THE INDEX OF A THREEFOLD CANONICAL SINGULARITY

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ABSTRACT. The index of a 3-fold canonical singularity at a crepant centre is at most 6.

1. INTRODUCTION

Let \( P \in X \) be a log canonical singularity. Shokurov asked if one can bound the index \( r_P \) of \( X \) at \( P \) in terms of the discrepancies of divisors over \( X \).

Suppose that \( X \) has log canonical singularities with \( P \) a log canonical centre. In \( \dim X = 2 \), \( r_P \) is 1, 2, 3, 4 or 6 by the classification of singularities. In an arbitrary dimension, Ishii [5] and Fujino [2] reduced the boundedness of \( r_P \) to a conjectural boundedness of a quotient of the birational automorphism group of a variety \( S \) with \( K_S \sim 0 \). In particular, they proved \( r_P \leq 66 \) in \( \dim X = 3 \).

Suppose that \( X \) has canonical singularities. In \( \dim X = 2 \), \( P \) is a rational double point, so \( r_P = 1 \). The purpose of this paper is to provide an affirmative answer in \( \dim X = 3 \).

**Theorem 1.1.** Let \( P \in X \) be a 3-fold canonical singularity such that \( P \) is a crepant centre. Then the index of \( X \) at \( P \) is at most 6.

**Remark 1.2.** We have such singularities \( P \) with \( r_P = 1, 2, 3, 4 \) in Example 4.3, but I do not know if there exists \( P \) with \( r_P = 5 \) or 6.

Note that no (even implicit) bound of \( r_P \) has been known before. Here a crepant centre means the centre of a divisor with discrepancy zero. The condition that \( P \) is a crepant centre is necessary even for a strictly canonical singularity, see Example 5.1. On the other hand, if once the minimal discrepancy at \( P \) is fixed, then one can bound \( r_P \) for an arbitrary 3-fold canonical singularity \( P \in X \) (Theorem 5.3).

We shall prove Theorem 1.1 by using the singular Riemann–Roch formula (singRR) [14], an orbifold version of Riemann–Roch formula, due to Reid. In Sect. 2, we build a tower \( Y \rightarrow X \) of crepant blow-ups with \( \mathbb{Q} \)-factorial terminal \( Y \), on which the singRR is applicable unconditionally. Then we construct a divisor \( F \) on \( Y \) which possesses the information on the index \( r_P \). The \( r_P \) is determined by the Euler characteristics \( \chi(iK_Y|F) \), which can be explicitly computed by the singRR (Sect. 3). We derive a numerical classification of the singularities on \( Y \) together with \( r_P \) in Sect. 4, by the method [6], [7] in the classification of 3-fold divisorial contractions. The boundedness of indices in terms of minimal discrepancies is discussed in Sect. 5.

We work over an algebraically closed field \( k \) of characteristic zero. A germ \( P \in X \) means an algebraic germ of a variety \( X \) at a closed point \( P \).

2. CREPANT BLOW-UPS

Let \( X \) be a normal \( \mathbb{Q} \)-Gorenstein variety.
Definition 2.1. The index of $X$ at a point $P$ is the smallest positive integer $r$ such that $rK_X$ is a Cartier divisor at $P$.

Consider a normal variety $Y$ with a proper birational morphism $f : Y \to X$. A prime divisor $E$ on any such $Y$ is called a divisor over $X$, and the image $f(E)$ is called the centre of $E$ on $X$ and denoted by $c_X(E)$. The valuation $v_E$ on the function field of $X$ given by such $E$ is called an algebraic valuation of $X$. If we write

$$K_Y = f^*K_X + \sum_E a_E(X)E \quad \text{with} \quad a_E(X) \in \mathbb{Q},$$

then $a_E(X)$ is called the discrepancy of $E$. We say that $X$ has log canonical, log terminal, canonical, terminal singularities if $a_E(X) \geq -1, > -1, \geq 0, > 0$ respectively for all exceptional divisors $E$ over $X$.

The notion of crepant is crucial in this paper.

Definition 2.2. (i) A crepant divisor over $X$ is an exceptional divisor $E$ over $X$ with $a_E(X) = 0$. A crepant valuation of $X$ is the algebraic valuation $v_E$ given by a crepant divisor $E$.

(ii) A crepant centre on $X$ is the centre $c_X(E)$ of a crepant divisor $E$.

(iii) A crepant blow-up $f : Y \to X$ is a projective birational morphism from a normal variety $Y$ such that $K_Y = f^*K_X$.

Remark 2.3. (i) Suppose that $X$ is canonical. Then every crepant valuation is realised as a divisor on any resolution of $X$. In particular, the number of crepant valuations of $X$ is finite. The complement of the union of all crepant centres is the largest terminal open subvariety of $X$.

(ii) If $Y \to X$ is a crepant blow-up, then $X$ is canonical if and only if so is $Y$.

We have a crepant blow-up by the LMMP.

Proposition 2.4. Let $X$ be a variety with canonical singularities and $v$ a crepant valuation of $X$. Then there exists a crepant blow-up $f : Y \to X$ such that

- $Y$ is $\mathbb{Q}$-factorial,
- $f$ has exactly one exceptional divisor $E$, and $v_E = v$,
- $-E$ is $f$-nef.

Proof. Take a projective resolution of singularities $g : Z \to X$, and denote by $E_Z$ the divisor on $Z$ with $v_{E_Z} = v$. Take a Cartier divisor $H > 0$ on $X$ whose support contains all the crepant centres. We write $g^*H = H_Z + F$ with the strict transform $H_Z$ of $H$, and $m$ for the coefficient of $E_Z$ in $F$. Fix $\epsilon > 0$ so that $(Z, \epsilon(H_Z + 2(F - mE_Z)))$ is klt, and run $(K_Z + \epsilon(H_Z + 2(F - mE_Z))$-LMMP over $X$ by [1] to get a log minimal model $f : Y \to X$.

By $K_Z + \epsilon(H_Z + 2(F - mE_Z)) \equiv_X K_Z + \epsilon(F - 2mE_Z)$, the negativity lemma [11, Lemma 2.19] shows that this LMMP contracts exactly all the $g$-exceptional divisors but $E_Z$, and $-E$ is $f$-nef for the strict transform $E$ of $E_Z$. Hence $f$ is a required crepant blow-up.

q.e.d.

Remark 2.5. If $X$ is $\mathbb{Q}$-factorial, then (ii) implies that $\rho(Y/X) = 1$ and $-E$ is $f$-ample.

Corollary 2.6. Let $X = X_0$ be a variety with canonical singularities and $Z$ a crepant centre on $X$. Then there exists a sequence of crepant blow-ups $f_i : X_i \to X_{i-1}$ for $1 \leq t \leq s$ such that
Then the inclusion $f^*O_X(\mathcal{X})$ satisfies $(u)_X + iK_X \geq 0$. If $r_p \nmid i$, then $(u)_X + iK_X$ is not Cartier at $P$, so there exists a divisor $D > 0$ passing through $P$ such that $(u)_X + iK_X - D$ is an effective Cartier divisor. Then $(u)_X + iK_X = f_1^*K_X - f_1^*D \geq 0$. By $f_1^*K_X = K_{X_1}$ and $F_1 \subset \text{Supp } f_1^*D$, we obtain $(u)_X + iK_X - F_1 \geq 0$, implying (1).

For $t > 1$, we set $F_t := \left[ f_t^*F_{t-1} \right]$ inductively. $F_t = f_t^*E_t + c_tE_t$ with some $c_t \in [0,1)$, so $-F_t$ is $f_t$-nef. The (4) is again by Kawamata–Viehweg vanishing theorem. If $c_t = 0$, then (3) is obvious. If $c_t > 0$, then the equality $iK_{X_{t-1}} - F_t = f_t^*(iK_{X_{t-1}} - F_{t-1}) - c_tE_t$ shows that $iK_{X_{t-1}} - F_{t-1}$ is not Cartier at every point in $f_t(E_t)$. Now we get (3) just as in the proof of (1) for $r_p \nmid i$. q.e.d.
3. The singular Riemann–Roch formula

We shall apply the singular Riemann–Roch formula due to Reid to our crepant blow-up, and use the method [6], [7] in the classification of 3-fold divisorial contractions. We briefly recall the formula on a canonical 3-fold.

**Theorem 3.1** ([14, Theorem 10.2]). Let $X$ be a projective 3-fold with canonical singularities and $D$ a divisor on $X$ such that $D \sim i_P K_X$ with $i_P \in \mathbb{Z}$ at each $P \in X$.

(i) There is a formula of the form

$$
\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D - K_X)(2D - K_X) + \frac{1}{12}D \cdot c_2(X) + \sum_p c_p(D),
$$

where the summation takes place over the singularities of $\mathcal{O}_X(D)$, and $c_p(D) \in \mathbb{Q}$ is a contribution due to the singularity at $P$, depending only on the analytic type.

(ii) For a terminal cyclic quotient singularity $P$ of type $\frac{1}{r_P}(1, -1, b_P)$,

$$
c_p(D) = -\frac{i_P r_P^2 - 1}{12r_P} + \sum_{j=1}^{\lfloor i_P r_P \rfloor} \frac{j b_P(r_P - j b_P)}{2r_P},
$$

where $i = i - \lfloor \frac{i}{r_P} \rfloor r_P$ denotes the residue of $i$ modulo $r_P$.

(iii) For an arbitrary terminal singularity $P$,

$$
c_p(D) = \sum_Q c_Q(D_Q),
$$

where $\{(Q, D_Q)\}_Q$ is a flat deformation of $(P, D)$ to the basket of terminal cyclic quotient singularities $Q$. Such $Q$ is called a fictitious singularity.

**Remark 3.2.** The condition $D \sim i_P K_X$ always holds if $X$ is $\mathbb{Q}$-factorial and terminal [8, Corollary 5.2].

Our object is a germ of a crepant blow-up $f : Y \to X$ with a divisor $F$ on $Y$ in Theorem 2.7 at a 3-fold canonical singularity $P \in X$ with index $r_P$. Shrinking and compactifying it, we may assume that $Y$ is projective and terminal ($f$ is merely a projective morphism outside a neighbourhood of $P$). We shall express the function $\delta_P(i)$ below.

**Definition 3.3.** We define the function $\delta_P(i)$ on $\mathbb{Z}$ as

$$
\delta_P(i) := \begin{cases} 
1 & \text{if } r_P | i, \\
0 & \text{otherwise.}
\end{cases}
$$

Applying (ii) in Theorem 2.7 and the vanishing $R^j f_* \mathcal{O}_Y(iK_Y) = 0$ for $j \geq 1$ to the exact sequence

$$
0 \to \mathcal{O}_Y(iK_Y - F) \to \mathcal{O}_Y(iK_Y) \to \mathcal{O}_F(iK_Y|_F) \to 0,
$$

we obtain

$$
\delta_P(i) = \dim_k f_* \mathcal{O}_Y(iK_Y)/f_* \mathcal{O}_Y(iK_Y - F)
= h^0(\mathcal{O}_F(iK_Y|_F))
= \chi(\mathcal{O}_F(iK_Y|_F))
= \chi(\mathcal{O}_Y(iK_Y)) - \chi(\mathcal{O}_Y(iK_Y - F)).
$$
Let $I_0 := \{ Q \text{ with type } \frac{1}{r_Q}(1,-1,b_Q) \}$ be the basket of fictitious singularities from singularities on $Y$. Note that $b_Q$ is co-prime to $r_Q$. For $Q \in I_0$, let $f_Q$ denote the smallest non-negative integer such that $F \sim f_QK_Y$ at $Q$. By replacing $b_Q$ with $r_Q - b_Q$ if necessary, we may assume $v_Q := f_Qb_Q \leq r_Q/2$. Set $I := \{ Q \in I_0 \mid f_Q \neq 0 \}$.

With this notation, the singular Riemann–Roch formula computes the right-hand side of (5), to provide

$$
\delta_P(i) = \frac{1}{6} F^3 + \frac{1}{12} F \cdot c_2(Y) + \sum_{Q \in I} (A_Q(i) - A_Q(i - f_Q)),
$$

where the contribution $A_Q(i)$ is given by

$$
A_Q(i) := -\frac{r_Q^2 - 1}{12r_Q} + \sum_{j=1}^{r_Q-1} \frac{12b_Q(r_Q - jb_Q)}{2r_Q}.
$$

The $A_Q(i)$ satisfies the formula

$$
A_Q(i+1) - A_Q(i) = -\frac{r_Q^2 - 1}{12r_Q} + B_Q(ib_Q)
$$

with

$$
B_Q(i) := \frac{7(r_Q - i)}{2r_Q}.
$$

Therefore by (6), we have

$$
\delta_P(i+1) - \delta_P(i) = \sum_{Q \in I} (B_Q(ib_Q) - B_Q(ib_Q - v_Q)).
$$

**Lemma 3.4.** The $r_P$ equals the l.c.m. of $r_Q$ for all $Q \in I$.

*Proof.* Since $r_PK_Y = r_Pf^*K_X$ is a Cartier divisor near $f^{-1}(P)$, $r_Q$ divides $r_P$ for all $Q \in I$. On the other hand, we see that $r_P$ divides the l.c.m. of $r_Q$ by (7) and the periodic properties of $\delta_P, B_Q$.

4. **BOUNDEDNESS OF INDICES**

We shall prove Theorem 1.1 in this section. Let $r_P$ denote the index of $X$ at $P$. We take a crepant blow-up $f : Y \to X$ with a divisor $F$ on $Y$ in Theorem 2.7. We restrict the possibilities of $J := \{ (r_Q, v_Q) \}_{Q \in I}$ using (7) for $i = 0$.

**Lemma 4.1.** $J$ is one of the types in Table 1.

<table>
<thead>
<tr>
<th>type</th>
<th>$J$</th>
<th>$r_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2, 1), (2, 1), (2, 1), (2, 1)</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>(2, 1), (2, 1), (4, 2)</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>(2, 1), (3, 1), (6, 1)</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>(2, 1), (4, 1), (4, 1)</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>(3, 1), (3, 1), (3, 1)</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>(4, 2), (4, 2)</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>(2, 1), (6, 3)</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>type</th>
<th>$J$</th>
<th>$r_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2, 1), (8, 2)</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>(3, 1), (6, 2)</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>(5, 1), (5, 2)</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>(8, 4)</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>(9, 3)</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>Ø</td>
<td>1</td>
</tr>
</tbody>
</table>
Proof. By Lemma 3.4, $r_p$ is determined by $J$, and $r_p = 1$ if and only if $J = \emptyset$. We assume $r_p > 1$ from now on. Then (7) for $i = 0$ is written as

\[(8) \sum_{Q \in I} B_Q(v_Q) = 1.\]

By the definition of $B_Q$ and $r_Q \geq 2v_Q$, we have

\[(9) v_Q/4 \leq B_Q(v_Q) < v_Q/2.\]

Then $J' := \{v_Q\}_{Q \in I}$, which satisfies (8) and (9), should be one of

\[
\{1, 1, 1, 1, \}, \{1, 1, 1, 2, \}, \{1, 1, 1, 1, 2, \}, \{1, 3, \}, \{1, 2, \}, \{3, \}, \{4, \}.
\]

For each of these candidates for $J'$, one can solve the equation (8) for $r_Q (\geq 2v_Q)$ explicitly. Every solution is in Table 1. For example, suppose $J' = \{1, 2, \}$. We set $J = \{(r_1, 1), (r_2, 2)\}$. Then (8) becomes $1/r_1 + 4/r_2 = 1$. Thus $(r_1, r_2) = (2, 8), (3, 6)$ or $(5, 5)$, so $J$ is of type 8, 9, 10 respectively.

q.e.d.

By Lemma 4.1, we have $r_p \leq 9$, and for Theorem 1.1 it is enough to exclude types 8, 11, 12. However, we derive a finer numerical classification by determining $\tilde{J} := \{(r_Q, v_Q, b_Q)\}_{Q \in I}$.

**Theorem 4.2.** $\tilde{J}$ is one of the types in Table 2.

<table>
<thead>
<tr>
<th>type</th>
<th>$J$</th>
<th>$r_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${2, 1, 1}, {2, 1, 1}, {2, 1, 1}, {2, 1, 1}$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>${2, 1, 1}, {3, 1, 2}, {6, 1, 5}$</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>${2, 1, 1}, {4, 1, 3}, {4, 1, 3}$</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>${3, 1, 2}, {3, 1, 2}, {3, 1, 2}$</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>${5, 2, 3}$</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>$\emptyset$</td>
<td>1</td>
</tr>
</tbody>
</table>

Proof. By Lemma 4.1, there exist only finitely many candidates for $\tilde{J}$. For each candidate, one can compute the right-hand side of (7) explicitly. It must coincide with $\delta_p(i+1) - \delta_p(i)$, but such a coincidence happens only if $\tilde{J}$ is one of the types in Table 2.

Here we demonstrate for type 3. $\tilde{J} = \{(2, 1, 1), (3, 1, b_2), (6, 1, b_3)\}$ with $b_2 = 1$ or 2 and $b_3 = 1$ or 5. The (7) for $i = 1$ is $\delta_p(2) - \delta_p(1) = 1.1/3, 2/3, 0$ when $(b_2, b_3) = (1, 1), (1, 5), (2, 1), (2, 5)$ respectively. Thus $(b_2, b_3)$ must be $(2, 5)$, and in this case (7) surely holds for any $i$. q.e.d.

**Example 4.3.** In simple cases, Theorem 1.1 is known by the classification.

(i) (Morrison [13], Ishida–Iwashita [4]) If $P$ is a cyclic quotient singularity, then $r_p = 1$ except $\delta_p(1, n+1, -2)$ $(n \geq 2)$, $\delta_p(1, 9, 11)$, $\delta_p(1, 4, 7)$, with $r_p = 2, 2, 3$ respectively.

(ii) (Hayakawa–Takeuchi [3]) If $P$ is an isolated singularity which is a cyclic quotient of a hypersurface singularity, then $r_p \leq 4$. The only case when $r_p = 4$ is $o \in (x_1x_2 + x_3^2 + x_4^2 = 0) \subset \mathbb{A}^4_{x_1x_2x_3x_4}/\mathbb{Z}_3(1, 5, 3, 7)$. 

5. Minimal discrepancies

To begin with, we provide an example which explains the need of $P$ being a crepant centre in Theorem 1.1 even for a strictly canonical singularity. A similar example exists also for a 3-fold strictly log canonical singularity [2, Example 6.1].

**Example 5.1.** Let $r \in \mathbb{N}$. Let $P \in X$ be the germ

$$o \in (x_1x_2 + x_3^2 = 0) \subset \mathbb{A}^4_{x_1,x_2,x_3} / \mathbb{Z}_r(1, -1, 0, 1),$$

which is singular along the $x_4$-axis $C$. Let $f : Y \to X$ be the weighted blow-up with weights $w(t(x_1, x_2, x_3, x_4)) = \frac{1}{r}(1, r - 1, r, 1)$. Then $K_Y = f^*K_X + \frac{1}{r}E$ with the exceptional divisor $E$, and $Y$ has 2 terminal quotient singularities of types $\frac{1}{r}(1, -1, 1)$ and $\frac{1}{r}(1, 1, 1)$ outside the strict transform $C_Y$ of $C$. Let $g : Z \to Y$ be the blow-up with centre $C_Y$. The $g$ is a crepant blow-up and $Z$ is smooth near $g^{-1}(C_Y)$. Hence $X$ has canonical singularities with a crepant centre $C$, but $P$ is not a crepant centre.

The index of $X$ at $P$ is $r$.

We focus on the minimal discrepancy to grasp this phenomenon. For a normal $\mathbb{Q}$-Gorenstein singularity $P \in X$, the minimal discrepancy $\text{md}_P X$ of $X$ at $P$ is the infimum of discrepancies $a_E(X)$ for all divisors $E$ over $X$ with $c_X(E) = P$. Note that $\text{md}_P X \in (-\infty) \cup [-1, \infty)$, and $P \in X$ is log canonical if and only if $\text{md}_P X \geq -1$.

In Example 5.1, we have $\text{md}_P X = 1/r$. Shokurov formulated a question on the boundedness of indices in terms of minimal discrepancies.

**Question 5.2 (Shokurov).** For each $(n, a) \in \mathbb{N} \times [-1, \infty)$, does there exist a number $r(n,a)$ such that the index of an arbitrary $n$-fold canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r(n,a)$?

He raised its weaker variant for canonical singularities.

**Question 5.2’.** For each $(n, a) \in \mathbb{N} \times [0, \infty)$, does there exist a number $r'(n,a)$ such that the index of an arbitrary $n$-fold canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r'(n,a)$?

The result of Ishii and Fujino gives $r(3, -1) = 66$ for Question 5.2. Theorem 1.1 gives $r'(3, 0) = 6$ for Question 5.2’. Further, we provide an affirmative answer to Question 5.2’ for $n = 3$.

**Theorem 5.3.** Question 5.2’ is true for $n = 3$. More precisely, the minimal discrepancy of a 3-fold canonical singularity is 0, $1/r$ ($r \in \mathbb{N}$) or 2, and one can take

$$r'(3, 0) = 6, \quad r'(3, 1/r) = r!, \quad r'(3, 2) = 1.$$

**Proof.** Let $P \in X$ be a 3-fold canonical singularity with index $r_P$. We shall verify the statement for any such $P$. We take a crepant blow-up $f : Y \to X$ with $Y$ terminal by Corollary 2.6.

Suppose $\dim f^{-1}(P) = 0$, that is, $P$ is terminal. Then it suffices to recall $\text{md}_P X = 1/r_P$ [9], [12] for terminal $P$ except for smooth $P$.

Suppose $\dim f^{-1}(P) = 1$. For any curve $C \subset f^{-1}(P)$, the blow-up of $Y$ with centre $C$ generates a divisor $E$ with $a_E(X) = 1$. Together with the mentioned result [9], [12], we see that $\text{md}_P X$ is the minimum of $1/r_Q$ for all $Q \in f^{-1}(P)$, where $r_Q$ denotes the index of $Y$ at $Q$. Hence $\text{md}_P X = 1/r$ with $r \in \mathbb{N}$ and $r_Q \leq r$ for all $Q \in f^{-1}(P)$. Thus $r!K_Y$ is a Cartier divisor near $f^{-1}(P)$, so $r_P | r!$ by [8, Corollary 1.5].
Suppose \( \dim f^{-1}(P) = 2 \). Then \( P \) is a crepant centre, that is, \( md_P X = 0 \). The statement holds by Theorem 1.1. q.e.d.

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