i) | 1 \leq j \leq i \leq d\}$$

where $\underline{n} = (n_1, \ldots, n_d)$ is an index and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\overline{K}^\times)^d$ is a non-trivial algebraic point such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. In [M1], we treated the case where $d = 2$, $n_1 = n_2$ and $\alpha_1 = \alpha_2$:

Theorem 2.1 ([M1]). *Let $n \geq 1$ be a positive integer and $\alpha \in \overline{K}^\times$ a non-trivial algebraic point such that $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. Suppose that $\widetilde{\pi}^n$ and $\mathrm{Li}_n(\alpha)$ are linearly independent over K. If $\widetilde{\pi}^{2n}$ and $\mathrm{Li}_n(\alpha)^2 - 2\,\mathrm{Li}_{n,n}(\alpha, \alpha) = \mathrm{Li}_{2n}(\alpha^2)$ are linearly independent over K, then $\widetilde{\pi}$, $\mathrm{Li}_n(\alpha)$ and $\mathrm{Li}_{n,n}(\alpha, \alpha)$ are algebraically independent over $\overline{K}$.*

Remark 2.2. Note that $\widetilde{\pi}^n \in K_\infty$ if and only if n is divisible by $q-1$, and $\mathrm{Li}_n(\alpha) \in K$ if $\alpha \in K$. Thus when n is not divisible by $q-1$ and $\alpha \in K^\times$, we can easily check the linear independence of $\widetilde{\pi}^n$ and $\mathrm{Li}_n(\alpha)$ over K.

When the depth one elements have no relations, we have the following theorem:

Theorem 2.3 ([M2]). *Let $\underline{n} = (n_1, \ldots, n_d)$ be an index and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (\overline{K}^\times)^d$ a non-trivial algebraic point such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. If $\widetilde{\pi}, \mathrm{Li}_{n_1}(\alpha_1), \ldots, \mathrm{Li}_{n_d}(\alpha_d)$ are algebraically independent over $\overline{K}$, then we have*

$$\mathrm{tr.deg}_{\overline{K}}\,\overline{K}(S(\underline{n}, \underline{\alpha})) = 1 + \frac{d(d+1)}{2}.$$

By using the result of Chang and Yu and Remark 2.2, the assumption in Theorem 2.3 can be checked in some cases. In particular, we have the following corollary:

Corollary 2.4. *Let $\underline{n} = (n_1, \ldots, n_d)$ be an index and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \in (K^\times)^d$ a non-trivial rational point such that n_i is not divisible by $q-1$ and $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i, and n_i/n_j is not an integral power of p for each $i \neq j$. Then we have*

$$\mathrm{tr.deg}_{\overline{K}}\,\overline{K}(S(\underline{n}, \underline{\alpha})) = 1 + \frac{d(d+1)}{2}.$$

3 Papanikolas' theory of pre-t-motives

In this section, we briefly review Papanikolas' theory ([P]) of pre-t-motives. The proofs of our theorems essentially depend on this theory.

Let $\mathbb{T}$ be the Tate algebra over $\mathbb{C}_\infty$ (= the subring of $\mathbb{C}_\infty[\![t]\!]$ consisting of the formal power series which converge on $|t|_\infty \leq 1$), and $\mathbb{L}$ the fraction field of $\mathbb{T}$. For each formal Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty(\!(t)\!)$, we set $\sigma(f) := \sum_i a_i^{q^{-1}} t^i$. The fields $\overline{K}(t)$ and $\mathbb{L}$ are stable under this action and their σ-fixed parts are $\mathbb{F}_q(t)$.

A *pre-t-motive* is a finite dimensional $\overline{K}(t)$-vector space M equipped with a σ-semilinear bijective map $\varphi\colon M \to M$. A morphism among two pre-t-motives is a $\overline{K}(t)$-linear map which is compatible with the φ's. Let $\mathcal{C}$ be the category of pre-t-motives M such that the natural map $\mathbb{L} \otimes_{\mathbb{F}_q(t)} \omega(M) \to \mathbb{L} \otimes_{\overline{K}(t)} M$ is an isomorphism, where

$$\omega(M) := \{x \in \mathbb{L} \otimes_{\overline{K}(t)} M \mid (\sigma \otimes \varphi)(x) = x\}$$

is the *Betti realization* of M. Then the category $\mathcal{C}$ forms a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor ω. For an object M in $\mathcal{C}$, we set G_M to be the fundamental group of the Tannakian subcategory of $\mathcal{C}$ generated by M with respect to ω. Then G_M can be naturally viewed as a subgroup scheme of $\mathrm{GL}(\omega(M))$. We define another group scheme over $\mathbb{F}_q(t)$ as follows. Let r be the dimension of M and we fix a $\overline{K}(t)$-basis $\mathbf{m}$ of M. By definition, there exists a matrix $\Psi = (\Psi_{ij}) \in \mathrm{GL}_r(\mathbb{L})$ such that $\Psi^{-1}\mathbf{m}$ forms an $\mathbb{F}_q(t)$-basis of $\omega(M)$. This is equivalent to the equality $\sigma(\Psi) = \Phi\Psi$, where $\Phi \in \mathrm{GL}_r(\overline{K}(t))$ is the matrix representing φ with respect to $\mathbf{m}$ and we set $\sigma(\Psi) := (\sigma(\Psi_{ij}))$. Note that such a matrix Φ gives an object of $\mathcal{C}$ conversely. We set $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$, where $(\Psi_1)_{ij} := \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} := 1 \otimes \Psi_{ij}$. The group scheme G_Ψ over $\mathbb{F}_q(t)$ is defined by

$$G_\Psi := \{(x_{ij}) \in \mathrm{GL}_r \,|\, f(x_{ij}) = 0 \text{ for } f \in \mathbb{F}_q(t)[X, 1/\det X] \text{ with } f(\widetilde{\Psi}_{ij}) = 0\},$$

where $X = (X_{ij})$ is a matrix of $r \times r$ variables. Then we have the inclusion

$$G_\Psi \hookrightarrow G_M;\ g \mapsto ((f_1, \dots, f_r) \mapsto (f_1, \dots, f_r)g^{-1}),$$

where we identify $\omega(M)$ with $\mathbb{F}_q(t)^r$ with respect to the basis $\Psi^{-1}\mathbf{m}$. Papanikolas proved that this inclusion is an isomorphism of smooth group schemes over $\mathbb{F}_q(t)$, and

$$\dim G_\Psi = \mathrm{tr.deg}_{\overline{K}(t)}\, \overline{K}(t)(\Psi_{ij}|i,j) = \mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\Psi_{ij}(\theta)|i,j)$$

if $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$, $\det\Phi/(t-\theta)^n \in \overline{K}^\times$ for some $n \geq 0$, and $\Psi \in \mathrm{GL}_r(\mathbb{T})$. Note that in this situation, each Ψ_{ij} converges at $t = \theta$ ([ABP]). The second equality also uses deep results in [ABP]. The values $\Psi_{ij}(\theta)$ are called *periods* of M.

Example 3.1. Let M, $\mathbf{m}$, Φ, Ψ be as above. Assume that the matrices Φ and Ψ are lower triangular matrices. For $r' \leq r$, let Φ' (resp. Ψ') be the lower right $r' \times r'$-submatrix of Φ (resp. Ψ). We consider the pre-t-motive M' defined by Φ'. Then M' is a quotient of M. Let $\mathbf{m}'$ be the standard basis of M', which is the image of $\mathbf{m}$. Then Φ' is the matrix representing the φ-action on M' with respect to the basis $\mathbf{m}'$. By Tannakian duality, we have a surjective map $G_M \twoheadrightarrow G_{M'}$. By the identifications $G_M \cong G_\Psi \subset \mathrm{GL}_r$ and $G_{M'} \cong G_{\Psi'} \subset \mathrm{GL}_{r'}$, this maps a matrix A to the lower right $r' \times r'$-submatrix of A. We also have similar calculations for subobjects.

Example 3.2. Let C be the pre-t-motive defined by $t - \theta \in \mathrm{GL}_1(\overline{K}(t))$. The formal power series

$$\Omega(t) := (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \overline{K_\infty}[\![t]\!]$$

is an element of $\mathbb{T}^\times$ and converges at $t = \theta$. Moreover, it satisfies $\sigma(\Omega) = (t-\theta)\Omega$ and $\Omega(\theta) = 1/\widetilde{\pi}$. Thus C is an object of $\mathcal{C}$. Since $\Omega \notin \overline{K}(t)$, we have $\dim G_\Omega = \mathrm{tr.deg}_{\overline{K}(t)}\, \overline{K}(t)(\Omega) = \mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}) = 1$ and $G_C \cong G_\Omega = \mathbb{G}_m$.

Example 3.3. Let $\underline{n} = (n_1, \dots, n_d)$ be an index and $\underline{\alpha} = (\alpha_1, \dots, \alpha_d) \in (\overline{K}^\times)^d$ a non-trivial algebraic point such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. We define a "lift of $\mathrm{Li}_{\underline{n}}(\underline{\alpha})$" by

$$L_{\underline{\alpha},\underline{n}}(t) := \sum_{i_1 > \cdots > i_d \geq 0} \frac{\alpha_1^{q^{i_1}} \cdots \alpha_d^{q^{i_d}}}{((t-\theta^q)\cdots(t-\theta^{q^{i_1}}))^{n_1} \cdots ((t-\theta^q)\cdots(t-\theta^{q^{i_d}}))^{n_d}} \in \mathbb{T},$$

which converges on $|t|_\infty < |\theta|_\infty^q$ and clearly $L_{\underline{\alpha},\underline{n}}(\theta) = \mathrm{Li}_{\underline{n}}(\underline{\alpha})$. Moreover, if we set $(d+1)\times(d+1)$-matrices

$$\Phi[\underline{\alpha},\underline{n}] := \begin{bmatrix} (t-\theta)^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ \alpha_1^{q^{-1}}(t-\theta)^{n_1+\cdots+n_d} & (t-\theta)^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ 0 & \alpha_2^{q^{-1}}(t-\theta)^{n_2+\cdots+n_d} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{n_d} & 0 \\ 0 & \cdots & 0 & \alpha_d^{q^{-1}}(t-\theta)^{n_d} & 1 \end{bmatrix}$$

and

$$\Psi[\underline{\alpha},\underline{n}] := \begin{bmatrix} \Omega^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d}L_{\alpha_1,n_1} & \Omega^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d}L_{\alpha_1,\alpha_2,n_1,n_2} & \Omega^{n_2+\cdots+n_d}L_{\alpha_2,n_2} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \Omega^{n_d} & 0 \\ \Omega^{n_1+\cdots+n_d}L_{\alpha_1,\ldots,\alpha_d,n_1,\ldots,n_d} & \Omega^{n_2+\cdots+n_d}L_{\alpha_2,\ldots,\alpha_d,n_2,\ldots,n_d} & \cdots & \Omega^{n_d}L_{\alpha_d,n_d} & 1 \end{bmatrix},$$

then they satisfy the equation $\sigma(\Psi[\underline{\alpha},\underline{n}]) = \Phi[\underline{\alpha},\underline{n}]\Psi[\underline{\alpha},\underline{n}]$. Thus the pre-$t$-motive $M[\underline{\alpha},\underline{n}]$ defined by $\Phi[\underline{\alpha},\underline{n}]$ is an object of $\mathcal{C}$. By Papanikolas' theory, we have the isomorphism $G_{\Psi[\underline{\alpha},\underline{n}]} \cong G_{M[\underline{\alpha},\underline{n}]}$ and the equalities

$$\begin{aligned} \dim G_{\Psi[\underline{\alpha},\underline{n}]} &= \mathrm{tr.deg}_{\overline{K}(t)}\,\overline{K}(t)(\Omega, L_{\alpha_j,\alpha_{j+1},\ldots,\alpha_i,n_j,n_{j+1},\ldots,n_i}|1\le j\le i\le d) \\ &= \mathrm{tr.deg}_{\overline{K}}\,\overline{K}(\widetilde{\pi}, \mathrm{Li}_{n_j,n_{j+1},\ldots,n_i}(\alpha_j,\alpha_{j+1},\ldots,\alpha_i)|1\le j\le i\le d). \end{aligned}$$

4 Outline of the proofs of Theorems 2.1 and 2.3

In this setion, we sketch the proofs of Theorems 2.1 and 2.3. We use the letters a, x, y, x_{ij} as coordinate variables of algebraic groups and they run over the elements of $\mathbb{F}_q(t)$-algebras R except $a \in R^\times$. For example, we use the following description of an algebraic group over $\mathbb{F}_q(t)$:

$$\left\{\begin{bmatrix} a & \\ x & 1\end{bmatrix}\right\} := \left(R \mapsto \left\{\begin{bmatrix} a & \\ x & 1\end{bmatrix}\middle| a\in R^\times, x\in R\right\}\right).$$

Proof of Theorem 2.1. By Papanikolas' theory, we have

$$G := G_{M[\alpha,\alpha,n,n]} \cong G_{\Psi[\alpha,\alpha,n,n]} \subset \overline{G} := \left\{\begin{bmatrix} a^2 & & \\ ax & a & \\ y & x & 1\end{bmatrix}\right\}$$

$$\left(\text{resp. } G' := G_{M[\alpha,n]} \cong G_{\Psi[\alpha,n]} \subset \overline{G'} := \left\{\begin{bmatrix} a & \\ x & 1\end{bmatrix}\right\}\right)$$

and

$$\dim G = \mathrm{tr.deg}_{\overline{K}(t)}\,\overline{K}(t)(\Omega, L_{\alpha,n}, L_{\alpha,\alpha,n,n}) = \mathrm{tr.deg}_{\overline{K}}\,\overline{K}(\widetilde{\pi}, \mathrm{Li}_n(\alpha), \mathrm{Li}_{n,n}(\alpha,\alpha))$$

$$\left(\text{resp. } \dim G' = \mathrm{tr.deg}_{\overline{K}(t)}\,\overline{K}(t)(\Omega, L_{\alpha,n}) = \mathrm{tr.deg}_{\overline{K}}\,\overline{K}(\widetilde{\pi}, \mathrm{Li}_n(\alpha))\right).$$

In terms of matrices, the surjection $G \twoheadrightarrow G'$ induced by Tannakian duality maps a matrix to its lower right 2×2-submatrix (see Example 3.1). By the assumption, we have $\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, \operatorname{Li}_n(\alpha)) = 2$ and hence $G' = \overline{G'}$. Thus the algebraic group G has dimension two or three and it has the property

$$\overline{G} \supset G \twoheadrightarrow \overline{G'}.$$

In characteristic 2, we can show that such G must have dimension three. Thus the transcendental degree is also three. Assume that $p \geq 3$. If $\dim G = 2$, we can show that

$$G = \left\{ \begin{bmatrix} a^2 & & \\ ax & a & \\ \frac{x^2}{2} - \frac{c_0}{2}(1-a^2) & x & 1 \end{bmatrix} \right\}$$

for some $c_0 \in \mathbb{F}_q(t)$. By the definition of $G_{\Psi[\alpha,\alpha,n,n]}$, this implies the equality

$$(\Omega^{2n} L_{\alpha,n}^2 - 2\Omega^{2n} L_{\alpha,\alpha,n,n} - c_0) \otimes \Omega^{2n} = \Omega^{2n} \otimes (\Omega^{2n} L_{\alpha,n}^2 - 2\Omega^{2n} L_{\alpha,\alpha,n,n} - c_0)$$

in $\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L}$. Thus there exists $f \in \overline{K}(t)$ such that

$$\Omega^{2n} L_{\alpha,n}^2 - 2\Omega^{2n} L_{\alpha,\alpha,n,n} - c_0 = f\Omega^{2n}.$$

By substituting $t = \theta^N$ for large N (see [C, Section 6.4]), we obtain

$$\operatorname{Li}_n(\alpha)^2 - 2\operatorname{Li}_{n,n}(\alpha,\alpha) = \widetilde{\pi}^{2n} c_0(\theta).$$

This is a contradiction. Thus we have $\dim G = 3$. $\square$

Proof of Theorem 2.3. Let M_1, M_2, M_3 and M_4 be the pre-t-motives defined by

$$\begin{aligned}
\Phi_1 &:= (t-\theta)^{n_2+n_3}\Phi[\alpha_1, n_1] \oplus (t-\theta)^{n_3}\Phi[\alpha_2, n_2] \oplus \Phi[\alpha_3, n_3], \\
\Phi_2 &:= (t-\theta)^{n_3}\Phi[\alpha_1, \alpha_2, n_1, n_2] \oplus \Phi[\alpha_3, n_3], \\
\Phi_3 &:= (t-\theta)^{n_3}\Phi[\alpha_1, \alpha_2, n_1, n_2] \oplus \Phi[\alpha_2, \alpha_3, n_2, n_3], \\
\Phi_4 &:= \Phi[\alpha_1, \alpha_2, \alpha_3, n_1, n_2, n_3],
\end{aligned}$$

respectively. We set

$$\begin{aligned}
\Psi_1 &:= \Omega^{n_2+n_3}\Psi[\alpha_1, n_1] \oplus \Omega^{n_3}\Psi[\alpha_2, n_2] \oplus \Psi[\alpha_3, n_3], \\
\Psi_2 &:= \Omega^{n_3}\Psi[\alpha_1, \alpha_2, n_1, n_2] \oplus \Psi[\alpha_3, n_3], \\
\Psi_3 &:= \Omega^{n_3}\Psi[\alpha_1, \alpha_2, n_1, n_2] \oplus \Psi[\alpha_2, \alpha_3, n_2, n_3], \\
\Psi_4 &:= \Psi[\alpha_1, \alpha_2, \alpha_3, n_1, n_2, n_3].
\end{aligned}$$

Then we have $\sigma(\Psi_k) = \Phi_k \Psi_k$ for each k. Hence each M_k is an object of $\mathcal{C}$. We set

$$G_k := G_{C \oplus M_k} \cong G_{[\Omega] \oplus \Psi_k} \subset \overline{G_k},$$

where the $\overline{G_k}$'s are as follows:

$$\overline{G_1} := \left\{ \begin{bmatrix} a & & & & & & \\ & a^{n_1+n_2+n_3} & & & & & \\ & x_{21} & a^{n_2+n_3} & & & & \\ & & & a^{n_2+n_3} & & & \\ & & & x_{32} & a^{n_3} & & \\ & & & & & a^{n_3} & \\ & & & & & x_{43} & 1 \end{bmatrix} \right\},$$

$$\overline{G_2} := \left\{ \begin{bmatrix} a & & & & & \\ & a^{n_1+n_2+n_3} & & & & \\ & x_{21} & a^{n_2+n_3} & & & \\ & x_{31} & x_{32} & a^{n_3} & & \\ & & & & a^{n_3} & \\ & & & & x_{43} & 1 \end{bmatrix} \right\},$$

$$\overline{G_3} := \left\{ \begin{bmatrix} a & & & & & & \\ & a^{n_1+n_2+n_3} & & & & & \\ & x_{21} & a^{n_2+n_3} & & & & \\ & x_{31} & x_{32} & a^{n_3} & & & \\ & & & & a^{n_2+n_3} & & \\ & & & & x_{32} & a^{n_3} & \\ & & & & x_{42} & x_{43} & 1 \end{bmatrix} \right\},$$

$$\overline{G_4} := \left\{ \begin{bmatrix} a & & & & \\ & a^{n_1+n_2+n_3} & & & \\ & x_{21} & a^{n_2+n_3} & & \\ & x_{31} & x_{32} & a^{n_3} & \\ & x_{41} & x_{42} & x_{43} & 1 \end{bmatrix} \right\}.$$

Since M_{k-1} is a direct sum of subquotients of M_k for each $k \geq 2$, we have the surjective maps

$$G_4 \overset{\psi_4}{\twoheadrightarrow} G_3 \overset{\psi_3}{\twoheadrightarrow} G_2 \overset{\psi_2}{\twoheadrightarrow} G_1$$

by Tannakian duality. In terms of coordinates, they are computed by

$$\begin{aligned}(a, x_{21}, x_{32}, x_{43}, x_{31}, x_{42}, x_{41}) &\mapsto (a, x_{21}, x_{32}, x_{43}, x_{31}, x_{42}) \\ &\mapsto (a, x_{21}, x_{32}, x_{43}, x_{31}) \mapsto (a, x_{21}, x_{32}, x_{43}).\end{aligned}$$

By Papanikolas' theory, it is enough to show that the equality $G_4 = \overline{G_4}$ holds. In fact, we show $G_k = \overline{G_k}$ $(1 \leq k \leq 4)$ by induction on k. By the assumption, we have

$$\dim G_1 = \operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, \operatorname{Li}_{n_1}(\alpha_1), \operatorname{Li}_{n_2}(\alpha_2), \operatorname{Li}_{n_3}(\alpha_3)) = 4 = \operatorname{tr.deg} \overline{G_1}.$$

Thus the equality holds for $k = 1$. Let $k \geq 2$ and assume that the equality holds for $k - 1$. Then the equality $G_k = \overline{G_k}$ is equivalent to the equality $\dim G_k = \dim G_{k-1} + 1$. We can check that the algebraic group G_k which satisfies

$$\overline{G_k} \supset G_k \twoheadrightarrow G_{k-1} = \overline{G_{k-1}}$$

must have dimension $\dim G_{k-1} + 1$. For example, let $k = 3$. We identify group schemes over $\mathbb{F}_q(t)$ with the set of their $\overline{\mathbb{F}_q(t)}$-valued points. If $\dim G_3 = \dim G_2$, it is clear that the natural surjection $V_3 \twoheadrightarrow V_2$ among the unipotent radicals of G_3 and G_2 becomes a bijective map. We take any elements

$$X=\begin{bmatrix}1&&&&&&\\&1&&&&&\\&x_{21}&1&&&&\\&x_{31}&x_{32}&1&&&\\&&&&1&&\\&&&&x_{32}&1&\\&&&&x_{42}&x_{43}&1\end{bmatrix},\ A=\begin{bmatrix}1&&&&&&\\&1&&&&&\\&a_{21}&1&&&&\\&a_{31}&a_{32}&1&&&\\&&&&1&&\\&&&&a_{32}&1&\\&&&&a_{42}&a_{43}&1\end{bmatrix}\in V_3.$$

Then we have

$$X^{-1}A^{-1}XA=\begin{bmatrix}1&&&&&&\\&1&&&&&\\&&1&&&&\\&a_{21}x_{32}-a_{32}x_{21}&&1&&&\\&&&&1&&\\&&&&&1&\\&&&&a_{32}x_{43}-a_{43}x_{32}&&1\end{bmatrix}.$$

Thus if the equality $a_{21}x_{32}-a_{32}x_{21}=0$ holds, then the equality $a_{32}x_{43}-a_{43}x_{32}=0$ also holds because $X^{-1}A^{-1}XA\in V_3\cap\operatorname{Ker}\varphi_3=\{1\}$. However, by the induction hypothesis and the surjectivity of $V_3\twoheadrightarrow V_2$, we can take $a_{21}=a_{32}=0$ and $a_{43}x_{32}\neq 0$. This is a contradiction. $\square$

References

[ABP] G. W. Anderson, W. D. Brownawell, M. A. Papanikolas, *Determination of the algebraic relations among special* Γ*-values in positive characteristic*, Ann. of Math. (2) **160** (2004), no. 1, 237–313.

[AT1] G. W. Anderson, D. S. Thakur, *Tensor powers of the Carlitz module and zeta values*, Ann. of Math. **132** (1990) 159–191.

[AT2] G. W. Anderson, D. S. Thakur, *Multizeta values for* $\mathbb{F}_q[t]$*, their period interpretation, and relations between them*, Int. Math. Res. Not. IMRN (2009), no. 11, 2038–2055.

[C] C.-Y. Chang, *Linear independence of monomials of multizeta values in positive characteristic*, preprint (2012), `http://arxiv.org/abs/1207.2326`

[CY] C.-Y. Chang, J. Yu, *Determination of algebraic relations among special zeta values in positive characteristic*, Adv. Math. **216** (2007) 321–345.

[Ha] M. Hata, *On the linear independence of the values of polylogarithmic functions*, J. Math. Pures Appl. (9) **69** (1990), 133–173.

[HO] N. Hirata-Kohno and H. Okada, *A note on linear independence of polylogarithms over the rationals*, Proc. Japan Acad. Ser. A Math. Sci. **88** (2012), 156–161.

[L] F. Lindemann, *Über die Zahl* π, Math. Ann. **20** (1882), 213–225.

[M1] Y. Mishiba, *Algebraic independence of the Carlitz period and the positive characteristic multizeta values at* n *and* (n, n), preprint (2013), `http://arxiv.org/abs/1307.3725`

[M2] Y. Mishiba, *On algebraic independence of certain multizeta values in characteristic* p, preprint (2014), `http://arxiv.org/abs/1401.3628`

[N] E. M. Nikišin, *Irrationality of values of functions* $F(x, s)$, Mat. Sb. (N.S.) **109 (151)** (1979), 410–417, 479.

[P] M. A. Papanikolas, *Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms*, Invent. Math. **171** (2008) 123–174.

[R] T. Rivoal, *Indépendance linéaire des valeurs des polylogarithmes*, J. Théor. Nombres Bordeaux **15** (2003), 551–559.