THE INDEX OF A THREEFOLD CANONICAL SINGULARITY

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ABSTRACT. The index of a 3-fold canonical singularity at a crepant centre is at most 6.

1. INTRODUCTION

Let $P \in X$ be a log canonical singularity. Shokurov asked if one can bound the index r_P of X at P in terms of the discrepancies of divisors over X.

Suppose that X has log canonical singularities with P a log canonical centre. In $\dim X = 2$, r_P is 1, 2, 3, 4 or 6 by the classification of singularities. In an arbitrary dimension, Ishii [5] and Fujino [2] reduced the boundedness of r_P to a conjectural boundedness of a quotient of the birational automorphism group of a variety S with $K_S \sim 0$. In particular, they proved $r_P \leq 66$ in $\dim X = 3$.

Suppose that X has canonical singularities. In $\dim X = 2$, P is a rational double point, so $r_P = 1$. The purpose of this paper is to provide an affirmative answer in $\dim X = 3$.

Theorem 1.1. *Let $P \in X$ be a 3-fold canonical singularity such that P is a crepant centre. Then the index of X at P is at most 6.*

Remark 1.2. We have such singularities P with $r_P = 1, 2, 3, 4$ in Example 4.3, but I do not know if there exists P with $r_P = 5$ or 6.

Note that no (even implicit) bound of r_P has been known before. Here a crepant centre means the centre of a divisor with discrepancy zero. The condition that P is a crepant centre is necessary even for a strictly canonical singularity, see Example 5.1. On the other hand, if once the minimal discrepancy at P is fixed, then one can bound r_P for an arbitrary 3-fold canonical singularity $P \in X$ (Theorem 5.3).

We shall prove Theorem 1.1 by using the singular Riemann–Roch formula (singRR) [14], an orbifold version of Riemann–Roch formula, due to Reid. In Sect. 2, we build a tower $Y \to X$ of crepant blow-ups with $\mathbb{Q}$-factorial terminal Y, on which the singRR is applicable unconditionally. Then we construct a divisor F on Y which possesses the information on the index r_P. The r_P is determined by the Euler characteristics $\chi(iK_Y|_F)$, which can be explicitly computed by the singRR (Sect. 3). We derive a numerical classification of the singularities on Y together with r_P in Sect. 4, by the method [6], [7] in the classification of 3-fold divisorial contractions. The boundedness of indices in terms of minimal discrepancies is discussed in Sect. 5.

We work over an algebraically closed field k of characteristic zero. A germ $P \in X$ means an algebraic germ of a variety X at a closed point P.

2. CREPANT BLOW-UPS

Let X be a normal $\mathbb{Q}$-Gorenstein variety.

Definition 2.1. The *index* of X at a point P is the smallest positive integer r such that rK_X is a Cartier divisor at P.

Consider a normal variety Y with a proper birational morphism $f\colon Y \to X$. A prime divisor E on any such Y is called a divisor *over* X, and the image $f(E)$ is called the *centre* of E on X and denoted by $c_X(E)$. The valuation v_E on the function field of X given by such E is called an *algebraic valuation* of X. If we write

$$K_Y = f^*K_X + \sum_E a_E(X)E \quad \text{with } a_E(X) \in \mathbb{Q},$$

then $a_E(X)$ is called the *discrepancy* of E. We say that X has *log canonical*, *log terminal*, *canonical*, *terminal* singularities if $a_E(X) \geq -1$, > -1, ≥ 0, > 0 respectively for all exceptional divisors E over X.

The notion of crepancy is crucial in this paper.

Definition 2.2.
- (i) A *crepant divisor* over X is an exceptional divisor E over X with $a_E(X) = 0$. A *crepant valuation* of X is the algebraic valuation v_E given by a crepant divisor E.
- (ii) A *crepant centre* on X is the centre $c_X(E)$ of a crepant divisor E.
- (iii) A *crepant blow-up* $f\colon Y \to X$ is a projective birational morphism from a normal variety Y such that $K_Y = f^*K_X$.

Remark 2.3.
- (i) Suppose that X is canonical. Then every crepant valuation is realised as a divisor on any resolution of X. In particular, the number of crepant valuations of X is finite. The complement of the union of all crepant centres is the largest terminal open subvariety of X.
- (ii) If $Y \to X$ is a crepant blow-up, then X is canonical if and only if so is Y.

We have a crepant blow-up by the LMMP.

Proposition 2.4. *Let X be a variety with canonical singularities and v a crepant valuation of X. Then there exists a crepant blow-up $f\colon Y \to X$ such that*
- (i) *Y is $\mathbb{Q}$-factorial,*
- (ii) *f has exactly one exceptional divisor E, and $v_E = v$,*
- (iii) *$-E$ is f-nef.*

Proof. Take a projective resolution of singularities $g\colon Z \to X$, and denote by E_Z the divisor on Z with $v_{E_Z} = v$. Take a Cartier divisor $H > 0$ on X whose support contains all the crepant centres. We write $g^*H = H_Z + F$ with the strict transform H_Z of H, and m for the coefficient of E_Z in F. Fix $\varepsilon > 0$ so that $(Z, \varepsilon(H_Z + 2(F - mE_Z)))$ is klt, and run $(K_Z + \varepsilon(H_Z + 2(F - mE_Z)))$-LMMP over X by [1] to get a log minimal model $f\colon Y \to X$.

By $K_Z + \varepsilon(H_Z + 2(F - mE_Z)) \equiv_X K_Z + \varepsilon(F - 2mE_Z)$, the negativity lemma [11, Lemma 2.19] shows that this LMMP contracts exactly all the g-exceptional divisors but E_Z, and $-E$ is f-nef for the strict transform E of E_Z. Hence f is a required crepant blow-up. q.e.d.

Remark 2.5. If X is $\mathbb{Q}$-factorial, then (ii) implies that $\rho(Y/X) = 1$ and $-E$ is f-ample.

Corollary 2.6. *Let $X = X_0$ be a variety with canonical singularities and Z a crepant centre on X. Then there exists a sequence of crepant blow-ups $f_t\colon X_t \to X_{t-1}$ for $1 \leq t \leq s$ such that*

(i) *X_t is $\mathbb{Q}$-factorial for $t \geq 1$ and X_s is terminal,*
(ii) *for $t \geq 1$, f_t has exactly one exceptional divisor E_t and $-E_t$ is f_t-nef,*
(iii) *$f_1(E_1) = Z$.*

We construct a divisor on X_s which possesses the information on the index of X.

Theorem 2.7. *Let $P \in X$ be a canonical singularity such that P is a crepant centre. Let r_P denote the index of X at P and $\mathfrak{m}_P$ the maximal ideal sheaf for P. Then there exist a crepant blow-up $f\colon Y \to X$ and an effective divisor F on Y supported in $f^{-1}(P)$ such that*

(i) *Y is $\mathbb{Q}$-factorial and terminal,*
(ii) *for $i \in \mathbb{Z}$,*

$$f_*\mathscr{O}_Y(iK_Y - F) = \begin{cases} \mathfrak{m}_P\mathscr{O}_X(iK_X) & \textit{if } r_P \mid i, \\ \mathscr{O}_X(iK_X) & \textit{otherwise,} \end{cases}$$
$$R^j f_*\mathscr{O}_Y(iK_Y - F) = 0 \quad \textit{for } j \geq 1.$$

Proof. We take a sequence of crepant blow-ups f_t in Corollary 2.6 with $Z = P$, and set $Y := X_s$. We will construct inductively divisors $F_t \geq 0$ on X_t such that

$$f_{1*}\mathscr{O}_{X_1}(iK_{X_1} - F_1) = \begin{cases} \mathfrak{m}_P\mathscr{O}_X(iK_X) & \text{if } r_P \mid i, \\ \mathscr{O}_X(iK_X) & \text{otherwise,} \end{cases} \tag{1}$$
$$R^j f_{1*}\mathscr{O}_{X_1}(iK_{X_1} - F_1) = 0 \quad \text{for } j \geq 1, \tag{2}$$

and for $t > 1$,

$$f_{t*}\mathscr{O}_{X_t}(iK_{X_t} - F_t) = \mathscr{O}_{X_{t-1}}(iK_{X_{t-1}} - F_{t-1}), \tag{3}$$
$$R^j f_{t*}\mathscr{O}_{X_t}(iK_{X_t} - F_t) = 0 \quad \text{for } j \geq 1. \tag{4}$$

Then Leray's spectral sequence induces that $F := F_s$ is a required divisor.

We set $F_1 := E_1$. The vanishing (2) follows from Kawamata–Viehweg vanishing theorem [10, Theorem 1.2.5, Remark 1.2.6]. If $r_P \mid i$, then (1) is by the projection formula. To see (1) for $r_P \nmid i$, we regard K_X as a fixed divisor (not a divisor class), and so $K_{X_1} = f_1^*K_X$. Denote by $\mathscr{K}_X$ the constant sheaf of the function field of X. Then the inclusion $f_{1*}\mathscr{O}_{X_1}(iK_{X_1} - F_1) \subset \mathscr{O}_X(iK_X)$ is interpreted by the expressions

$$f_{1*}\mathscr{O}_{X_1}(iK_{X_1} - F_1) = \{u \in \mathscr{K}_X \mid (u)_{X_1} + if_1^*K_X - F_1 \geq 0\},$$
$$\mathscr{O}_X(iK_X) = \{u \in \mathscr{K}_X \mid (u)_X + iK_X \geq 0\}.$$

Suppose $u \in \mathscr{K}_X$ satisfies $(u)_X + iK_X \geq 0$. If $r_P \nmid i$, then $(u)_X + iK_X$ is not Cartier at P, so there exists a divisor $D > 0$ passing through P such that $(u)_X + iK_X - D$ is an effective Cartier divisor. Then $(u)_{X_1} + if_1^*K_X - f_1^*D \geq 0$. By $f_1^*K_X = K_{X_1}$ and $F_1 \subset \operatorname{Supp} f_1^*D$, we obtain $(u)_{X_1} + if_1^*K_X - F_1 \geq 0$, implying (1).

For $t > 1$, we set $F_t := \lceil f_t^*F_{t-1} \rceil$ inductively. $F_t = f_t^*F_{t-1} + c_tE_t$ with some $c_t \in [0,1)$, so $-F_t$ is f_t-nef. The (4) is again by Kawamata–Viehweg vanishing theorem. If $c_t = 0$, then (3) is obvious. If $c_t > 0$, then the equality $iK_{X_t} - F_t = f_t^*(iK_{X_{t-1}} - F_{t-1}) - c_tE_t$ shows that $iK_{X_{t-1}} - F_{t-1}$ is not Cartier at every point in $f_t(E_t)$. Now we get (3) just as in the proof of (1) for $r_P \nmid i$. q.e.d.

3. The singular Riemann–Roch formula

We shall apply the singular Riemann–Roch formula due to Reid to our crepant blow-up, and use the method [6], [7] in the classification of 3-fold divisorial contractions. We briefly recall the formula on a canonical 3-fold.

Theorem 3.1 ([14, Theorem 10.2]). *Let X be a projective 3-fold with canonical singularities and D a divisor on X such that $D \sim i_P K_X$ with $i_P \in \mathbb{Z}$ at each $P \in X$.*

(i) *There is a formula of the form*

$$\chi(\mathscr{O}_X(D)) = \chi(\mathscr{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}D\cdot c_2(X) + \sum_P c_P(D),$$

where the summation takes place over the singularities of $\mathscr{O}_X(D)$, and $c_P(D) \in \mathbb{Q}$ is a contribution due to the singularity at P, depending only on the analytic type.

(ii) *For a terminal cyclic quotient singularity P of type $\frac{1}{r_P}(1,-1,b_P)$,*

$$c_P(D) = -\overline{i_P}\frac{r_P^2-1}{12r_P} + \sum_{j=1}^{\overline{i_P}-1} \frac{\overline{jb_P}(r_P-\overline{jb_P})}{2r_P},$$

where $\overline{i} = i - \lfloor \frac{i}{r_P} \rfloor r_P$ denotes the residue of i modulo r_P.

(iii) *For an arbitrary terminal singularity P,*

$$c_P(D) = \sum_Q c_Q(D_Q),$$

where $\{(Q,D_Q)\}_Q$ is a flat deformation of (P,D) to the basket of terminal cyclic quotient singularities Q. Such Q is called a fictitious singularity.

Remark 3.2. The condition $D \sim i_P K_X$ always holds if X is $\mathbb{Q}$-factorial and terminal [8, Corollary 5.2].

Our object is a germ of a crepant blow-up $f\colon Y \to X$ with a divisor F on Y in Theorem 2.7 at a 3-fold canonical singularity $P \in X$ with index r_P. Shrinking and compactifying it, we may assume that Y is projective and terminal (f is merely a projective morphism outside a neighbourhood of P). We shall express the function $\delta_P(i)$ below.

Definition 3.3. We define the function $\delta_P(i)$ on $\mathbb{Z}$ as

$$\delta_P(i) := \begin{cases} 1 & \text{if } r_P \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Applying (ii) in Theorem 2.7 and the vanishing $R^j f_*\mathscr{O}_Y(iK_Y) = 0$ for $j \geq 1$ to the exact sequence

$$0 \to \mathscr{O}_Y(iK_Y - F) \to \mathscr{O}_Y(iK_Y) \to \mathscr{O}_F(iK_Y|_F) \to 0,$$

we obtain

$$\begin{aligned} \delta_P(i) &= \dim_k f_*\mathscr{O}_Y(iK_Y)/f_*\mathscr{O}_Y(iK_Y - F) \\ &= h^0(\mathscr{O}_F(iK_Y|_F)) \\ &= \chi(\mathscr{O}_F(iK_Y|_F)) \\ &= \chi(\mathscr{O}_Y(iK_Y)) - \chi(\mathscr{O}_Y(iK_Y - F)). \end{aligned} \tag{5}$$

Let $I_0 := \{Q \text{ with type } \frac{1}{r_Q}(1,-1,b_Q)\}$ be the basket of fictitious singularities from singularities on Y. Note that b_Q is co-prime to r_Q. For $Q \in I_0$, let f_Q denote the smallest non-negative integer such that $F \sim f_Q K_Y$ at Q. By replacing b_Q with $r_Q - b_Q$ if necessary, we may assume $v_Q := \overline{f_Q b_Q} \le r_Q/2$. Set $I := \{Q \in I_0 \mid f_Q \neq 0\}$.

With this notation, the singular Riemann–Roch formula computes the right-hand side of (5), to provide

$$\delta_P(i) = \frac{1}{6}F^3 + \frac{1}{12}F \cdot c_2(Y) + \sum_{Q\in I}(A_Q(i) - A_Q(i - f_Q)), \tag{6}$$

where the contribution $A_Q(i)$ is given by

$$A_Q(i) := -\bar{i}\frac{r_Q^2 - 1}{12r_Q} + \sum_{j=1}^{\bar{i}-1}\frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}.$$

The $A_Q(i)$ satisfies the formula

$$A_Q(i+1) - A_Q(i) = -\frac{r_Q^2 - 1}{12r_Q} + B_Q(ib_Q)$$

with

$$B_Q(i) := \frac{\bar{i}(r_Q - \bar{i})}{2r_Q}.$$

Therefore by (6), we have

$$\delta_P(i+1) - \delta_P(i) = \sum_{Q\in I}(B_Q(ib_Q) - B_Q(ib_Q - v_Q)). \tag{7}$$

Lemma 3.4. *The r_P equals the l.c.m. of r_Q for all $Q \in I$.*

Proof. Since $r_P K_Y = r_P f^* K_X$ is a Cartier divisor near $f^{-1}(P)$, r_Q divides r_P for all $Q \in I$. On the other hand, we see that r_P divides the l.c.m. of r_Q by (7) and the periodic properties of δ_P, B_Q. q.e.d.

4. Boundedness of indices

We shall prove Theorem 1.1 in this section. Let r_P denote the index of X at P. We take a crepant blow-up $f\colon Y \to X$ with a divisor F on Y in Theorem 2.7. We restrict the possibilities of $J := \{(r_Q, v_Q)\}_{Q\in I}$ using (7) for $i = 0$.

Lemma 4.1. *J is one of the types in Table 1.*

Table 1

type	J	r_P
1	$(2,1),(2,1),(2,1),(2,1)$	2
2	$(2,1),(2,1),(4,2)$	4
3	$(2,1),(3,1),(6,1)$	6
4	$(2,1),(4,1),(4,1)$	4
5	$(3,1),(3,1),(3,1)$	3
6	$(4,2),(4,2)$	4
7	$(2,1),(6,3)$	6
8	$(2,1),(8,2)$	8
9	$(3,1),(6,2)$	6
10	$(5,1),(5,2)$	5
11	$(8,4)$	8
12	$(9,3)$	9
13	$\emptyset$	1

Proof. By Lemma 3.4, r_P is determined by J, and $r_P = 1$ if and only if $J = \emptyset$. We assume $r_P > 1$ from now on. Then (7) for $i = 0$ is written as

$$\sum_{Q\in I} B_Q(v_Q) = 1. \tag{8}$$

By the definition of B_Q and $r_Q \geq 2v_Q$, we have

$$v_Q/4 \leq B_Q(v_Q) < v_Q/2. \tag{9}$$

Then $J' := \{v_Q\}_{Q\in I}$, which satisfies (8) and (9), should be one of

$$\{1,1,1,1\},\{1,1,2\},\{1,1,1\},\{2,2\},\{1,3\},\{1,2\},\{3\},\{4\}.$$

For each of these candidates for J', one can solve the equation (8) for r_Q ($\geq 2v_Q$) explicitly. Every solution is in Table 1. For example, suppose $J' = \{1,2\}$. We set $J = \{(r_1,1),(r_2,2)\}$. Then (8) becomes $1/r_1 + 4/r_2 = 1$. Thus $(r_1,r_2) = (2,8)$, $(3,6)$ or $(5,5)$, so J is of type 8, 9, 10 respectively. q.e.d.

By Lemma 4.1, we have $r_P \leq 9$, and for Theorem 1.1 it is enough to exclude types 8, 11, 12. However, we derive a finer numerical classification by determining $\tilde{J} := \{(r_Q,v_Q,b_Q)\}_{Q\in I}$.

Theorem 4.2. *$\tilde{J}$ is one of the types in Table 2.*

TABLE 2

type	$\tilde{J}$	r_P
1	$(2,1,1),(2,1,1),(2,1,1),(2,1,1)$	2
3	$(2,1,1),(3,1,2),(6,1,5)$	6
4	$(2,1,1),(4,1,3),(4,1,3)$	4
5	$(3,1,2),(3,1,2),(3,1,2)$	3
10	$(5,1,4),(5,2,3)$	5
13	$\emptyset$	1

Proof. By Lemma 4.1, there exist only finitely many candidates for $\tilde{J}$. For each candidate, one can compute the right-hand side of (7) explicitly. It must coincide with $\delta_P(i+1) - \delta_P(i)$, but such a coincidence happens only if $\tilde{J}$ is one of the types in Table 2.

Here we demonstrate for type 3. $\tilde{J} = \{(2,1,1),(3,1,b_2),(6,1,b_3)\}$ with $b_2 = 1$ or 2 and $b_3 = 1$ or 5. The (7) for $i = 1$ is $\delta_P(2) - \delta_P(1) = 1,1/3,2/3,0$ when $(b_2,b_3) = (1,1),(1,5),(2,1),(2,5)$ respectively. Thus (b_2,b_3) must be $(2,5)$, and in this case (7) surely holds for any i. q.e.d.

Example 4.3. In simple cases, Theorem 1.1 is known by the classification.

(i) (Morrison [13], Ishida–Iwashita [4]) If P is a cyclic quotient singularity, then $r_P = 1$ except $\frac{1}{4n}(1,2n+1,-2)$ ($n \geq 2$), $\frac{1}{14}(1,9,11)$, $\frac{1}{9}(1,4,7)$, with $r_P = 2,2,3$ respectively.

(ii) (Hayakawa–Takeuchi [3]) If P is an isolated singularity which is a cyclic quotient of a hypersurface singularity, then $r_P \leq 4$. The only case when $r_P = 4$ is $o \in (x_1x_2 + x_3^2 + x_4^2 = 0) \subset \mathbb{A}^4_{x_1x_2x_3x_4}/\mathbb{Z}_8(1,5,3,7)$.

5. MINIMAL DISCREPANCIES

To begin with, we provide an example which explains the need of P being a crepant centre in Theorem 1.1 even for a strictly canonical singularity. A similar example exists also for a 3-fold strictly log canonical singularity [2, Example 6.1].

Example 5.1. Let $r \in \mathbb{N}$. Let $P \in X$ be the germ

$$o \in (x_1x_2 + x_3^2 = 0) \subset \mathbb{A}^4_{x_1x_2x_3x_4}/\mathbb{Z}_r(1,-1,0,1),$$

which is singular along the x_4-axis C. Let $f\colon Y \to X$ be the weighted blow-up with weights $\mathrm{wt}(x_1,x_2,x_3,x_4) = \frac{1}{r}(1,r-1,r,1)$. Then $K_Y = f^*K_X + \frac{1}{r}E$ with the exceptional divisor E, and Y has 2 terminal quotient singularities of types $\frac{1}{r-1}(1,-1,1)$ and $\frac{1}{r}(1,-1,1)$ outside the strict transform C_Y of C. Let $g\colon Z \to Y$ be the blow-up with centre C_Y. The g is a crepant blow-up and Z is smooth near $g^{-1}(C_Y)$. Hence X has canonical singularities with a crepant centre C, but P is not a crepant centre. The index of X at P is r.

We focus on the minimal discrepancy to grasp this phenomenon. For a normal $\mathbb{Q}$-Gorenstein singularity $P \in X$, the *minimal discrepancy* $\mathrm{md}_P X$ of X at P is the infimum of discrepancies $a_E(X)$ for all divisors E over X with $c_X(E) = P$. Note that $\mathrm{md}_P X \in \{-\infty\} \cup [-1,\infty)$, and $P \in X$ is log canonical if and only if $\mathrm{md}_P X \geq -1$.

In Example 5.1, we have $\mathrm{md}_P X = 1/r$. Shokurov formulated a question on the boundedness of indices in terms of minimal discrepancies.

Question 5.2 (Shokurov)**.** *For each $(n,a) \in \mathbb{N} \times [-1,\infty)$, does there exist a number $r(n,a)$ such that the index of an arbitrary n-fold log canonical singularity $P \in X$ with $\mathrm{md}_P X = a$ is at most $r(n,a)$?*

He raised its weaker variant for canonical singularities.

Question 5.2′. *For each $(n,a) \in \mathbb{N} \times [0,\infty)$, does there exist a number $r'(n,a)$ such that the index of an arbitrary n-fold canonical singularity $P \in X$ with $\mathrm{md}_P X = a$ is at most $r'(n,a)$?*

The result of Ishii and Fujino gives $r(3,-1) = 66$ for Question 5.2. Theorem 1.1 gives $r'(3,0) = 6$ for Question 5.2′. Further, we provide an affirmative answer to Question 5.2′ for $n = 3$.

Theorem 5.3. *Question 5.2′ is true for $n = 3$. More precisely, the minimal discrepancy of a 3-fold canonical singularity is 0, $1/r$ ($r \in \mathbb{N}$) or 2, and one can take*

$$r'(3,0) = 6, \qquad r'(3,1/r) = r!, \qquad r'(3,2) = 1.$$

Proof. Let $P \in X$ be a 3-fold canonical singularity with index r_P. We shall verify the statement for any such P. We take a crepant blow-up $f\colon Y \to X$ with Y terminal by Corollary 2.6.

Suppose $\dim f^{-1}(P) = 0$, that is, P is terminal. Then it suffices to recall $\mathrm{md}_P X = 1/r_P$ [9], [12] for terminal P except for smooth P.

Suppose $\dim f^{-1}(P) = 1$. For any curve $C \subset f^{-1}(P)$, the blow-up of Y with centre C generates a divisor E with $a_E(X) = 1$. Together with the mentioned result [9], [12], we see that $\mathrm{md}_P X$ is the minimum of $1/r_Q$ for all $Q \in f^{-1}(P)$, where r_Q denotes the index of Y at Q. Hence $\mathrm{md}_P X = 1/r$ with $r \in \mathbb{N}$ and $r_Q \leq r$ for all $Q \in f^{-1}(P)$. Thus $r!K_Y$ is a Cartier divisor near $f^{-1}(P)$, so $r_P \mid r!$ by [8, Corollary 1.5].

Suppose $\dim f^{-1}(P) = 2$. Then P is a crepant centre, that is, $\mathrm{md}_P X = 0$. The statement holds by Theorem 1.1. q.e.d.

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