THE INDEX OF A THREEFOLD CANONICAL SINGULARITY

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ABSTRACT. The index of a 3-fold canonical singularity at a crepant centre is at most 6.

1. INTRODUCTION

Let \( P \in X \) be a log canonical singularity. Shokurov asked if one can bound the index \( r_P \) of \( X \) at \( P \) in terms of the discrepancies of divisors over \( X \).

Suppose that \( X \) has log canonical singularities with \( P \) a log canonical centre. In \( \dim X = 2 \), \( r_P \) is 1, 2, 3, 4 or 6 by the classification of singularities. In an arbitrary dimension, Ishii [5] and Fujino [2] reduced the boundedness of \( r_P \) to a conjectural boundedness of a quotient of the birational automorphism group of a variety \( S \) with \( K_S \sim 0 \). In particular, they proved \( r_P \leq 66 \) in \( \dim X = 3 \).

Suppose that \( X \) has canonical singularities. In \( \dim X = 2 \), \( P \) is a rational double point, so \( r_P = 1 \). The purpose of this paper is to provide an affirmative answer in \( \dim X = 3 \).

**Theorem 1.1.** Let \( P \in X \) be a 3-fold canonical singularity such that \( P \) is a crepant centre. Then the index of \( X \) at \( P \) is at most 6.

**Remark 1.2.** We have such singularities \( P \) with \( r_P = 1, 2, 3, 4 \) in Example 4.3, but I do not know if there exists \( P \) with \( r_P = 5 \) or 6.

Note that no (even implicit) bound of \( r_P \) has been known before. Here a crepant centre means the centre of a divisor with discrepancy zero. The condition that \( P \) is a crepant centre is necessary even for a strictly canonical singularity, see Example 5.1. On the other hand, if once the minimal discrepancy at \( P \) is fixed, then one can bound \( r_P \) for an arbitrary 3-fold canonical singularity \( P \in X \) (Theorem 5.3).

We shall prove Theorem 1.1 by using the singular Riemann–Roch formula (singRR) [14], an orbifold version of Riemann–Roch formula, due to Reid. In Sect. 2, we build a tower \( Y \to X \) of crepant blow-ups with \( \mathbb{Q} \)-factorial terminal \( Y \), on which the singRR is applicable unconditionally. Then we construct a divisor \( F \) on \( Y \) which possesses the information on the index \( r_P \). The \( r_P \) is determined by the Euler characteristics \( \chi(iK_Y|_F) \), which can be explicitly computed by the singRR (Sect. 3). We derive a numerical classification of the singularities on \( Y \) together with \( r_P \) in Sect. 4, by the method [6], [7] in the classification of 3-fold divisorial contractions. The boundedness of indices in terms of minimal discrepancies is discussed in Sect. 5.

We work over an algebraically closed field \( k \) of characteristic zero. A germ \( P \in X \) means an algebraic germ of a variety \( X \) at a closed point \( P \).

2. CREPANT BLOW-UPS

Let \( X \) be a normal \( \mathbb{Q} \)-Gorenstein variety.
Definition 2.1. The index of $X$ at a point $P$ is the smallest positive integer $r$ such that $rK_X$ is a Cartier divisor at $P$.

Consider a normal variety $Y$ with a proper birational morphism $f: Y \to X$. A prime divisor $E$ on any such $Y$ is called a divisor over $X$, and the image $f(E)$ is called the centre of $E$ on $X$ and denoted by $c_X(E)$. The valuation $v_E$ on the function field of $X$ given by such $E$ is called an algebraic valuation of $X$. If we write

$$K_Y = f^*K_X + \sum_E a_E(X)E \quad \text{with } a_E(X) \in \mathbb{Q},$$

then $a_E(X)$ is called the discrepancy of $E$. We say that $X$ has log canonical, log terminal, canonical, terminal singularities if $a_E(X) \geq -1, > -1, \geq 0, > 0$ respectively for all exceptional divisors $E$ over $X$.

The notion of crepacy is crucial in this paper.

Definition 2.2. (i) A crepant divisor over $X$ is an exceptional divisor $E$ over $X$ with $a_E(X) = 0$. A crepant valuation of $X$ is the algebraic valuation $v_E$ given by a crepant divisor $E$.

(ii) A crepant centre on $X$ is the centre $c_X(E)$ of a crepant divisor $E$.

(iii) A crepant blow-up $f: Y \to X$ is a projective birational morphism from a normal variety $Y$ such that $K_Y = f^*K_X$.

Remark 2.3. (i) Suppose that $X$ is canonical. Then every crepant valuation is realised as a divisor on any resolution of $X$. In particular, the number of crepant valuations of $X$ is finite. The complement of the union of all crepant centres is the largest terminal open subvariety of $X$.

(ii) If $Y \to X$ is a crepant blow-up, then $X$ is canonical if and only if so is $Y$.

We have a crepant blow-up by the LMMP.

Proposition 2.4. Let $X$ be a variety with canonical singularities and $v$ a crepant valuation of $X$. Then there exists a crepant blow-up $f: Y \to X$ such that

(i) $Y$ is $\mathbb{Q}$-factorial,

(ii) $f$ has exactly one exceptional divisor $E$, and $v_E = v$,

(iii) $-E$ is $f$-nef.

Proof. Take a projective resolution of singularities $g: Z \to X$, and denote by $E_Z$ the divisor on $Z$ with $v_{E_Z} = v$. Take a Cartier divisor $H > 0$ on $X$ whose support contains all the crepant centres. We write $g^*H = H_Z + F$ with the strict transform $H_Z$ of $H$, and $m$ for the coefficient of $E_Z$ in $F$. Fix $\varepsilon > 0$ so that $(Z, \varepsilon(H_Z + 2(F - mE_Z)))$ is klt, and run $(K_Z + \varepsilon(H_Z + 2(F - mE_Z)))$-LMMP over $X$ by [1] to get a log minimal model $f: Y \to X$.

By $K_Z + \varepsilon(H_Z + 2(F - mE_Z)) \equiv_X K_Z + \varepsilon(F - 2mE_Z)$, the negativity lemma [11, Lemma 2.19] shows that this LMMP contracts exactly all the $g$-exceptional divisors but $E_Z$, and $-E$ is $f$-nef for the strict transform $E$ of $E_Z$. Hence $f$ is a required crepant blow-up.

q.e.d.

Remark 2.5. If $X$ is $\mathbb{Q}$-factorial, then (ii) implies that $\rho(Y/X) = 1$ and $-E$ is $f$-ample.

Corollary 2.6. Let $X = X_0$ be a variety with canonical singularities and $Z$ a crepant centre on $X$. Then there exists a sequence of crepant blow-ups $f_t: X_t \to X_{t-1}$ for $1 \leq t \leq s$ such that
Then the inclusion

$$f_r^*\mathcal{O}_Y(iK_Y-F) = \begin{cases}
\mathcal{O}_X(iK_X) & \text{if } r_F | i, \\
\mathcal{O}_X(iK_X) & \text{otherwise},
\end{cases}$$

and for \( t > 1 \),

$$(1) \quad f_t^*\mathcal{O}_X(iK_X - F_t) = \mathcal{O}_{X_{t-1}}(iK_{X_{t-1}} - F_{t-1}),$$

$$(2) \quad R^j f_t^*\mathcal{O}_X(iK_X - F_t) = 0 \quad \text{for } j \geq 1,$$

$$(3) \quad f_r^*\mathcal{O}_X(iK_X - F_t) = \mathcal{O}_{X_{r-1}}(iK_{X_{r-1}} - F_{r-1}),$$

$$(4) \quad R^j f_r^*\mathcal{O}_X(iK_X - F_t) = 0 \quad \text{for } j \geq 1.$$
3. THE SINGULAR RIEMANN–ROCH FORMULA

We shall apply the singular Riemann–Roch formula due to Reid to our crepant blow-up, and use the method [6], [7] in the classification of 3-fold divisorial contractions. We briefly recall the formula on a canonical 3-fold.

**Theorem 3.1** ([14, Theorem 10.2]). Let $X$ be a projective 3-fold with canonical singularities and $D$ a divisor on $X$ such that $D \sim i_P K_X$ with $i_P \in \mathbb{Z}$ at each $P \in X$.

(i) There is a formula of the form

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12} D(D - K_X)(2D - K_X) + \frac{1}{12} D \cdot c_2(X) + \sum_P c_P(D),$$

where the summation takes place over the singularities of $\mathcal{O}_X(D)$, and $c_P(D) \in \mathbb{Q}$ is a contribution due to the singularity at $P$, depending only on the analytic type.

(ii) For a terminal cyclic quotient singularity $P$ of type $\frac{1}{r_P}(1, -1, b_P)$,

$$c_P(D) = -i_P r_P^2 - 1 + \frac{\overline{j_P r_P}}{2 r_P},$$

where $\overline{i} = i - \lfloor \frac{i}{r_P} \rfloor r_P$ denotes the residue of $i$ modulo $r_P$.

(iii) For an arbitrary terminal singularity $P$,

$$c_P(D) = \sum_Q c_Q(D_Q),$$

where $\{(Q, D_Q)\}_Q$ is a flat deformation of $(P, D)$ to the basket of terminal cyclic quotient singularities $Q$. Such $Q$ is called a fictitious singularity.

**Remark 3.2.** The condition $D \sim i_P K_X$ always holds if $X$ is $\mathbb{Q}$-factorial and terminal [8, Corollary 5.2].

Our object is a germ of a crepant blow-up $f : Y \to X$ with a divisor $F$ on $Y$ in Theorem 2.7 at a 3-fold canonical singularity $P \in X$ with index $r_P$. Shrinking and compactifying it, we may assume that $Y$ is projective and terminal ($f$ is merely a projective morphism outside a neighbourhood of $P$). We shall express the function $\delta_P(i)$ below.

**Definition 3.3.** We define the function $\delta_P(i)$ on $\mathbb{Z}$ as

$$\delta_P(i) := \begin{cases} 1 & \text{if } r_P | i, \\ 0 & \text{otherwise}. \end{cases}$$

Applying (ii) in Theorem 2.7 and the vanishing $R^j f_* \mathcal{O}_Y(iK_Y) = 0$ for $j \geq 1$ to the exact sequence

$$0 \to \mathcal{O}_Y(iK_Y - F) \to \mathcal{O}_Y(iK_Y) \to \mathcal{O}_F(iK_Y|_F) \to 0,$$

we obtain

$$\delta_P(i) = \dim_k f_* \mathcal{O}_Y(iK_Y)/f_* \mathcal{O}_Y(iK_Y - F)$$

$$= h^0(\mathcal{O}_F(iK_Y|_F))$$

$$= \chi(\mathcal{O}_F(iK_Y|_F))$$

$$= \chi(\mathcal{O}_Y(iK_Y)) - \chi(\mathcal{O}_Y(iK_Y - F)).$$
Let \( I_0 := \{ Q \text{ with type } \frac{1}{r}(1, -1, b_Q) \} \) be the basket of fictitious singularities from singularities on \( Y \). Note that \( b_Q \) is co-prime to \( r_Q \). For \( Q \in I_0 \), let \( f_Q \) denote the smallest non-negative integer such that \( F \sim f_Q K_Y \) at \( Q \). By replacing \( b_Q \) with \( r_Q - b_Q \) if necessary, we may assume \( v_Q := f_Q b_Q \leq r_Q/2 \). Set \( I := \{ Q \in I_0 \mid f_Q \neq 0 \} \).

With this notation, the singular Riemann–Roch formula computes the right-hand side of (5), to provide

(6) \[ \delta_p(i) = \frac{1}{6} F^3 + \frac{1}{12} F \cdot c_2(Y) + \sum_{Q \in I} (A_Q(i) - A_Q(i - f_Q)), \]

where the contribution \( A_Q(i) \) is given by

\[ A_Q(i) := -\frac{r_Q^2 - 1}{12r_Q} + \sum_{j=1}^{7-1} \frac{j b_Q(r_Q - j b_Q)}{2r_Q}. \]

The \( A_Q(i) \) satisfies the formula

\[ A_Q(i + 1) - A_Q(i) = -\frac{r_Q^2 - 1}{12r_Q} + B_Q(i b_Q) \]

with

\[ B_Q(i) := \frac{I(r_Q - I)}{2r_Q}. \]

Therefore by (6), we have

(7) \[ \delta_p(i + 1) - \delta_p(i) = \sum_{Q \in I} (B_Q(i b_Q) - B_Q(i b_Q - v_Q)). \]

**Lemma 3.4.** The \( r_p \) equals the l.c.m. of \( r_Q \) for all \( Q \in I \).

**Proof.** Since \( r_p K_Y = r_p f^* K_X \) is a Cartier divisor near \( f^{-1}(P) \), \( r_Q \) divides \( r_p \) for all \( Q \in I \). On the other hand, we see that \( r_p \) divides the l.c.m. of \( r_Q \) by (7) and the periodic properties of \( \delta_p, B_Q \).

q.e.d.

4. **Boundedness of indices**

We shall prove Theorem 1.1 in this section. Let \( r_p \) denote the index of \( X \) at \( P \).

We take a crepant blow-up \( f : Y \to X \) with a divisor \( F \) on \( Y \) in Theorem 2.7. We restrict the possibilities of \( J := \{(r_Q, v_Q)\}_{Q \in I} \) using (7) for \( i = 0 \).

**Lemma 4.1.** \( J \) is one of the types in Table 1.

**Table 1**

<table>
<thead>
<tr>
<th>( r_p )</th>
<th>( J )</th>
<th>( r_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( (2, 1) )</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>( (2, 1), (2, 1), (2, 1) )</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>( (2, 1), (2, 1), (4, 2) )</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>( (2, 1), (3, 1), (6, 1) )</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>( (2, 1), (4, 1), (4, 1) )</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>( (3, 1), (3, 1), (3, 1) )</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>( (4, 2), (4, 2) )</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>( (2, 1), (6, 3) )</td>
<td>( \emptyset )</td>
</tr>
</tbody>
</table>
Proof. By Lemma 3.4, $r_P$ is determined by $J$, and $r_P = 1$ if and only if $J = \emptyset$. We assume $r_P > 1$ from now on. Then (7) for $i = 0$ is written as

$$(8) \quad \sum_{Q \in I} B_Q(v_Q) = 1.$$  

By the definition of $B_Q$ and $r_Q \geq 2v_Q$, we have

$$(9) \quad v_Q/4 \leq B_Q(v_Q) < v_Q/2.$$  

Then $J' := \{v_Q\}_{Q \in I}$, which satisfies (8) and (9), should be one of

$$\{1,1,1,1\}, \{1,1,2\}, \{1,1,1\}, \{2,2\}, \{1,3\}, \{1,2\}, \{3\}, \{4\}.$$  

For each of these candidates for $J'$, one can solve the equation (8) for $r_Q (\geq 2v_Q)$ explicitly. Every solution is in Table 1. For example, suppose $J' = \{1,2\}$. We set $J = \{(r_1,1), (r_2,2)\}$. Then (8) becomes $1/r_1 + 4/r_2 = 1$. Thus $(r_1, r_2) = (2,8), (3,6)$ or $(5,5)$, so $J$ is of type 8, 9, 10 respectively. q.e.d.

By Lemma 4.1, we have $r_P \leq 9$, and for Theorem 1.1 it is enough to exclude types 8, 11, 12. However, we derive a finer numerical classification by determining $\tilde{J} := \{\{r_Q, v_Q, b_Q\}\}_{Q \in I}$.  

**Theorem 4.2.** $J$ is one of the types in Table 2.

<table>
<thead>
<tr>
<th>Type</th>
<th>$J$</th>
<th>$r_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${(2,1,1), (2,1,1), (2,1,1), (2,1,1)}$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>${(2,1,1), (2,1,1), (2,1,2), (2,1,2)}$</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>${(2,1,1), (2,1,1), (2,1,3), (2,1,3)}$</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>${(2,1,1), (2,1,1), (2,1,1), (2,1,2)}$</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>${(2,1,1), (2,1,1), (2,1,1), (2,1,2)}$</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>${(2,1,1), (2,1,1), (2,1,2), (2,1,2)}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Proof. By Lemma 4.1, there exist only finitely many candidates for $\tilde{J}$. For each candidate, one can compute the right-hand side of (7) explicitly. It must coincide with $\delta_P(i+1) - \delta_P(i)$, but such a coincidence happens only if $\tilde{J}$ is one of the types in Table 2.

Here we demonstrate for type 3. $J = \{(2,1,1), (3,1,b_2), (6,1,b_3)\}$ with $b_2 = 1$ or 2 and $b_3 = 1$ or 5. The (7) for $i = 1$ is $\delta_P(2) - \delta_P(1) = 1.1/3, 2/3, 0$ when $(b_2, b_3) = (1,1), (1,5), (2,1), (2,5)$ respectively. Thus $(b_2, b_3)$ must be $(2,5)$, and in this case (7) surely holds for any $i$. q.e.d.

**Example 4.3.** In simple cases, Theorem 1.1 is known by the classification. 

(i) (Morrison [13], Ishida–Iwashita [4]) If $P$ is a cyclic quotient singularity, then $r_P = 1$ except $1/4(1, 2n + 1, -2)$ ($n \geq 2$), $1/3(1, 9, 11)$, $1/2(1, 4, 7)$, with $r_P = 2, 2, 3$ respectively.

(ii) (Hayakawa–Takeuchi [3]) If $P$ is an isolated singularity which is a cyclic quotient of a hypersurface singularity, then $r_P \leq 4$. The only case when $r_P = 4$ is $o \in (x_1x_2 + x_3^2 + x_4^2 = 0) \subset \mathbb{A}^4_{x_1,x_2,x_3,x_4}/\mathbb{Z}_8(1,5,3,7)$.
5. Minimal discrepancies

To begin with, we provide an example which explains the need of $P$ being a crepant centre in Theorem 1.1 even for a strictly canonical singularity. A similar example exists also for a 3-fold strictly log canonical singularity [2, Example 6.1].

Example 5.1. Let $r \in \mathbb{N}$. Let $P \in X$ be the germ
\[ \sigma \in \langle x_1x_2 + x_3^2 = 0 \rangle \subset \mathbb{A}^4_{x_1,x_2,x_3,x_4}/\mathbb{Z}_r(1,-1,0,1), \]
which is singular along the $x_4$-axis $C$. Let $f: Y \to X$ be the weighted blow-up with weights $\text{wt}(x_1,x_2,x_3,x_4) = \frac{1}{r}(1,r-1,r,1)$. Then $K_Y = f^*K_X + \frac{1}{r}E$ with the exceptional divisor $E$, and $Y$ has at least terminal quotient singularities of types $\frac{1}{r}(1,-1,1)$ and $\frac{1}{r}(1,1,1)$ outside the strict transform $C_Y$ of $C$. Let $g: Z \to Y$ be the blow-up with centre $C_Y$. Then $g$ is a crepant blow-up and $Z$ is smooth near $g^{-1}(C_Y)$. Hence $X$ has canonical singularities with a crepant centre $C$, but $P$ is not a crepant centre.

The index of $X$ at $P$ is $r$.

We focus on the minimal discrepancy to grasp this phenomenon. For a normal $\mathbb{Q}$-Gorenstein singularity $P \in X$, the minimal discrepancy $\text{md}_P X$ of $X$ at $P$ is the infimum of discrepancies $a_E(X)$ for all divisors $E$ over $X$ with $c_X(E) = P$. Note that $\text{md}_P X \in \{-\infty\} \cup [-1,\infty)$, and $P \in X$ is log canonical if and only if $\text{md}_P X \geq -1$.

In Example 5.1, we have $\text{md}_P X = 1/r$. Shokurov formulated a question on the boundedness of indices in terms of minimal discrepancies.

Question 5.2 (Shokurov). For each $(n,a) \in \mathbb{N} \times [-1,\infty)$, does there exist a number $r(n,a)$ such that the index of an arbitrary $n$-fold canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r(n,a)$?

He raised its weaker variant for canonical singularities.

Question 5.2’. For each $(n,a) \in \mathbb{N} \times [0,\infty)$, does there exist a number $r'(n,a)$ such that the index of an arbitrary $n$-fold canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r'(n,a)$?

The result of Ishii and Fujino gives $r(3,-1) = 66$ for Question 5.2. Theorem 1.1 gives $r'(3,0) = 6$ for Question 5.2’. Further, we provide an affirmative answer to Question 5.2’ for $n = 3$.

Theorem 5.3. Question 5.2’ is true for $n = 3$. More precisely, the minimal discrepancy of a 3-fold canonical singularity is 0, 1/r ($r \in \mathbb{N}$) or 2, and one can take $r'(3,0) = 6$, $r'(3,1/r) = r'$, $r'(3,2) = 1$.

Proof. Let $P \in X$ be a 3-fold canonical singularity with index $r_P$. We shall verify the statement for any such $P$. We take a crepant blow-up $f: Y \to X$ with $Y$ terminal by Corollary 2.6.

Suppose $\dim f^{-1}(P) = 0$, that is, $P$ is terminal. Then it suffices to recall $\text{md}_P X = 1/r_P$ [9], [12] for terminal $P$ except for smooth $P$.

Suppose $\dim f^{-1}(P) = 1$. For any curve $C \subset f^{-1}(P)$, the blow-up of $Y$ with centre $C$ generates a divisor $E$ with $a_E(X) = 1$. Together with the mentioned result [9], [12], we see that $\text{md}_P X$ is the minimum of $1/r_Q$ for all $Q \in f^{-1}(P)$, where $r_Q$ denotes the index of $Y$ at $Q$. Hence $\text{md}_P X = 1/r$ with $r \in \mathbb{N}$ and $r_Q \leq r$ for all $Q \in f^{-1}(P)$. Thus $rK_Y$ is a Cartier divisor near $f^{-1}(P)$, so $r_P | r!$ by [8, Corollary 1.5].
Suppose \( \dim f^{-1}(P) = 2 \). Then \( P \) is a crepant centre, that is, \( \text{md}_P X = 0 \). The statement holds by Theorem 1.1. q.e.d.

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