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<thead>
<tr>
<th>Title</th>
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</tr>
</thead>
<tbody>
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<td>Kato, Hiroyuki</td>
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Control of the Inflation Rate through Central Banks in an Equilibrium Model

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Faculty of Management and Economics,
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1 Introduction

In a continuous time stochastic equilibrium model, we describe a mechanism in which the central bank affects the inflation rates through ordinary banks, and studies under what conditions the change in the inflation rates alters the equilibrium consumption, investment and production. Every transaction is executed through bank accounts that bear a nominal interest rate and the bank accounts are called money in this paper. The central bank lends money to the ordinary banks by depositing the money into the reserve which is set in the ordinary banks. In this setting, the equilibrium processes are indeterminate. Since the real interest rates (nominal interest rates minus inflation rates) are not determined uniquely, we can study the effects of monetary policy on the real interest rates and therefore the consumption, investment and production. Because the real interest rates do not equal to the productivity of capital, the divergent capital paths can be called equilibrium unlike the existing dynamic stochastic equilibrium models.

2 The economy

The model we want to consider in the subsequent paper is based on the typical dynamic stochastic general equilibrium models in continuous time with production (see e.g. Duffie (2001), Dana and Jeanblanc-Picque (2003), Espino and Hintermaiser (2009)) except that in this paper the banking system is included in the setting.

The mathematical preliminaries are presented first. We work on the infinite interval of time \([0, \infty)\) and on a probability space \((\Omega, \mathcal{F}, P)\). We assume
that a Brownian motion \((B_t = (B^1_t, B^2_t); t \geq 0)\) is constructed on the probability space and set a filtration by \(\mathcal{F}_t = \sigma(B_s; s \leq t)\) that is assumed to include the \(P\)-null sets. \((\mathcal{F}_t, t \geq 0)\) describes the flow of information available to every agent at \(t\).

Although the analysis in this paper is executed in continuous time, in order to make the economic meaning clear, let us first illustrate our model in discrete time. The economy consists of four entities: consumers, firms, ordinary banks and the central bank. The following steps show the procedure in which transactions among the consumers and firms occur through the bank accounts that bear an interest rate \(r_t\), and accompanying behavior of the central bank.

We start from the following state of the ordinary banks’ balance sheet. The right hand of balance sheet (credit side) means that consumers have the deposit the quantity of which is \(b_0\) and firms \(p_0f(0, k_{-1}, l)\) at time 0. The left hand (debit side) stands for the asset whose contents are lending to the consumers and firms by issuing the ‘deposit bond’ the quantity of which is \(\theta_0^c + \theta_0^f\) to the consumers and \(\theta_0^f\) the firms, at the price of \(X_0\) at time 0. The central bank lends money to the banks, the quantity of which, called \(\Phi_0\), is deposited to the central bank as the reserve.

### Ordinary Banks’ Balance Sheet (B/S) (1)

<table>
<thead>
<tr>
<th>Reserve (to the central bank) (\Phi_0)</th>
<th>Consumers (b_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lend (X_0(\theta_0^c + \theta_0^f))</td>
<td>Firms (p_0f(0, k_{-1}, l))</td>
</tr>
<tr>
<td>Borrow (from the central bank) (\Phi_0)</td>
<td></td>
</tr>
</tbody>
</table>

Second, the banks pay the interest rate \(r_0\) which accrues on the deposits, and receive the interest rate \(\Delta_0\) paid by the borrowers (consumers and firms).

### Ordinary Banks’ Balance Sheet (B/S) (2)

<table>
<thead>
<tr>
<th>Reserve (to the central bank) (\Phi_0)</th>
<th>Consumers (b_0 + r_0b_0 - \Delta_0\theta_0^c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lend (X_0(\theta_0^c + \theta_0^f))</td>
<td>Firms (p_0f(0, k_{-1}, l) + r_0p_0f(0, k_{-1}, l) - \Delta_0\theta_0^f)</td>
</tr>
<tr>
<td>Profit (\Delta_0(\theta_0^c + \theta_0^f))</td>
<td>Borrow (from the central bank) (\Phi_0 + r_0\Phi_0)</td>
</tr>
<tr>
<td>Loss (-{r_0b_0 + r_0p_0f(0, k_{-1}, l) + r_0\Phi_0})</td>
<td>Equity capital</td>
</tr>
</tbody>
</table>

\[
\Phi_0^B := \Delta_0(\theta_0^c + \theta_0^f) - \{r_0b_0 + r_0p_0f(0, k_{-1}, l) + r_0\Phi_0\}
\]

which is paid as the wage to workers in the banks.
Ordinary Banks' Balance Sheet (B/S) (3)

| reserve (to the central bank) $\Phi_0$ | consumers $b_0 + r_0b_0 - \Delta_0 \theta_0^c + \delta_0a_0 + w_0l + r_0\Phi_0 + \Phi_0^B$
| lend $X_0(\theta_0^c + \theta_0^f)$ | firms 0
| | borrow (from the central bank) $\Phi_0$

where

$$p_0f(0, k_{-1}, l) + r_0p_0f(0, k_{-1}, l) - \Delta_0 \theta_0^c - w_0l$$

$$a_0 =: \delta_0$$

which means the dividend per share at time 0.

Ordinary Banks' Balance Sheet (B/S) (4)

| reserve (to the central bank) $\Phi_1$ | consumers $b_0 + r_0b_0 - \Delta_0 \theta_0^c + \delta_0a_0 + w_0l + r_0\Phi_0 + \Phi_0^B$
| lend $X_1(\theta_0^c + \theta_0^f)$ | $+ X_1(\theta_1^c - \theta_0^c) + X_1(\theta_1^f - \theta_0^f)$
| | firms $X_1(\theta_1^f - \theta_0^f)$
| | borrow (from the central bank) $\Phi_1$
| | equity capital (latent gain)
| | $$(X_1 - X_0)(\theta_0^c + \theta_0^f)$$

Next, consumers select $c_1, a_1$ and $b_1$ under the budget constraint

$$p_1c_1 + S_1(a_1 - a_0) + b_1 = b_0 + r_0b_0 - \Delta_0 \theta_0^c + \delta_0a_0 + w_0l + r_0\Phi_0 + \Phi_0^B + X_1(\theta_1^f - \theta_0^f)$$

(B)

where $p_1$ and $S_1$ represent the price of the good and the stock price at time 1 respectively. Thus,

Ordinary Banks' Balance Sheet (B/S) (5)

| reserve (to the central bank) $\Phi_1$ | consumers $b_1$
| lend $X_1(\theta_1^c + \theta_1^f)$ | firms $p_1c_1 + S_1(a_1 - a_0) + X_1(\theta_1^f - \theta_0^f)$
| | borrow (from the central bank) $\Phi_1$
| | equity capital (latent gain)
| | $$(X_1 - X_0)(\theta_0^c + \theta_0^f)$$

The firms conduct the investment by making use of $S_1(a_1 - a_0) + X_1(\theta_1^f - \theta_0^f)$. So letting $p_1(k_1 - k_0)$ stand for the quantity of the investment, it holds

$$p_1(k_1 - k_0) = S_1(a_1 - a_0) + X_1(\theta_1^f - \theta_0^f).$$
If we assume that the whole products are absorbed by consumption and investment, we see

\[ p_1 f(1, k_0, l) = p_1 c_1 + S_1 (a_1 - a_0) + X_1 (\theta_1^f - \theta_0^f). \]

<table>
<thead>
<tr>
<th>Ordinary Banks' Balance Sheet (B/S) (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>reserve (to the central bank) ( \Phi_1 )</td>
</tr>
<tr>
<td>lend ( X_1 (\theta_1^c + \theta_1^f) )</td>
</tr>
<tr>
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</table>

The above phase (6) is back to (1) replacing time 1 with 0 except the equity capital part. The ensuing process following (6) proceeds similarly to (1)-(6). The equity capital as latent gains at time \( t \) is represented as, by writing in continuous time,

\[ \int_0^t (\theta_u^c + \theta_u^f) dX_u. \]

<table>
<thead>
<tr>
<th>Ordinary Banks' Balance Sheet (B/S) ( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>reserve (to the central bank) ( \Phi_t )</td>
</tr>
<tr>
<td>lend ( X_t (\theta_t^c + \theta_t^f) )</td>
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<td></td>
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</tr>
</tbody>
</table>

The central bank obtains the interest rate \( r_t \) that accrues on \( \Phi_t \) and pays it to consumers (e.g. workers to the central bank) as the wage at time \( t \geq 0 \). Then its balance sheet is represented as follows.

<table>
<thead>
<tr>
<th>The Central Banks' Balance Sheet (B/S) ( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lend ( \Phi_t )</td>
</tr>
<tr>
<td>profit ( r_t \Phi_t )</td>
</tr>
<tr>
<td>loss ( -r_t \Phi_t )</td>
</tr>
</tbody>
</table>

### 2.1 Consumers' problem

The problem that homogenous consumers face is the following;

\[ \max_{(a_t, b_t, \theta_t^c, \theta_t^f) \geq 0} E_P \left[ \int_0^\infty e^{-\int_0^t \rho_u du} u(c_t) dt \right] \tag{2.1.1} \]
subject to
\[ p_t c_t dt + S_t da_t + db_t - X_t d\theta_t^c = r_t b_t dt - \Delta_t \theta_t^c dt + \delta_t a_t dt + w_t l_t dt + r_t \Phi_t dt + \Phi_t^B dt \quad t \geq 0, \quad (2.1.2) \]

\[ a_0, b_0 > 0, \text{ and } \theta_0^c > 0 \text{ are given} \quad (2.1.3) \]

\[ \exp \left( - \int_0^t \frac{\Delta_u}{X_u} du \right) \theta_t^c = o(e^{-\nu t}), \text{ a.e.} \quad (2.1.4) \]

\[ (a, b, \theta^c, c) \in (C^+)^4 \quad (2.1.5) \]

where \( c_t \) is consumption at period \( t \), \( a_t \) is the quantity of stock selected up to \( t \), \( \delta_t \) means the quantity of bank accounts which remains by \( t \), \( \theta_t^c \) is the existing quantity of deposit bond which is bought from banks by \( t \), \( S_t \) stands for the price of stock at \( t \), \( w_t \) represents the nominal wage recievied at \( t \), \( \Phi_t \) is the lending from the central bank. We assume that the labors are supplied inelastically with respect to the wage, the interpretation of which is, for example, that if consumers do not work, they will die.

The first constraint (2.1.2) comes from (B). (2.1.4) reflects the condition that the debt should not grow faster than the interest rate (the no-Ponzi condition), but this is stronger requirement.

### 2.2 Firms’ problem

The firms’ maximization problem is synonymous with the stock holders’ one whose aim is to maximize the expected rate of revenue which is defined as the discount rate under which the expectation of the discounted integration of all the net profits equals the current value of firms’ equity capital, \( p_0 k_0 - X_0 \theta_0^f \).

Then the firms’ problem is described as follows;

\[ \max_{(a_t, \theta_t^f, k_t, l_t)_{t \geq 0}} \{ \phi_t \}_{t \geq 0} \text{ maximize by the order } \phi_t \geq \phi_t' \text{ a.e. for all } t, \phi, \phi' \in L^+ \quad (2.2.1) \]

subject to
\[ p_0 k_0 - X_0 \theta_0^f = E_p \left[ \int_0^\infty e^{-\int_0^t \phi_s ds} \left\{ \left( (1 + \tau_t) p_t f(t, k_t, l_t) - \Delta_t \theta_t^f - w_t l_t \right) dt - S_t da_t \right\} \right], \quad (2.2.2) \]
\[ k_t = k_0 + \int_0^t p_u^{-1} S_u da_u + \int_0^t p_u^{-1} X_u d\theta^f_u \quad (2.2.3) \]

\[ a_0, \theta^f_0 > 0 \text{ and } k_0 > 0 \text{ are given} \quad (2.2.4) \]

\[ \exp\left(-\int_0^t \frac{\Delta_u}{X_u} du\right) \theta^f_t = o(e^{-v't}), \text{ a.e., for some constant } v' > 0, \quad (2.2.5) \]

\[ (a, \theta^f, k, l) \in (C^+)^4 \quad (2.2.6) \]

where \( k_t \) is capital stock accumulated up to \( t \), \( a_t \) is the quantity of stock issued up to \( t \), \( \theta^f_t \) is the existing quantity of deposit bond which is bought from banks by \( t \), \( S_t \) stands for the price of the stock at \( t \), \( w_t \) represents the nominal wage at \( t \). Firms’ all profit is assumed to be distributed to stock holders as dividends which is defined as;

\[ \delta_t := \frac{(1+r_t)p_t f(t, k_t, l_t) - \Delta_t \theta^f_t - w_t l_t}{a_t} \]

\[ \delta_t = 0 \text{ in the case of } a_t = 0. \]

Firms’ balance sheet is then described as follows:

<table>
<thead>
<tr>
<th>Firms’ B/S at ( t )</th>
<th>capital stock ( p_t k_t )</th>
<th>debt ( X_t \theta^f_t )</th>
<th>equity capital</th>
</tr>
</thead>
</table>

2.3 Ordinary banks’ problem

In this paper, we assume the reserve requirement system in which a quantity of money needs to be deposited to the central bank as the reserve that is determined by [‘the reserve ratio’ \( \times \) ‘all deposit owed by ordinary banks’]. We write the reserve ratio as \( 0 < \epsilon_t < 1 \). The ordinary banks control \( \theta^c_t \) and \( \theta^f_t \) subject to

\[ b_t + \Phi_t + p_t f(t, k_t, l) dt + \int_0^t \theta_u dX_u - X_t (\theta^c_t + \theta^f_t) \geq \epsilon_t (b_t + \Phi_t + p_t f(t, k_t, l) dt) \]

namely

\[ b_t + \Phi_t + \int_0^t \theta_u dX_u - X_t (\theta^c_t + \theta^f_t) \geq \epsilon_t (b_t + \Phi_t) \quad t \geq 0 \quad (2.3.1) \]

so as to maximize

\[ \Phi^B_t = \Delta_t (\theta^c_t + \theta^f_t) - \{r_t b_t + r_t p_t f(t, k_t, l) + r_t \Phi_t \} \quad t \geq 0. \quad (2.3.2) \]
Thus we can see that the optimal solutions are always the corner's ones in which (2.3.1) holds binding. So we see in this paper

\[ b_t + \Phi_t + \int_0^t \theta_u dX_u - X_t(\theta^c_t + \theta^f_t) = \varepsilon_t(b_t + \Phi_t) \quad t \geq 0. \] (2.3.3)

Note (2.3.3) is equivalent to

\[-X_0(\theta^c_0 + \theta^f_0) + (1 - \varepsilon_t)(b_t + \Phi_t) = \int_0^t X_u d(\theta^c_t + \theta^f_t) \quad t \geq 0. \] (2.3.4)

### 3 Results

In this paper, we focus on the equilibrium which is defined as follows.

**The definition of the equilibrium.** An equilibrium of this economy is a set of stochastic processes

\[ \{(a_t, b_t, \theta^c_t, \theta^f_t), (c_t, k_t, l_t), (p_t, S_t, X_t, r_t, \triangle_t, w_t), \varepsilon_t, \Phi_t\}_{t \geq 0} \]

such that

1. given \(\{p_t, S_t, X_t, r_t\}\), \(\{a_t, b_t, \theta^c_t, c_t\}\) solves the consumers' problem;
2. given \(\{p_t, S_t, X_t, r_t\}\), \(\{a_t, \theta^f_t, k_t, l_t\}\) solves the firms' problem;
3. given \(\{X_t, \Delta_t, r_t\}\), \(\{\theta^c_t, \theta^f_t\}\) solves the banks' problem;
4. the good market clears;
5. the stock market clears;
6. the deposit bond market clears;
7. the labor market clears;
8. \(\Phi_t\) and \(\varepsilon_t\) are exogenous variables determined by the central bank.

We assume that the utility function is CRRA class.

**Assumption 1.**

\[ u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad c \geq 0 \]

where \(0 < \gamma < 1\).

We can see

\[ \frac{u''(c)c}{u'(c)} = \gamma, \quad \frac{u''(c)c^2}{u'(c)} = \gamma(\gamma + 1). \]
The production function is assumed to be linear with respect to both variables.

**Assumption 2.**

\[ f(t, x_t, l_t) = \alpha(t, B_t)x_t + \beta(t, B_t)l_t \]

\[ 0 < \inf_{(t,\omega)} \alpha(t, B_t), \quad 0 < \inf_{(t,\omega)} \beta(t, B_t) \]

where \( \alpha(t, B_t), \beta(t, B_t) \in C^{2}([0, \infty) \times \mathbb{R}^2) \) (i.e., \( \alpha, \beta \) is twice continuously differentiable on \([0, \infty) \times \mathbb{R}^2\).

Put

\[ p_t := p_0 \exp \left( \int_0^t \pi_u du - \frac{1}{2} \int_0^t \{ (\sigma_u^{p1})^2 + (\sigma_u^{p2})^2 \} du - \int_0^t \{ \sigma_u^p dB_u^1 + \sigma_u^a dB_u^2 \} \right) \]

(3.1)

and

\[ c_t \]

\[ := c_0 \exp \left( \int_0^t \frac{1}{\gamma} \left[ r_u - \pi_u - (\sigma_u^{p1} \lambda_u^1 + \sigma_u^{a1} \lambda_u^2) - \rho_u + \frac{1}{2} \{ (\sigma_u^{p1} + \lambda_u^1)^2 + (\sigma_u^{a1} + \lambda_u^2)^2 \} \right] du + \int_0^t \left\{ \frac{((\sigma_u^{p1} + \lambda_u^1)/\gamma) dB_u^1 + ((\sigma_u^{a1} + \lambda_u^2)/\gamma) dB_u^2} \right\} \right) \]

(3.2)

so that it holds that

\[ p_0 = u'(c_0). \]  

(3.3)

Put

\[ k_t = \exp \left( \int_0^t \alpha_u du \right) \left[ k_0 + \int_0^t \exp \left( - \int_0^u \alpha_{\tau} d\tau \right) (\beta_u l - c_u) du \right]. \]

(3.4)

Set

\[ S_t = \exp \left( \int_0^t \mu_u du - \frac{1}{2} \int_0^t \{ (\sigma_u^1)^2 + (\sigma_u^2)^2 \} du + \int_0^t \{ \sigma_u^1 dB_u^1 + \sigma_u^2 dB_u^2 \} \right) \]

(3.5)

for some \( \mu, \sigma^1, \sigma^2 \in L. \)
Put
\[ X_t = \exp \left( \int_0^t \hat{r}_u du - \frac{1}{2} \int_0^t \{ (\zeta_u^1)^2 + (\zeta_u^2)^2 \} du + \int_0^t \{ \zeta_u^1 dB_u^1 + \zeta_u^2 dB_u^2 \} \right) \] (3.6)
for some \( \hat{r}, \zeta^1, \zeta^2 \in L \).

Select \( a, \theta^f \in C^+ \) so as to satisfy
\[ k_t = k_0 + \int_0^t p_u^{-1} S_u da_u + \int_0^t p_u^{-i} X_u d\theta^f_u \] (3.7)
and put
\[ \delta_t = \frac{\{ \hat{r}_t + (\Delta_t/X_t) - \pi_t \} p_t k_t - \Delta_t \theta^f_t}{a_t}. \] (3.8)

Choose \( r_t \) so that it holds
\[ \alpha_t = \frac{\hat{r}_t + (\Delta_t/X_t) - \pi_t}{1 + r_t}. \] (3.9)

Solve \( (\lambda_t^1, \lambda_t^2) \) as the solution of the following equation;
\[ \begin{pmatrix} \sigma_t^1 & \sigma_t^2 \\ \zeta_t^1 & \zeta_t^2 \end{pmatrix} \begin{pmatrix} \lambda_t^1 \\ \lambda_t^2 \end{pmatrix} = \begin{pmatrix} \mu_t - r_t + (\delta_t/S_t) \\ \hat{r}_t - r_t + (\Delta_t/X_t) \end{pmatrix} \] (3.10)

Determine the above parameters so that \( \lambda_t^1 \) and \( \lambda_t^2 \in L \) satisfy the following Novikof condition;
\[ E_P \left[ \exp \left( \frac{1}{2} \int_0^t \{ (\lambda_u^1)^2 + (\lambda_u^2)^2 \} du \right) \right] < \infty, \quad t \geq 0. \] (3.11)

Then, by the Girsanov theorem, \( \tilde{B}_t^i := B_t^i + \int_0^t \lambda_u^i du, i = 1, 2, \) is a standard Brownian motion under the new measure \( Q \) defined by
\[ dQ := \exp \left( - \frac{1}{2} \int_0^t \{ (\lambda_u^1)^2 + (\lambda_u^2)^2 \} du - \int_0^t \{ \lambda_u^1 dB_u^1 + \lambda_u^2 dB_u^2 \} \right) dP. \]
Under these settings we prove the following lemma.

**Lemma 1.** There exists a set of parameters \( \{r, \hat{r}, \zeta^1, \zeta^2, \mu, \sigma^1, \sigma^2, \lambda^1, \lambda^2, \pi, \sigma^{p1}, \sigma^{p2}, \Delta, a, \theta^f \} \) which simultaneously satisfies (3.1) – (3.11).

(Proof) Set first the parameters so as to satisfy (3.1) – (3.6), (3.9) – (3.11). Determine \( \theta^f \) by

\[
p_t^{-1}S_t d\left( \frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\} p_t k_t - \Delta_t \theta^f_t}{\delta_t} \right) + p_t^{-1}X_t d\theta^f_t = dk_t,
\]

which is equivalent to

\[
\left( \frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\} p_t^{-1}S_t}{\delta_t} - 1 \right) dk_t + d\left( \frac{\{\hat{r}_t + (\Delta_t/X_t) - \pi_t\} p_t}{\delta_t} \right) k_t
\]

\[
= \left( \frac{\Delta_t}{\delta_t} - p_t^{-1}X_t \right) d\theta^f_t + d\left( \frac{\Delta_t}{\delta_t} \right) \theta^f_t
\]

(3.12)

Lastly, \( a_t \) is determined by (3.8). \( \square \)

In the subsequent lemmata, we prove that the parameters found in the lemma 1 consist of the equilibrium.

Let \( \Phi_t, b_t \) and \( \theta_t^c \) satisfy (2.3.4). Define \( \Phi_t^B \) by (2.3.2).

**Lemma 2.** Let \( \{(a_t, b_t, \theta_t^c, c_t)\}_{t \geq 0} \) meet the consumers' budget constraint (2.1.2) – (2.1.5). Then \( c_t, t \geq 0 \) satisfies

\[
E_Q \left[ \int_{0}^{\infty} R_t p_t c_t dt \right] \leq S_0a_0 + b_0 - X_0 \theta_0^c + E_Q \left[ \int_{0}^{\infty} (R_t w_t l + r_t R_t \Phi_t dt + R_t \Phi_t^B) dt \right]
\]

(3.13)

where

\[
R_t := \exp \left( - \int_{0}^{t} r_u du \right).
\]

(Proof) From the condition in this claim, we see

\[
p_t c_t dt + S t dt + db_t - X_t d\theta_t^c
\]

\[
= r_t b_t dt - \Delta_t \theta_t^c dt + \delta_t a_t dt + w_t dt + r_t \Phi_t dt + \Phi_t^B \quad t \geq 0,
\]

(2.1.2)
Multiplying both sides by \( R_t \) and calculating a little yield that
\[
R_t p_t c_t dt + d(R_t S_t a_t) + d(R_t b_t) - d(R_t X_t \theta_t^c)
\]
\[
= a_t d(R_t S_t) - \theta_t^f d(R_t X_t) - R_t \Delta_t \theta_t^c dt + R_t \delta_t + R_t d\Phi_t dt + R_t \Phi_t^B
\]
\[
= \sigma_t^1 R_t S_t a_t d\hat{B}_1^t + \sigma_t^2 R_t S_t a_t d\hat{B}_2^t + \zeta_t^1 R_t X_t \theta_t^c d\hat{B}_1^t + \zeta_t^2 R_t S_t \theta_t^c d\hat{B}_2^t
\]
\[
+ R_t w_t l dt + R_t \Phi_t dt + R_t \Phi_t^B.
\]

Integrating both sides in time and taking expectations by the measure \( Q \) generate that
\[
E_Q \left[ \int_0^t R_t p_t c_t du \right] + E_Q \left[ R_t S_t a_t + R_t b_t - R_t X_t \theta_t^c \right]
\]
\[
= S_0 a_0 + b_0 - X_0 \theta_0^c + E_Q \left[ \int_0^t \left( R_t w_t l + R_a \Phi_u + R_a \Phi_u^B \right) du \right].
\]

Note
\[
R_t X_t = \exp \left( -\int_0^t \frac{\Delta_u}{X_u} du - \frac{1}{2} \int_0^t \left\{ (\zeta_t^1)^2 + (\zeta_t^2)^2 \right\} du + \int_0^t \left\{ \zeta_t^1 d\hat{B}_1^t + \zeta_t^2 d\hat{B}_2^t \right\} \right).
\]

So by the condition (2.2.5), we find that
\[
\lim_{t \to \infty} E_Q [R_t X_t \theta_t^c] = 0.
\]

That completes the proof. \( \square \)

**Lemma 3.** If \( c \) and \( k \in C^+ \) satisfy
\[
k_t = \exp \left( \int_0^t \alpha_u du \right) \left[ k_0 + \int_0^t \exp \left( -\int_0^u \alpha_r dr \right) (\beta_u l - c_u) du \right] \quad (3.4)
\]
and \( a \) and \( \theta^f \in C^+ \) are determined so that
\[
k_t = k_0 + \int_0^t p_u^{-1} S_u da_u + \int_0^t p_u^{-1} X_u d\theta_u^f, \quad (3.7)
\]
holds. Let \( b \) and \( \theta^c \in C^+ \) are chosen so that it holds
\[
\Phi_t^B = \Delta_t (\theta_t^c + \theta_t^f) - \{ r_t b_t + r_t p_t f(t, k_t, l) + r_t \Phi_t \} \quad t \geq 0. \quad (3.14)
\]
and that the money market clears, namely,
\[ b_t = X_0(\theta_0^c + \theta_0^f) + \int_0^t X_u d(\theta_u^c + \theta_u^f) \quad t \geq 0. \]  \hfill (3.15)

Then it follows that
\[ p_tC_t dt + S_t da_t + db_t - X_t d\theta_{t}^{c} = r_t b_t dt - \Delta_t \theta_{t}^{c} dt + \delta_t a_t dt + w_t l dt + r_t \Phi_t dt + \Phi_t^{B} t \geq 0. \]  \hfill (2.1.2)

(Proof) Firstly note that (3.4) is equivalent to
\[ \dot{k}_{t} = f(t, k_{t}, l) - c_{t}. \]

Then by multiplying both sides by \( p_t \)
\[ p_tC_t dt + S_t da_t + X_t d\theta_{t}^{f} = \left[ p_t f(t, k_{t}, l) - \Delta_t \theta_{t}^{f} - w_t l \right] dt + \Delta_t \theta_{t}^{f} dt + w_t l dt = \left[ (1 + r_t)p_t f(t, k_{t}, l) - \Delta_t \theta_{t}^{f} - w_t l \right] dt - r_t p_t f(t, k_{t}, l) dt + \Delta_t \theta_{t}^{f} dt + w_t l dt. \]

From (3.14) and the definition of \( \delta_t \)
\[ \delta_t = \Delta_0 a_t + r_t b_t - \Delta_0 \theta_{t}^{c} dt + r_t \Phi_t dt + \Phi_t^{B} dt + w_t l dt. \]

On the left hand side, we obtain from (3.15)
\[ p_tC_t dt + S_t da_t + X_t d\theta_{t}^{f} = p_tC_t dt + S_t da_t + X_t d(\theta_{t}^{c} + \theta_{t}^{f}) - X_t d\theta_{t}^{c}. \]

Thus we find (2.1.2) holds. \( \square \)

**Lemma 4.** The solution of the firms’ problem
\[ \max_{(a_t, \theta_t^c, \theta_t^f), t \geq 0} \{\phi_t\}_{t \geq 0} \quad \text{maximize by the order } \phi_t \geq \phi_t' \ a.e. \text{ for all } t, \phi, \phi' \in L^{+} \]  \( \text{(2.2.1)} \)

subject to (2.2.2)-(2.2.6) is
\[ \left\{ \hat{r}_t + \frac{\Delta_t}{X_t} \right\}_{t \geq 0}. \]

(Proof) Let
\[ \hat{R}_t = \exp \left( - \int_0^t \left\{ \hat{r}_u + \frac{\Delta_u}{X_u} \right\} du \right). \]
\[(1 + r_i)\hat{R}_t p_t f(t, k_t, l_t) - \Delta_t \hat{R}_t \theta_t^f - \hat{R}_t w_t l_t \] dt - \hat{R}_t S_t da_t

\[= \{\hat{r}_t + (\Delta_t / X_t)\} \hat{R}_t p_t k_t - (\Delta_t / X_t) \hat{R}_t X_t \theta_t^f \] dt - \hat{R}_t S_t da_t

\[= -k_t d(\hat{R}_t p_t) - (\hat{R}_t p_t) d\theta_t^f - (\Delta_t / X_t) \hat{R}_t X_t \theta_t^f dt
\]

\[= -d(\hat{R}_t p_t k_t) + d(\hat{R}_t X_t \theta_t^f) + \sigma_t^1 \hat{R}_t p_t k_t dB_t^1 + \sigma_t^2 \hat{R}_t p_t k_t dB_t^2 + \zeta_t^1 \hat{R}_t X_t \theta_t^f dB_t^1 + \zeta_t^2 \hat{R}_t X_t \theta_t^f dB_t^2.

Thus it follows by integrating and taking expectations of both sides

\[p_0 k_0 - X_0 \theta_0^f \]

\[= E_P [\int_0^t \exp \left( \frac{\hat{r}_t + (\Delta_t / X_t)}{1 + r_t} \right) dt \left\{ (1 + r_u) p_u f(t, k_u, l_u - \Delta_u \theta_u^f - w_u l_u) du - S_u da_u \right\}]

Because for any \((a, \theta^f, k, l)\) that satisfies \(\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0\) it holds that

\[p_0 k_0 - X_0 \theta_0^f \]

\[= E_P \left[ \int_0^\infty \exp \left( - \int_0^t \{\hat{r}_u + (\Delta_u / X_u)\} du \right) \left\{ (1 + r_t) p_t f(t, k_t, l_t - \Delta_t \theta_t^f - w_t l_t) dt - S_t da_t \right\} \right],

it suffices to prove that \(\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0\) follows for the selected \((a, \theta^f, k, l)\).

Recall

\[\alpha_t = \frac{\hat{r}_t + (\Delta_t / X_t) - \pi_t}{1 + r_t}.

Because

\[\alpha_t < \hat{r}_t + (\Delta_t / X_t) - \pi_t,

we can see

\[\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t] = 0.

From (2.2.5)

\[\lim_{t \to \infty} E_p[\hat{R}_t X_t \theta_t^f] = 0.

Then this lemma is proved. \(\square\)

**Lemma 5.** If \(c^* \in C^+\) satisfies

\[R_t p_t \exp \left( -\frac{1}{2} \int_0^t \{ (\lambda_u^1)^2 + (\lambda_u^2)^2 \} du - \int_0^t \{ \lambda_u^1 dB_u^1 + \lambda_u^2 dB_u^2 \} \right) = \exp \left( -\int_0^t \rho_u du \right) u'(c_u^*),

\[p_0 k_0 - X_0 \theta_0^f \]

\[= E_P [\int_0^t \exp \left( -\frac{\hat{r}_t + (\Delta_t / X_t)}{1 + r_t} \right) dt \left\{ (1 + r_u) p_u f(t, k_u, l_u) - \Delta_u \theta_u^f - w_u l_u \right\} du - S_u da_u]

\[+ E_P [\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f].

Because any \((a, \theta^f, k, l)\) that satisfies \(\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0\) it holds that

\[p_0 k_0 - X_0 \theta_0^f \]

\[= E_P \left[ \int_0^\infty \exp \left( -\int_0^t \{\hat{r}_u + (\Delta_u / X_u)\} du \right) \left\{ (1 + r_t) p_t f(t, k_t, l_t) - \Delta_t \theta_t^f - w_t l_t \right\} dt - S_t da_t \right],

it suffices to prove that \(\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0\) follows for the selected \((a, \theta^f, k, l)\).

Recall

\[\alpha_t = \frac{\hat{r}_t + (\Delta_t / X_t) - \pi_t}{1 + r_t}.

Because

\[\alpha_t < \hat{r}_t + (\Delta_t / X_t) - \pi_t,

we can see

\[\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t] = 0.

From (2.2.5)

\[\lim_{t \to \infty} E_p[\hat{R}_t X_t \theta_t^f] = 0.

Then this lemma is proved. \(\square\)

**Lemma 5.** If \(c^* \in C^+\) satisfies

\[R_t p_t \exp \left( -\frac{1}{2} \int_0^t \{ (\lambda_u^1)^2 + (\lambda_u^2)^2 \} du - \int_0^t \{ \lambda_u^1 dB_u^1 + \lambda_u^2 dB_u^2 \} \right) = \exp \left( -\int_0^t \rho_u du \right) u'(c_u^*),

\[p_0 k_0 - X_0 \theta_0^f \]

\[= E_P [\int_0^t \exp \left( -\frac{\hat{r}_t + (\Delta_t / X_t)}{1 + r_t} \right) dt \left\{ (1 + r_u) p_u f(t, k_u, l_u) - \Delta_u \theta_u^f - w_u l_u \right\} du - S_u da_u]

\[+ E_P [\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f].

Because any \((a, \theta^f, k, l)\) that satisfies \(\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0\) it holds that

\[p_0 k_0 - X_0 \theta_0^f \]

\[= E_P \left[ \int_0^\infty \exp \left( -\int_0^t \{\hat{r}_u + (\Delta_u / X_u)\} du \right) \left\{ (1 + r_t) p_t f(t, k_t, l_t) - \Delta_t \theta_t^f - w_t l_t \right\} dt - S_t da_t \right],

it suffices to prove that \(\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t - \hat{R}_t X_t \theta_t^f] = 0\) follows for the selected \((a, \theta^f, k, l)\).

Recall

\[\alpha_t = \frac{\hat{r}_t + (\Delta_t / X_t) - \pi_t}{1 + r_t}.

Because

\[\alpha_t < \hat{r}_t + (\Delta_t / X_t) - \pi_t,

we can see

\[\lim_{t \to \infty} E_p[\hat{R}_t p_t k_t] = 0.

From (2.2.5)

\[\lim_{t \to \infty} E_p[\hat{R}_t X_t \theta_t^f] = 0.

Then this lemma is proved. \(\square\)
$c^*$ is the solution of the problem:

$$\max_{c \in C^+} E_p \left[ \int_0^\infty e^{-\int_0^t \rho_u du} u(c_t) dt \right]$$

subject to

$$E_Q \left[ \int_0^\infty R_t p_t c_t dt \right] \leq S_0 a_0 + b_0 - X_0 \theta_0^c + E_Q \left[ \int_0^\infty \left( R_t w_t + \rho_t \Phi_t dt \right) \right].$$

The proof of lemma 5 is almost the same as the standard argument (see e.g. Dana and Jeanblanc-Picque (2003)), so we omitted here.

Note that under

$$p_t = p_0 \exp \left( \int_0^t \pi_u du - \frac{1}{2} \int_0^t \left\{ \left( \sigma_u^{p1} \right)^2 + \left( \sigma_u^{p2} \right)^2 \right\} du \right)$$

we can deduce that

$$c_t^* := c_0 \exp \left( \int_0^t \left( \frac{1}{\gamma} \left[ r_u - \pi_u - \sigma_u^{p1} \lambda_u^1 + \sigma_u^{p2} \lambda_u^2 \right] - \rho_u + \frac{1}{2} \left\{ \left( \sigma_u^{p1} + \lambda_u^1 \right)^2 + \left( \sigma_u^{p2} + \lambda_u^2 \right)^2 \right\} \right) du \right)$$

$$+ \int_0^t \left\{ \left( \sigma_u^{p1} + \lambda_u^1 \right) / \gamma \right\} dB_u^1 + \left( \sigma_u^{p2} + \lambda_u^2 \right) / \gamma dB_u^2 \right\}.$$  (3.2)

### 4 Policy Implications

We consider the case in which the central bank attempts to ease the quantity of money by reducing the reserve rate, $\varepsilon_t$, or increasing the lending, $\Phi_t$. The change in these parameters leads to the change of the quantity $\theta_t^c$ and $\theta_t^f$ from (2.3.4). By (3.2) and (3.4), we can see that the increase of the inflation rate $\pi_t$ means the increase of the capital path $k_t$. Thus recalling (3.12)

$$\left( \frac{\hat{r}_t + (\Delta_t / X_t) - \pi_t \left\{ \sigma_t^{-1} S_t \right\} \delta_t}{\delta_t} - 1 \right) dk_t + d\left( \frac{\hat{r}_t + (\Delta_t / X_t) - \pi_t \left\{ \delta_t \right\} p_t}{\delta_t} \right) k_t$$

$$= \left( \frac{\Delta_t}{\delta_t} - p_t^{-1} X_t \right) d\theta_t^f + d\left( \frac{\Delta_t}{\delta_t} \right) \theta_t^f,$$  (3.12)

the increase of $\theta_t^f$ accompanies the increase of the inflation rate $\pi_t$ provided

$$\text{sign} \left( \frac{\hat{r}_t + (\Delta_t / X_t) - \pi_t \left\{ \sigma_t^{-1} S_t \right\} \delta_t}{\delta_t} - 1 \right) dt = \text{sign} \left( \frac{\hat{r}_t + (\Delta_t / X_t) - \pi_t \left\{ \delta_t \right\} p_t}{\delta_t} \right)$$

$$= \text{sign} \left( \frac{\Delta_t}{\delta_t} - p_t^{-1} X_t \right) dt = \text{sign} \left( \frac{\Delta_t}{\delta_t} \right).$$
The case of tightening the money can also be studied in the similar manner so we omitted the discussion.

References

