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THE INDEX OF A THREEFOLD CANONICAL SINGULARITY

MASAYUKI KAWAKITA

ABSTRACT. The index of a 3-fold canonical singularity at a crepant centre is at most 6.

1. INTRODUCTION

Let \( P \in X \) be a log canonical singularity. Shokurov asked if one can bound the index \( r_P \) of \( X \) at \( P \) in terms of the discrepancies of divisors over \( X \).

Suppose that \( X \) has log canonical singularities with \( P \) a log canonical centre. In \( \dim X = 2 \), \( r_P \) is 1, 2, 3, 4 or 6 by the classification of singularities. In an arbitrary dimension, Ishii [5] and Fujino [2] reduced the boundedness of \( r_P \) to a conjectural boundedness of a quotient of the birational automorphism group of a variety \( S \) with \( K_S \sim 0 \). In particular, they proved \( r_P \leq 66 \) in \( \dim X = 3 \).

Suppose that \( X \) has canonical singularities. In \( \dim X = 2 \), \( P \) is a rational double point, so \( r_P = 1 \). The purpose of this paper is to provide an affirmative answer in \( \dim X = 3 \).

Theorem 1.1. Let \( P \in X \) be a 3-fold canonical singularity such that \( P \) is a crepant centre. Then the index of \( X \) at \( P \) is at most 6.

Remark 1.2. We have such singularities \( P \) with \( r_P = 1, 2, 3, 4 \) in Example 4.3, but I do not know if there exists \( P \) with \( r_P = 5 \) or 6.

Note that no (even implicit) bound of \( r_P \) has been known before. Here a crepant centre means the centre of a divisor with discrepancy zero. The condition that \( P \) is a crepant centre is necessary even for a strictly canonical singularity, see Example 5.1. On the other hand, if once the minimal discrepancy at \( P \) is fixed, then one can bound \( r_P \) for an arbitrary 3-fold canonical singularity \( P \in X \) (Theorem 5.3).

We shall prove Theorem 1.1 by using the singular Riemann–Roch formula (singRR) [14], an orbifold version of Riemann–Roch formula, due to Reid. In Sect. 2, we build a tower \( Y \to X \) of crepant blow-ups with \( \mathbb{Q} \)-factorial terminal \( Y \), on which the singRR is applicable unconditionally. Then we construct a divisor \( F \) on \( Y \) which possesses the information on the index \( r_P \). The \( r_P \) is determined by the Euler characteristics \( \chi(iK_Y|F) \), which can be explicitly computed by the singRR (Sect. 3). We derive a numerical classification of the singularities on \( Y \) together with \( r_P \) in Sect. 4, by the method [6], [7] in the classification of 3-fold divisorial contractions. The boundedness of indices in terms of minimal discrepancies is discussed in Sect. 5.

We work over an algebraically closed field \( k \) of characteristic zero. A germ \( P \in X \) means an algebraic germ of a variety \( X \) at a closed point \( P \).

2. CREPANT BLOW-UPS

Let \( X \) be a normal \( \mathbb{Q} \)-Gorenstein variety.
Definition 2.1. The index of $X$ at a point $P$ is the smallest positive integer $r$ such that $rK_X$ is a Cartier divisor at $P$.

Consider a normal variety $Y$ with a proper birational morphism $f: Y \to X$. A prime divisor $E$ on any such $Y$ is called a divisor over $X$, and the image $f(E)$ is called the centre of $E$ on $X$ and denoted by $c_X(E)$. The valuation $v_E$ on the function field of $X$ given by such $E$ is called an algebraic valuation of $X$. If we write

$$K_Y = f^*K_X + \sum_E a_E(X)E \quad \text{with} \quad a_E(X) \in \mathbb{Q},$$

then $a_E(X)$ is called the discrepancy of $E$. We say that $X$ has log canonical, log terminal, canonical, terminal singularities if $a_E(X) \geq -1, > -1, \geq 0, > 0$ respectively for all exceptional divisors $E$ over $X$.

The notion of crepant is crucial in this paper.

Definition 2.2. (i) A crepant divisor over $X$ is an exceptional divisor $E$ over $X$ with $a_E(X) = 0$. A crepant valuation of $X$ is the algebraic valuation $v_E$ given by a crepant divisor $E$.

(ii) A crepant centre on $X$ is the centre $c_X(E)$ of a crepant divisor $E$.

(iii) A crepant blow-up $f: Y \to X$ is a projective birational morphism from a normal variety $Y$ such that $K_Y = f^*K_X$.

Remark 2.3. (i) Suppose that $X$ is canonical. Then every crepant valuation is realised as a divisor on any resolution of $X$. In particular, the number of crepant valuations of $X$ is finite. The complement of the union of all crepant centres is the largest terminal open subvariety of $X$.

(ii) If $Y \to X$ is a crepant blow-up, then $X$ is canonical if and only if so is $Y$.

We have a crepant blow-up by the LMMP.

Proposition 2.4. Let $X$ be a variety with canonical singularities and $v$ a crepant valuation of $X$. Then there exists a crepant blow-up $f: Y \to X$ such that

(i) $Y$ is $\mathbb{Q}$-factorial,
(ii) $f$ has exactly one exceptional divisor $E$, and $v_E = v$,
(iii) $-E$ is $f$-nef.

Proof. Take a projective resolution of singularities $g: Z \to X$, and denote by $E_Z$ the divisor on $Z$ with $v_{E_Z} = v$. Take a Cartier divisor $H > 0$ on $X$ whose support contains all the crepant centres. We write $g^*H = H_Z + F$ with the strict transform $H_Z$ of $H$, and $m$ for the coefficient of $E_Z$ in $F$. Fix $e > 0$ so that $(Z, e(H_Z + 2(F - mE_Z)))$ is klt, and run $(K_Z + e(H_Z + 2(F - mE_Z)))-\text{LMMP}$ over $X$ by [1] to get a log minimal model $f: Y \to X$.

By $K_Z + e(H_Z + 2(F - mE_Z)) \equiv_X K_Z + e(F - 2mE_Z)$, the negativity lemma [11, Lemma 2.19] shows that this LMMP contracts exactly all the $g$-exceptional divisors but $E_Z$, and $-E$ is $f$-nef for the strict transform $E$ of $E_Z$. Hence $f$ is a required crepant blow-up.

q.e.d.

Remark 2.5. If $X$ is $\mathbb{Q}$-factorial, then (ii) implies that $\rho(Y/X) = 1$ and $-E$ is $f$-ample.

Corollary 2.6. Let $X = X_0$ be a variety with canonical singularities and $Z$ a crepant centre on $X$. Then there exists a sequence of crepant blow-ups $f_t: X_t \to X_{t-1}$ for $1 \leq t \leq s$ such that
Then the inclusion $u$ and for $(\text{an effective Cartier divisor. Then})$
\begin{align*}
P & \text{at } P \\
& \text{Theorem 2.7. Let } P \in X \text{ be a canonical singularity such that } P \text{ is a crepant centre. Let } r_p \text{ denote the index of } X \text{ at } P \text{ and } m_p \text{ the maximal ideal sheaf for } P. \text{ Then there exist a crepant blow-up } f : Y \rightarrow X \text{ and an effective divisor } F \text{ on } Y \text{ supported in } f^{-1}(P) \text{ such that} \\
& \text{(i) } Y \text{ is } \mathbb{Q}\text{-factorial and terminal,} \\
& \text{(ii) for } i \in \mathbb{Z}, \\
& \quad f_* \mathcal{O}_Y(iK_Y - F) = \begin{cases} \\
m_p \mathcal{O}_X(iK_X) & \text{if } r_p \mid i, \\
\mathcal{O}_X(iK_X) & \text{otherwise}, \\
\end{cases} \\
& \quad R^jf_* \mathcal{O}_Y(iK_Y - F) = 0 \quad \text{for } j \geq 1.
\end{align*}

\textbf{Proof.} We take a sequence of crepant blow-ups $f_i$ in Corollary 2.6 with $Z = P$, and set $Y := X$. We will construct inductively divisors $F_i \geq 0$ on $X$ such that

\begin{align*}
& f_{*i} \mathcal{O}_{X_i}(iK_{X_i} - F_i) = \begin{cases} \\
m_p \mathcal{O}_X(iK_X) & \text{if } r_p \mid i, \\
\mathcal{O}_X(iK_X) & \text{otherwise}, \\
\end{cases} \\
& R^jf_{*i} \mathcal{O}_{X_i}(iK_{X_i} - F_i) = 0 \quad \text{for } j \geq 1,
\end{align*}

and for $t > 1$,

\begin{align*}
& f_{*}\mathcal{O}_{X_t}(iK_{X_t} - F_t) = \mathcal{O}_{X_{t-1}}(iK_{X_{t-1}} - F_{t-1}), \\
& R^jf_{*}\mathcal{O}_{X_t}(iK_{X_t} - F_t) = 0 \quad \text{for } j \geq 1.
\end{align*}

Then Leray’s spectral sequence induces that $F := F_i$ is a required divisor.

We set $F_1 := E_1$. The vanishing (2) follows from Kawamata–Viehweg vanishing theorem [10, Theorem 1.2.5, Remark 1.2.6]. If $r_p \mid i$, then (1) is by the projection formula. To see (1) for $r_p \nmid i$, we regard $K_X$ as a fixed divisor (not a divisor class), and so $K_X = f_1^*K_X$. Denote by $\mathcal{N}_X$ the constant sheaf of the function field of $X$. Then the inclusion $f_{*i} \mathcal{O}_{X_i}(iK_{X_i} - F_i) \subset \mathcal{O}_X(iK_X)$ is interpreted by the expressions

\begin{align*}
f_{*i} \mathcal{O}_{X_t}(iK_{X_t} - F_t) & = \{ u \in \mathcal{N}_X \mid (u)_X + if^*_1K_X - F_1 \geq 0 \}, \\
\mathcal{O}_X(iK_X) & = \{ u \in \mathcal{N}_X \mid (u)_X + iK_X \geq 0 \}.
\end{align*}

Suppose $u \in \mathcal{N}_X$ satisfies $(u)_X + iK_X \geq 0$. If $r_p \nmid i$, then $(u)_X + iK_X$ is not Cartier at $P$, so there exists a divisor $D > 0$ passing through $P$ such that $(u)_X + iK_X - D$ is an effective Cartier divisor. Then $(u)_X + if^*_1K_X - f_1^*D \geq 0$. By $f_1^*K_X = K_X$ and $\text{Supp } f_1^*D$, we obtain $(u)_X + if^*_1K_X - F_1 \geq 0$, implying (1).

For $t > 1$, we set $F_t := [f_t^*F_{t-1} - c_tE_t]$ inductively. $F_t = f_t^*F_{t-1} + c_tE_t$ with some $c_t \in [0, 1)$, so $-F_t$ is $f_t$-nef. The (4) is again by Kawamata–Viehweg vanishing theorem. If $c_t = 0$, then (3) is obvious. If $c_t > 0$, then the inequality $iK_{X_{t-1}} - F_t = f_t^*(iK_{X_{t-1}} - F_{t-1}) - c_tE_t$ shows that $iK_{X_{t-1}} - F_{t-1}$ is not Cartier at every point in $f_t(E_t)$. Now we get (3) just as in the proof of (1) for $r_p \nmid i$. q.e.d.
3. THE SINGULAR RIEMANN–ROCH FORMULA

We shall apply the singular Riemann–Roch formula due to Reid to our crepant blow-up, and use the method [6], [7] in the classification of 3-fold divisorial contractions. We briefly recall the formula on a canonical 3-fold.

Theorem 3.1 ([14, Theorem 10.2]). Let $X$ be a projective 3-fold with canonical singularities and $D$ a divisor on $X$ such that $D \sim i_P K_X$ with $i_P \in \mathbb{Z}$ at each $P \in X$.

(i) There is a formula of the form

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12} D(D - K_X)(2D - K_X) + \frac{1}{12} D \cdot c_2(X) + \sum_P c_P(D),$$

where the summation takes place over the singularities of $\mathcal{O}_X(D)$, and $c_P(D) \in \mathbb{Q}$ is a contribution due to the singularity at $P$, depending only on the analytic type.

(ii) For a terminal cyclic quotient singularity $P$ of type $\frac{1}{r_P}(1, -1, b_P)$,

$$c_P(D) = -\frac{i_P r_P^2}{12r_P} + \frac{i_{r_P} - 1}{2r_P} \sum_{j=1}^{i_{r_P} - 1} \frac{\tilde{t} - j}{r_P} (r_P - jb_P),$$

where $\tilde{t} = i - \frac{i}{r_P} r_P$ denotes the residue of $i$ modulo $r_P$.

(iii) For an arbitrary terminal singularity $P$,

$$c_P(D) = \sum_Q c_Q(D_Q),$$

where $\{(Q, D_Q)\}_Q$ is a flat deformation of $(P, D)$ to the basket of terminal cyclic quotient singularities $Q$. Such $Q$ is called a fictitious singularity.

Remark 3.2. The condition $D \sim i_P K_X$ always holds if $X$ is $\mathbb{Q}$-factorial and terminal [8, Corollary 5.2].

Our object is a germ of a crepant blow-up $f: Y \to X$ with a divisor $F$ on $Y$ in Theorem 2.7 at a 3-fold canonical singularity $P \in X$ with index $r_P$. Shrinking and compactifying it, we may assume that $Y$ is projective and terminal (if $f$ is merely a projective morphism outside a neighbourhood of $P$). We shall express the function $\delta_p(i)$ below.

Definition 3.3. We define the function $\delta_p(i)$ on $\mathbb{Z}$ as

$$\delta_p(i) := \begin{cases} 1 & \text{if } r_P | i, \\ 0 & \text{otherwise.} \end{cases}$$

Applying (ii) in Theorem 2.7 and the vanishing $R^j f_* \mathcal{O}_Y(iK_Y) = 0$ for $j \geq 1$ to the exact sequence

$$0 \to \mathcal{O}_Y(iK_Y - F) \to \mathcal{O}_Y(iK_Y) \to \mathcal{O}_F(iK_Y|_F) \to 0,$$

we obtain

$$\delta_p(i) = \dim_k f_* \mathcal{O}_Y(iK_Y)/f_* \mathcal{O}_Y(iK_Y - F)\quad (5)$$

$$= h^0(\mathcal{O}_F(iK_Y|_F))$$

$$= \chi(\mathcal{O}_F(iK_Y|_F))$$

$$= \chi(\mathcal{O}_Y(iK_Y)) - \chi(\mathcal{O}_Y(iK_Y - F)).$$
Let $I_0 := \{Q$ with type $\frac{1}{r_0^*}(1,-1,b_Q)\}$ be the basket of fictitious singularities from singularities on $Y$. Note that $b_Q$ is co-prime to $r_Q$. For $Q \in I_0$, let $f_Q$ denote the smallest non-negative integer such that $F \sim f_Q K_Y$ at $Q$. By replacing $b_Q$ with $r_Q - b_Q$ if necessary, we may assume $v_Q := f_Q b_Q \leq r_Q / 2$. Set $I := \{Q \in I_0 \mid f_Q \neq 0\}$.

With this notation, the singular Riemann–Roch formula computes the right-hand side of (5), to provide

$$\delta_p(i) = \frac{1}{6} F^3 + \frac{1}{12} F \cdot c_2(Y) + \sum_{Q \in I} (A_Q(i) - A_Q(i - f_Q)),$$

where the contribution $A_Q(i)$ is given by

$$A_Q(i) := -\frac{1}{12 r_Q} + \sum_{j=1}^{7} \frac{j b_Q (r_Q - j b_Q)}{2 r_Q}.$$

The $A_Q(i)$ satisfies the formula

$$A_Q(i + 1) - A_Q(i) = -\frac{1}{12 r_Q} + B_Q(i b_Q)$$

with

$$B_Q(i) := \frac{7 (r_Q - i)}{2 r_Q}.$$

Therefore by (6), we have

$$\delta_p(i + 1) - \delta_p(i) = \sum_{Q \in I} (B_Q(i b_Q) - B_Q(i b_Q - v_Q)).$$

**Lemma 3.4.** The $r_p$ equals the l.c.m. of $r_Q$ for all $Q \in I$.

**Proof.** Since $r_p K_Y = r_p f^* K_X$ is a Cartier divisor near $f^{-1}(P)$, $r_Q$ divides $r_p$ for all $Q \in I$. On the other hand, we see that $r_p$ divides the l.c.m. of $r_Q$ by (7) and the periodic properties of $\delta_p, B_Q$. q.e.d.

### 4. Boundedness of Indices

We shall prove Theorem 1.1 in this section. Let $r_p$ denote the index of $X$ at $P$. We take a crepant blow-up $f : Y \rightarrow X$ with a divisor $F$ on $Y$ in Theorem 2.7. We restrict the possibilities of $J := \{(r_Q, v_Q)\}_{Q \in I}$ using (7) for $i = 0$.

**Lemma 4.1.** $J$ is one of the types in Table 1.

<table>
<thead>
<tr>
<th>type</th>
<th>$J$</th>
<th>$r_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(2, 1), (2, 1), (2, 1)$, $(2, 1)$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$(2, 1), (2, 1), (4, 2)$</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>$(2, 1), (3, 1), (6, 1)$</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>$(2, 1), (4, 1), (4, 1)$</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>$(3, 1), (3, 1), (3, 1)$</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>$(4, 2), (4, 2)$</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>$(2, 1), (6, 3)$</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>type</th>
<th>$J$</th>
<th>$r_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$(2, 1), (8, 2)$</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>$(3, 1), (6, 2)$</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>$(5, 1), (5, 2)$</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>$(8, 4)$</td>
<td>8</td>
</tr>
<tr>
<td>12</td>
<td>$(9, 3)$</td>
<td>9</td>
</tr>
<tr>
<td>13</td>
<td>$\emptyset$</td>
<td>1</td>
</tr>
</tbody>
</table>
For each of these candidates for $J$, and $r_p = 1$ if and only if $J = \emptyset$. We assume $r_p > 1$ from now on. Then (7) for $i = 0$ is written as

$$\sum_{Q \in I} B_Q(v_Q) = 1.$$  

By the definition of $B_Q$ and $r_Q \geq 2v_Q$, we have

$$v_Q/4 \leq B_Q(v_Q) < v_Q/2.$$  

Then $J' := \{v_Q\}_{Q \in I}$, which satisfies (8) and (9), should be one of

$$\{1,1,1,1\}, \{1,1,2\}, \{1,1,1\}, \{2,2\}, \{1,3\}, \{1,2\}, \{3\}, \{4\}.$$  

For each of these candidates for $J'$, one can solve the equation (8) for $r_Q \geq 2v_Q$ explicitly. Every solution is in Table 1. For example, suppose $J' = \{1,2\}$. We set $J = \{(r_1,1),(r_2,2)\}$. Then (8) becomes $1/r_1 + 4/r_2 = 1$. Thus $(r_1,r_2) = (2,8)$, $(3,6)$ or $(5,5)$, so $J$ is of type 8, 9, 10 respectively.

By Lemma 4.1, we have $r_p \leq 9$, and for Theorem 1.1 it is enough to exclude types 8, 11, 12. However, we derive a finer numerical classification by determining $J' := \{(r_Q,v_Q,b_Q)\}_{Q \in I}$.  

**Theorem 4.2.** $J$ is one of the types in Table 2.

<table>
<thead>
<tr>
<th>type</th>
<th>$J$</th>
<th>$r_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(2,1,1),(2,1,1),(2,1,1),(2,1,1)$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$(2,1,1),(3,1,2),(6,1,5)$</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>$(2,1,1),(4,1,3),(4,1,3)$</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>$(3,1,2),(3,1,2),(3,1,2)$</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>$(5,1,4),(5,2,3)$</td>
<td>5</td>
</tr>
<tr>
<td>13</td>
<td>$\emptyset$</td>
<td>1</td>
</tr>
</tbody>
</table>

*Proof.** By Lemma 4.1, there exist only finitely many candidates for $J$. For each candidate, one can compute the right-hand side of (7) explicitly. It must coincide with $\delta_p(i+1) - \delta_p(i)$, but such a coincidence happens only if $J$ is one of the types in Table 2.

Here we demonstrate for type 3. $J = \{(2,1,1),(3,1,b_2),(6,1,b_3)\}$ with $b_2 = 1$ or 2 and $b_3 = 1$ or 5. The (7) for $i = 1$ is $\delta_p(2) - \delta_p(1) = 1/1, 1/3, 2/3, 0$ when $(b_2,b_3) = (1,1),(1,5),(2,1),(2,5)$ respectively. Thus $(b_2,b_3)$ must be $(2,5)$, and in this case (7) surely holds for any $i$. q.e.d.

**Example 4.3.** In simple cases, Theorem 1.1 is known by the classification.

(i) (Morrison [13], Ishida–Iwashita [4]) If $P$ is a cyclic quotient singularity, then $r_p = 1$ except $\frac{1}{2n}(1,2n+1,-2)$ $(n \geq 2)$, $\frac{1}{14}(1,9,11)$, $\frac{1}{3}(1,4,7)$, with $r_p = 2,2,3$ respectively.

(ii) (Hayakawa–Takeuchi [3]) If $P$ is an isolated singularity which is a cyclic quotient of a hypersurface singularity, then $r_p \leq 4$. The only case when $r_p = 4$ is $o \in (x_1x_2 + x_3^2 + x_4^2 = 0) \subset \mathbb{A}^4_{x_1x_2,x_3,x_4}/\mathbb{Z}_8(1,5,3,7)$. 


5. Minimal discrepancies

To begin with, we provide an example which explains the need of $P$ being a crepant centre in Theorem 1.1 even for a strictly canonical singularity. A similar example exists also for a 3-fold strictly log canonical singularity [2, Example 6.1].

Example 5.1. Let $r \in \mathbb{N}$. Let $P \in X$ be the germ
\[ o \in (x_1x_2 + x_3^2 = 0) \subset \mathbb{A}^4_{x_1,x_2,x_3}/\mathbb{Z}_r(1,-1,0,1), \]
which is singular along the $x_3$-axis $C$. Let $f : Y \to X$ be the weighted blow-up with weights $\text{wt}(x_1,x_2,x_3,x_4) = \frac{1}{r}(1,r-1,r,1)$. Then $K_Y = f^*K_X + \frac{1}{r}E$ with the exceptional divisor $E$, and $Y$ has 2 terminal quotient singularities of types $\frac{1}{r}(1,-1,1)$ and $\frac{1}{r}(1,-1,1)$ outside the strict transform $C_Y$ of $C$. Let $g : Z \to Y$ be the blow-up with centre $C_Y$. The $g$ is a crepant blow-up and $Z$ is smooth near $g^{-1}(C_Y)$. Hence $X$ has canonical singularities with a crepant centre $C$, but $P$ is not a crepant centre. The index of $X$ at $P$ is $r$.

We focus on the minimal discrepancy to grasp this phenomenon. For a normal $\mathbb{Q}$-Gorenstein singularity $P \in X$, the minimal discrepancy $\text{md}_P X$ at $P$ is the infimum of discrepancies $a_E(X)$ for all divisors $E$ over $X$ with $c_X(E) = P$. Note that $\text{md}_P X \in \{-\infty\} \cup [-1,\infty)$, and $P \in X$ is log canonical if and only if $\text{md}_P X \geq -1$.

In Example 5.1, we have $\text{md}_P X = 1/r$. Shokurov formulated a question on the boundedness of indices in terms of minimal discrepancies.

Question 5.2 (Shokurov). For each $(n,a) \in \mathbb{N} \times [-1,\infty)$, does there exist a number $r(n,a)$ such that the index of an arbitrary $n$-fold log canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r(n,a)$?

He raised its weaker variant for canonical singularities.

Question 5.2'. For each $(n,a) \in \mathbb{N} \times [0,\infty)$, does there exist a number $r'(n,a)$ such that the index of an arbitrary $n$-fold canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r'(n,a)$?

The result of Ishii and Fujino gives $r(3,-1) = 66$ for Question 5.2, Theorem 1.1 gives $r'(3,0) = 6$ for Question 5.2'. Further, we provide an affirmative answer to Question 5.2’ for $n = 3$.

Theorem 5.3. Question 5.2’ is true for $n = 3$. More precisely, the minimal discrepancy of a 3-fold canonical singularity is $0$, $1/r$ ($r \in \mathbb{N}$) or $2$, and one can take
\[ r'(3,0) = 6, \quad r'(3,1/r) = r!, \quad r'(3,2) = 1. \]

Proof. Let $P \in X$ be a 3-fold canonical singularity with index $r_P$. We shall verify the statement for any such $P$. We take a crepant blow-up $f : Y \to X$ with $Y$ terminal by Corollary 2.6. Suppose $\dim f^{-1}(P) = 0$, that is, $P$ is terminal. Then it suffices to recall $\text{md}_P X = 1/r_P$ [9], [12] for terminal $P$ except for smooth $P$.

Suppose $\dim f^{-1}(P) = 1$. For any curve $C \subset f^{-1}(P)$, the blow-up of $Y$ with centre $C$ generates a divisor $E$ with $a_E(X) = 1$. Together with the mentioned result [9], [12], we see that $\text{md}_P X$ is the minimum of $1/r_Q$ for all $Q \in f^{-1}(P)$, where $r_Q$ denotes the index of $Y$ at $Q$. Hence $\text{md}_P X = 1/r$ with $r \in \mathbb{N}$ and $r_Q \leq r$ for all $Q \in f^{-1}(P)$. Thus $r!K_Y$ is a Cartier divisor near $f^{-1}(P)$, so $r_P \mid r!$ by [8, Corollary 1.5].
Suppose $\dim f^{-1}(P) = 2$. Then $P$ is a crepant centre, that is, $m_d P X = 0$. The statement holds by Theorem 1.1. q.e.d.

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