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The law of the iterated logarithm for discrepancies of three variations of geometric progressions

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Abstract

We prove the law of the iterated logarithm for discrepancies for three sequences: a sequence determined by products of random numbers, a sequence given by products of periodic numbers, and a sequence given by an arrangement in increasing order of the union of finitely many geometric progressions.

§ 1. Main Result

In this note we consider three variations of positive geometric progressions and prove the law of the iterated logarithm for discrepancies $D_N\{n_k x\}$ and star discrepancies $D_N^*\{n_k x\}$ of these sequences $\{n_k x\}$, i.e.,

$$\overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma, \quad \text{a.e.}$$

for some constant Σ. As to a sequence satisfying Hadamard's gap condition,

$$n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \dots), \tag{1.1}$$

it was conjectured by Erdős-Gál that the limsups above are bounded. By applying methods due to Erdős-Gál [4], Takahashi [17], and Gál-Gál [11], Philipp [13] proved the bounded law of the iterated logarithm below and solved the conjecture:

$$\frac{1}{4\sqrt{2}} \leq \overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} \leq \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} \leq K_q < \infty, \quad \text{a.e.} \tag{1.2}$$

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For a positive geometric progression $\{\theta^k x\}$, the limsup value depends on algebraic nature of θ as below:

Theorem 1.1 ([6, 7]). *For any $\theta > 1$, there exists a real number Σ_θ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{\theta^k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta, \quad a.e. \tag{1.3}$$

We have

$$\Sigma_\theta = \frac{1}{2}$$

if and only if θ satisfies

$$\theta^r \notin \mathbf{Q} \quad (r \in \mathbf{N}). \tag{1.4}$$

In other cases, θ can be written uniquely by

$$\theta = \sqrt[r]{p/q}, \quad r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\},\ p, q \in \mathbf{N}, \quad \gcd(p, q) = 1. \tag{1.5}$$

In this case Σ_θ does not depend on r and satisfies

$$1/2 < \Sigma_\theta \le \sqrt{(pq+1)/(pq-1)}/2.$$

Moreover, we can evaluate it in the following cases:

$$\Sigma_\theta = \begin{cases} \sqrt{(pq+1)/(pq-1)}/2, & \text{if } p \text{ and } q \text{ are both odd;} \\ \sqrt{(p+1)/(p-1)}/2, & \text{especially if } p \text{ is odd and } q = 1; \\ \sqrt{(p+1)p(p-2)/(p-1)^3}/2, & \text{if } p \ge 4 \text{ is even and } q = 1; \\ \sqrt{42}/9, & \text{if } p = 2 \text{ and } q = 1; \\ \sqrt{22}/9, & \text{if } p = 5 \text{ and } q = 2. \end{cases}$$

In this paper, we consider three variations of the above result.

The first variation is given by randomizing the ratio θ in a geometric progression.

Theorem 1.2. *Assume that sets A and B of positive integers satisfy*

$$b/a > q > 1 \quad \text{and} \quad \gcd(a, b) = 1 \quad \text{for all} \quad a \in A, \quad \text{and} \quad b \in B. \tag{1.6}$$

Let $\{(X_k, Y_k)\}$ be an $A \times B$-valued i.i.d. and define $\{n_k\}$ by

$$n_k = \prod_{j=1}^{k} \frac{Y_j}{X_j}. \tag{1.7}$$

Then there exists a constant $\Sigma_{\mathcal{L}(X_1,Y_1)}$ *depending only on the law* $\mathcal{L}(X_1,Y_1)$ *of* (X_1,Y_1) *such that*

$$P\Big(\overline{\lim_{N\to\infty}}\frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}}=\overline{\lim_{N\to\infty}}\frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}}=\Sigma_{\mathcal{L}(X_1,Y_1)},\quad a.e.\ x.\Big)=1. \tag{1.8}$$

When A *and* B *both consist of odd numbers, we have*

$$\Sigma_{\mathcal{L}(X_1,Y_1)}=\frac{1}{2}\sqrt{\frac{\Upsilon_{\mathcal{L}(X_1,Y_1)}+1}{\Upsilon_{\mathcal{L}(X_1,Y_1)}-1}},\quad \textit{where}\quad \Upsilon_{\mathcal{L}(X_1,Y_1)}=\Big(E\Big(\frac{1}{X_1Y_1}\Big)\Big)^{-1}. \tag{1.9}$$

Because $\Upsilon_{\mathcal{L}(X_1,Y_1)}$ can be written as

$$\Upsilon_{\mathcal{L}(X_1,Y_1)}=\Big(\sum_{(a,b)\in A\times B}\frac{p_{(a,b)}}{ab}\Big)^{-1}$$

where $p_{(a,b)}=P((X_1,Y_1)=(a,b))$, it can be regarded as a generalized harmonic mean of ab over $A\times B$.

The second variation is given by changing the ratio θ periodically.

Theorem 1.3. *For* $\theta_1,\ \dots,\ \theta_\tau>1$*, we define a sequence* $\{n_k\}$ *by*

$$n_0=1,\quad n_{k+1}=\theta_{j+1}n_k \textit{ if } k=j \bmod \tau \textit{ and } j=0,\dots,\tau-1.$$

Then there exists a constant $\Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{periodic}}$ *such that*

$$\overline{\lim_{N\to\infty}}\frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}}=\overline{\lim_{N\to\infty}}\frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}}=\Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{periodic}},\quad a.e. \tag{1.10}$$

We have permutation invariance below when $\tau=2,\ 3$.

$$\Sigma_{\theta_1,\theta_2;\mathrm{periodic}}=\Sigma_{\theta_2,\theta_1;\mathrm{periodic}}, \tag{1.11}$$

and

$$\begin{aligned}&\Sigma_{\theta_1,\theta_2,\theta_3;\mathrm{periodic}}=\Sigma_{\theta_2,\theta_3,\theta_1;\mathrm{periodic}}=\Sigma_{\theta_3,\theta_1,\theta_2;\mathrm{periodic}}\\&=\Sigma_{\theta_1,\theta_3,\theta_2;\mathrm{periodic}}=\Sigma_{\theta_3,\theta_2,\theta_1;\mathrm{periodic}}=\Sigma_{\theta_2,\theta_1,\theta_3;\mathrm{periodic}}.\end{aligned} \tag{1.12}$$

Let A *and* B *be sets of positive integers satisfying (1.6). If* A *and* B *both consist of odd numbers, and if* $p_j\in B$ *and* $q_j\in A$ $(j=1,\dots,\tau)$*, then* $\Sigma_{p_1/q_1,\dots,p_\tau/q_\tau;\mathrm{periodic}}$ *equals to*

$$\frac{1}{2}\sqrt{\frac{1}{(s_1\dots s_\tau-1)}\Big(1+s_1\dots s_\tau+\frac{2}{\tau}\sum_{1\le j<k\le\tau}\big(s_j\dots s_{k-1}+s_1\dots s_{j-1}s_k\dots s_\tau\big)\Big)}, \tag{1.13}$$

where $s_j=p_jq_j$.

We can explain the reason why we have the invariance (1.11): If we replace x by $\theta_2 x$ in

$$\theta_1 x, \theta_1\theta_2 x, \theta_1^2\theta_2 x, \theta_1^2\theta_2^2 x, \theta_1^3\theta_2^2 x, \dots$$

and add one term $\theta_2 x$ at the top, we have

$$\theta_2 x, \theta_1\theta_2 x, \theta_1\theta_2^2 x, \theta_1^2\theta_2^2 x, \theta_1^2\theta_2^3 x, \theta_1^3\theta_2^3 x, \dots.$$

Since the law of the iterated logarithm holds for a.e. x, and since adding finitely many terms does not affect on the law of the iterated logarithm, we have (1.11).

As to the invariance (1.12), while invariances among circular permutations

$$\Sigma_{\theta_1,\theta_2,\theta_3;\mathrm{periodic}} = \Sigma_{\theta_2,\theta_3,\theta_1;\mathrm{periodic}} = \Sigma_{\theta_3,\theta_1,\theta_2;\mathrm{periodic}}$$

and

$$\Sigma_{\theta_1,\theta_3,\theta_2;\mathrm{periodic}} = \Sigma_{\theta_3,\theta_2,\theta_1;\mathrm{periodic}} = \Sigma_{\theta_2,\theta_1,\theta_3;\mathrm{periodic}}$$

are explained in the same way, we could not find any easy explanation for the fact that these two values are identical.

We must also mention that we cannot expect such an invariance when $\tau \geq 4$. Actually, we have

$$\Sigma_{3,5,7,11;\mathrm{periodic}} = \sqrt{\frac{423}{1154}}, \quad \text{while} \quad \Sigma_{3,7,5,11;\mathrm{periodic}} = \sqrt{\frac{421}{1154}}.$$

The third variation is the arrangement in increasing order of the union of finitely many positive geometric progressions. Although it no longer satisfies Hadamard's gap condition, it still has good arithmetic structure to have the following limit theorems. It is proved in [5] that sequences $\{f(\theta_1^k x)\}$ and $\{g(\theta_2^k x)\}$ are asymptotically independent in some sense when $\log\theta_1/\log\theta_2$ is irrational. We can find a similar phenomenon in case of metric studies on discrepancies.

Theorem 1.4. *Suppose that $\theta_1, \dots, \theta_\tau > 1$ are given and that geometric progressions $\{\theta_1^k\}, \dots, \{\theta_\tau^k\}$ are mutually disjoint from each other, i.e.,*

$$\log\theta_i/\log\theta_j \notin \mathbf{Q}, \quad (i \neq j). \tag{1.14}$$

Let $\{n_k\}$ be the arrangement in increasing order of $\{\theta_1^k\} \cup \dots \cup \{\theta_\tau^k\}$. Then there exists a real number $\Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{union}}$ such that

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{union}}, \quad a.e.$$

If each θ_j satisfies (1.4) or given by (1.5) with odd p and q, then

$$\Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{union}} = \sqrt{\Big(\frac{\Sigma_{\theta_1}^2}{\log\theta_1} + \dots + \frac{\Sigma_{\theta_\tau}^2}{\log\theta_\tau}\Big) \Big/ \Big(\frac{1}{\log\theta_1} + \dots + \frac{1}{\log\theta_\tau}\Big)},$$

where $\Sigma_{\theta_1}, \dots, \Sigma_{\theta_\tau}$ *are defined by (1.3). In particular when all* $\theta_1, \dots, \theta_\tau$ *satisfy (1.4), then*

$$\Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{union}} = \frac{1}{2}.$$

§ 2. Preliminaries

For $x, y, \xi, \eta \in [0,1)$, put $V(x,\xi) = x \wedge \xi - x\xi$ and $\widetilde{V}(x,y,\xi,\eta) = V(x,\xi) + V(y,\eta) - V(x,\eta) - V(y,\xi)$. Note $V(x,\xi) = \widetilde{V}(0,x,0,\xi)$. By $0 \le V(x,\xi) \le 1/4$, clearly we have $|\widetilde{V}(x,y,\xi,\eta)| \le 1$. The proof of the next lemma can be found in [6].

Lemma 2.1. *Let* $0 \le a < b < 1$. *For any positive integers* P *and* Q *with* $\gcd(P,Q) = 1$, *we have*

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx = \frac{1}{PQ}\widetilde{V}(\langle Pa\rangle, \langle Pb\rangle, \langle Qa\rangle, \langle Qb\rangle), \tag{2.1}$$

$$\widetilde{V}(\langle Pa\rangle, \langle Pb\rangle, \langle Qa\rangle, \langle Qb\rangle) \le V(\langle P(b-a)\rangle, \langle Q(b-a)\rangle) \le \frac{1}{4}. \tag{2.2}$$

For a bounded measurable function g, we define *the mean value* $\int_{\mathbf{R}} g(x)\,\mu_R(dx)$ by

$$\int_{\mathbf{R}} g(x)\,\mu_R(dx) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(x)\,dx$$

if the limit on the right hand side exists. For a trigonometric polynomial g with period 1 satisfying $\int_0^1 g = 0$, we have

$$\int_{\mathbf{R}} g(\Theta x)g(x)\,\mu_R(dx) = 0$$

for $\Theta \notin \mathbf{Q}$, and

$$\int_{\mathbf{R}} g((P/Q)x)g(x)\,\mu_R(dx) = \int_{\mathbf{R}} g(Px)g(Qx)\,\mu_R(dx) = \int_0^1 g(Px)g(Qx)\,dx$$

for non-zero integers P and Q.

The next lemma controls asymptotic behavior of variances. It is a variation of the key lemma of [9]. Denote the d-th subsum of the Fourier series of $\widetilde{\mathbf{1}}_{[a,b)}$ by $\widetilde{\mathbf{1}}_{[a,b);d}$.

Lemma 2.2. *For a sequence* $\{n_k\}$ *of positive numbers satisfying Hadamard's gap condition (1.1) and for arbitrary* $d \ge 3$, *we have*

$$\int_{\mathbf{R}} \Big(\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\Big)^2 \mu_R(dx) \le \int_{\mathbf{R}} \Big(\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[0,b-a);d}(n_k x)\Big)^2 \mu_R(dx) + NC_q\frac{\log d}{d},$$

$$\int_{\mathbf{R}} \Big(\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\Big)^2 \mu_R(dx) \le C_q N,$$

where C_q is a constant depending only on q.

Proof. Because we have

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(Px)\widetilde{\mathbf{1}}_{[a,b);d}(Qx)\,dx = \int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b);d}(Qx)\,dx$$

for coprime positive integers $Q \le P$, we obtain

$$\begin{aligned}&\left|\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx - \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(Px)\widetilde{\mathbf{1}}_{[a,b);d}(Qx)\,dx\right|\\ &= \left|\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)(\widetilde{\mathbf{1}}_{[a,b)} - \widetilde{\mathbf{1}}_{[a,b);d})(Qx)\,dx\right| \le \left|\sum_{|\nu|\ge d/P} \widehat{\widetilde{\mathbf{1}}}_{[a,b)}(Q\nu)\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(-P\nu)\right|\\ &\le \frac{2}{\pi^2 PQ}\sum_{\nu\ge d/P}\frac{1}{\nu^2} \le \frac{2}{\pi^2 PQ}\Big(2\wedge\frac{2P}{d}\Big) \le \frac{Q}{P}\Big(1\wedge\frac{1}{d}\frac{P}{Q}\Big).\end{aligned}$$

We prove

$$\begin{aligned}&\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(n_jx)\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\,\mu_R(dx)\\ &\qquad\le \int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[0,b-a);d}(n_jx)\widetilde{\mathbf{1}}_{[0,b-a);d}(n_kx)\,\mu_R(dx) + 2\frac{n_j}{n_k}\Big(1\wedge\frac{1}{d}\frac{n_k}{n_j}\Big)\end{aligned} \tag{2.3}$$

for arbitrary $j \le k$. Because we have

$$\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(n_jx)\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\,\mu_R(dx) = \int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}\Big(\frac{n_k}{n_j}x\Big)\,\mu_R(dx),$$

we see that it equals to 0 if $n_k/n_j \notin \mathbf{Q}$, and the inequality (2.3) holds in this case. Otherwise, we can write

$$\frac{n_k}{n_j} = \frac{P}{Q} \tag{2.4}$$

where $P \ge Q$ are coprime positive integers. In this case we have

$$\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(n_jx)\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\,\mu_R(dx) = \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(Px)\widetilde{\mathbf{1}}_{[a,b);d}(Qx)\,dx$$

and thereby

$$\left|\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(n_jx)\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\,\mu_R(dx) - \int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx\right| \le \frac{n_j}{n_k}\Big(1\wedge\frac{1}{d}\frac{n_k}{n_j}\Big). \tag{2.5}$$

By applying (2.1) and (2.2), we have

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx \le \int_0^1 \widetilde{\mathbf{1}}_{[0,b-a)}(Px)\widetilde{\mathbf{1}}_{[0,b-a)}(Qx)\,dx.$$

These together with

$$\int_0^1 \widetilde{\mathbf{1}}_{[0,b-a)}(Px)\widetilde{\mathbf{1}}_{[0,b-a)}(Qx)\,dx \le \int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[0,b-a);d}(n_jx)\widetilde{\mathbf{1}}_{[0,b-a);d}(n_kx)\,\mu_R(dx) + \frac{n_j}{n_k}\Big(1\wedge\frac{1}{d}\frac{n_k}{n_j}\Big)$$

imply (2.3).

By applying (2.3), we have

$$\begin{aligned}
&\int_{\mathbf{R}}\Big(\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\Big)^2\mu_R(dx)\\
&=\sum_{k=M+1}^{M+N}\Big\{\int_{\mathbf{R}}\widetilde{\mathbf{1}}^2_{[a,b);d}(n_kx)\mu_R(dx)+2\sum_{j=M+1}^{k-1}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\widetilde{\mathbf{1}}_{[a,b);d}(n_jx)\mu_R(dx)\Big\}\\
&\le\sum_{k=M+1}^{M+N}\Big\{\int_{\mathbf{R}}\widetilde{\mathbf{1}}^2_{[0,b-a);d}(n_kx)\mu_R(dx)+\frac{2}{d}\\
&\qquad+2\sum_{j=M+1}^{k-1}\Big(\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[0,b-a);d}(n_kx)\widetilde{\mathbf{1}}_{[0,b-a);d}(n_jx)\mu_R(dx)+2\frac{n_j}{n_k}\Big(1\wedge\frac{1}{d}\frac{n_k}{n_j}\Big)\Big)\Big\}\\
&\le\int_{\mathbf{R}}\Big(\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[0,b-a);d}(n_kx)\Big)^2\mu_R(dx)+\frac{2N}{d}+4\sum_{k=M+1}^{M+N}\sum_{j=M+1}^{k-1}\frac{n_j}{n_k}\Big(1\wedge\frac{1}{d}\frac{n_k}{n_j}\Big).
\end{aligned}$$

To have the first inequality of Lemma 2.2, it is enough to bound the last summation of the above formula. Fix j arbitrarily and let L be the largest l such that $n_{j+l}/n_j \le d$. Since $q^L \le n_{j+L}/n_j \le d$, we have $L \le (\log d)/(\log q)$. For $l \le L$, we have

$$\frac{n_j}{n_{j+l}}\Big(1\wedge\frac{1}{d}\frac{n_{j+l}}{n_j}\Big)=\frac{1}{d},$$

and for $l \ge L+1$,

$$\frac{n_j}{n_{j+l}}\Big(1\wedge\frac{1}{d}\frac{n_{j+l}}{n_j}\Big)=\frac{n_j}{n_{j+l}}=\frac{n_j}{n_{j+L+1}}\frac{n_{j+L+1}}{n_{j+l}}\le\frac{1}{d}\frac{1}{q^{l-L-1}},$$

since $n_{j+l}/n_j \ge n_{j+L+1}/n_j > d$. Hence we have a desirable estimate below

$$\sum_{l=1}^{\infty}\frac{n_j}{n_{j+l}}\Big(1\wedge\frac{1}{d}\frac{n_{j+l}}{n_j}\Big)\le\sum_{l\le L}\frac{1}{d}+\sum_{l\ge L+1}\frac{1}{d}\frac{1}{q^{l-L-1}}\le\frac{\log d}{d\log q}+\frac{q}{d(q-1)}.$$

Let us prove the second inequality of Lemma 2.2. First we prove

$$\Big|\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(n_jx)\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\,\mu_R(dx)\Big|\le\frac{n_j}{n_k}+\frac{n_j}{n_k}\Big(1\wedge\frac{1}{d}\frac{n_k}{n_j}\Big),\quad (k\ge j). \tag{2.6}$$

When n_k/n_j is irrational, the left hand side integral equals to 0 and (2.6) is clearly valid. Otherwise, by expressing n_k/n_j by (2.4), and by applying the expression (2.1), we have

$$\left|\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b)}(n_j x)\widetilde{\mathbf{1}}_{[a,b)}(n_k x)\,\mu_R(dx)\right| = \left|\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx\right| \le \frac{1}{PQ} \le \frac{Q}{P} = \frac{n_j}{n_k}.$$

This together with (2.5), implies (2.6). By applying (2.6), we have

$$\begin{aligned}
&\int_{\mathbf{R}}\Big(\sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\Big)^2 \mu_R(dx)\\
&= \sum_{k=M+1}^{M+N}\Big\{\int_{\mathbf{R}} \widetilde{\mathbf{1}}^2_{[a,b);d}(n_k x)\mu_R(dx) + 2\sum_{j=M+1}^{k-1}\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\widetilde{\mathbf{1}}_{[a,b);d}(n_j x)\mu_R(dx)\Big\}\\
&\le \sum_{k=M+1}^{M+N}\Big\{\frac{1}{4}+\frac{1}{d}+2\sum_{j=M+1}^{k-1}\Big(\frac{n_j}{n_k}+\frac{n_j}{n_k}\Big(1\wedge\frac{1}{d}\frac{n_k}{n_j}\Big)\Big)\Big\}\\
&\le N\Big(\frac{1}{4}+\frac{1}{d}+\frac{2}{q-1}+\frac{2\log d}{d\log q}+\frac{2q}{d(q-1)}\Big) \le N\Big(\frac{1}{4}+\frac{1}{3}+\frac{2}{q-1}+\frac{2}{\log q}+\frac{2q}{3(q-1)}\Big).
\end{aligned}$$

□

The next two lemmas are very convenient to give an error estimate while approximating a sum of functions by a martingale. Proofs can be found in [6, 9, 8].

Lemma 2.3. *If g is a bounded measurable function with period 1 satisfying $\int_0^1 g = 0$, then for all $a < b$ and $\lambda > 0$, we have*

$$\left|\int_a^b g(\lambda x)\,dx\right| \le \frac{\|g\|_\infty}{\lambda}.$$

Lemma 2.4. *Let g be a trigonometric polynomial with period 1 and degree d satisfying $\int_0^1 g = 0$. There exists a constant $\widehat{C}_q$ depending only on q such that, for any integer L and for a sequence $\{\lambda_k\}$ of real numbers satisfying Hadamard's gap condition $\lambda_{k+1}/\lambda_k \ge q > 1$ and $\lambda_1 \ge 1$,*

$$\int_L^{L+1}\Big(\sum_{k=M+1}^{M+N} g(\lambda_k x)\Big)^4 dx \le \widehat{C}_q\Big(\sum_{|\nu|\le d}|\widehat{g}(\nu)|\Big)^4 N^2.$$

Final lemma we present here will be used to control asymptotic behaviors of two gaussian processes. The proof is based on the path properties of gaussian processes and can be found in [9].

Lemma 2.5. *Let $\{Z_k\}$ and $\{Z'_k\}$ be standard normal i.i.d. Suppose that $\{v_k\}$ and $\{v'_k\}$ are sequence of positive numbers satisfying $c_1 i \le v_i \le c_2 i$, $d_1 i \le v'_i \le d_2 i$, and $v_i \le v'_i + \gamma i$ for some $0 < c_1 < c_2 < \infty$, $0 < d_1 < d_2 < \infty$, and $0 < \gamma < \infty$. Put $l_M = M(M+1)/2$, $\beta_M = v_1 + \cdots + v_M$, $\beta'_M = v'_1 + \cdots + v'_M$, and $\phi(x) = \sqrt{2x \log\log x}$. Then we have*

$$\sqrt{c_1} \le \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \left| \sum_{k \le \beta_M} Z_k \right| \le \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)} \left| \sum_{k \le \beta'_M} Z'_k \right| + \sqrt{\gamma} \le \sqrt{d_2} + \sqrt{\gamma}, \quad a.s.,$$

where both of limsups above are constants a.s.

§ 3. Law of the iterated logarithm

In this section we prove the following proposition, which gives sufficient conditions to have the law of the iterated logarithm for lacunary series. It is very easy to verify these conditions for some variations of geometric progression.

Proposition 3.1. *Suppose that a sequence $\{n_k\}$ of positive numbers satisfies Hadamard's gap condition (1.1). Let $d \ge 3$ and suppose that the condition*

$$\inf\Big\{ |n_k\nu - n_l\nu'|/n_l \;\Big|\; 1 \le l \le k,\ 1 \le \nu, \nu' \le d,\ n_k\nu - n_l\nu' \ne 0 \Big\} > 0 \tag{3.1}$$

is satisfied. Then there exists a real number $C(a,b;d)$ such that

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x) \right| = C(a,b;d), \quad a.e. \tag{3.2}$$

and

$$C(a,b;d) \le C(0, b-a; d) + \sqrt{C_q \frac{\log d}{d}}. \tag{3.3}$$

By putting

$$\overline{\sigma}^2_{a,b;d} = \varlimsup_{N\to\infty} \frac{1}{N} \int_{\mathbf{R}} \left(\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x) \right)^2 \mu_R(x) \quad \text{and}$$

$$\underline{\sigma}^2_{a,b;d} = \varliminf_{N\to\infty} \frac{1}{N} \int_{\mathbf{R}} \left(\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x) \right)^2 \mu_R(x),$$

we have

$$\underline{\sigma}_{a,b;d} \le C(a,b;d) \le \overline{\sigma}_{a,b;d}. \tag{3.4}$$

If (3.1) holds for all $d \ge 3$, then there exists a real number Σ such that

$$\varlimsup_{N\to\infty} \frac{N D_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{N D_N\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma, \quad a.e. \tag{3.5}$$

and

$$\sup_{0\le a<b<1} \overline{\lim_{d\to\infty}}\, \underline{\sigma}_{a,b;d} \le \Sigma \le \sup_{0\le a<b<1} \underline{\lim_{d\to\infty}}\, \overline{\sigma}_{a,b;d} \tag{3.6}$$

Especially, if there exist non-negative numbers $\sigma_{a,b;d}$ *and* $\sigma_{a,b}$ *such that*

$$\lim_{N\to\infty} \frac{1}{N} \int_{\mathbf{R}} \left(\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x) \right)^2 \mu_R(x) = \sigma^2_{a,b;d} \quad \text{and} \quad \lim_{d\to\infty} \sigma^2_{a,b;d} = \sigma^2_{a,b}, \tag{3.7}$$

then we have

$$\Sigma = \sup_{0\le a<b<1} \sigma_{a,b}. \tag{3.8}$$

The next lemma gives an easy sufficient condition to have the condition (3.1) of the previous proposition. The most typical example satisfying this condition is a positive diverging geometric progression.

Lemma 3.2. *Suppose that a sequence* $\{n_k\}$ *of positive numbers satisfies Hadamard's gap condition (1.1). If* $\{n_k\}$ *satisfies the condition*

$$\#\Big(\big\{ n_{j+1}/n_j \mid j \in \mathbf{N} \big\} \cap [\,q, Q\,] \Big) < \infty \quad \text{for all } Q > q, \tag{3.9}$$

then it satisfies (3.1) for all $d \ge 3$.

Proof. Take $d \ge 3$, $1 \le \nu$, $\nu' \le d$, and $1 \le l \le k$ arbitrarily. We use the following estimate:

$$n_k\nu - n_l\nu' = \left(\nu \prod_{j=l}^{k-1} \frac{n_{j+1}}{n_j} - \nu' \right) n_l \ge \left(\prod_{j=l}^{k-1} \frac{n_{j+1}}{n_j} - d \right) n_l.$$

If $k - l \ge \log_q(d+1)$, then

$$\prod_{j=l}^{k-1} \frac{n_{j+1}}{n_j} - d \ge q^{k-l} - d \ge 1$$

and hence we have

$$n_k\nu - n_l\nu' \ge n_l.$$

If $n_{j+1}/n_j > d+1$ for some $j \in \{l, \dots, k-1\}$, then we have

$$\prod_{j=l}^{k-1} \frac{n_{j+1}}{n_j} - d \ge 1,$$

and hence we have the same conclusion. Let us consider the case when $k-l < \log_q(d+1)$ and $n_{j+1}/n_j \le d+1$ for $j = l, \ldots, k-1$. Because of the condition (3.9), only finitely many values of n_{j+1}/n_j can belong to $[\,q, d+1]$. Hence

$$\left|\prod_{j=l}^{k-1} \frac{n_{j+1}}{n_j}\nu - \nu'\right|$$

can takes only finitely many values in this case, and thereby

$$D := \min\left\{ \left|\prod_{j=l}^{k-1} \frac{n_{j+1}}{n_j}\nu - \nu'\right| \neq 0 \;\middle|\; \frac{n_{j+1}}{n_j} \le d+1 \ (j = l, \ldots, k-1),\ k-l < \log_q(d+1) \right\}$$

is positive. Hence we have $|n_k\nu - n_l\nu'| \ge Dn_l$ in this case. □

To prove Proposition 3.1, we follow the method of martingale approximation given in [1], which originated with Berkes [3] and Philipp [14].

We take an arbitrary integer L and prove (3.2) for a.e. $x \in [\,L, L+1)$. Let us divide $\mathbf{N}$ into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \ldots$ satisfying ${}^{\#}\Delta_i' = [\,1 + 9\log_q i\,]$ and ${}^{\#}\Delta_i = i$. Denote $i^- = \min \Delta_i$ and $i^+ = \max \Delta_i$. We have

$$n_{i^-}/n_{(i-1)^+} \ge q^{9\log_q i} = i^9.$$

Put $\mu(i) = [\,\log_2 i^4 n_{i^+}\,] + 1$ and let $\mathcal{F}_i$ be a σ-field on $[\,L, L+1)$ defined by

$$\mathcal{F}_i = \sigma\{[\,L + j2^{-\mu(i)}, L + (j+1)2^{-\mu(i)}) \mid j = 0, \ldots, 2^{\mu(i)} - 1\}.$$

Note that $i^4 n_{i^+} \le 2^{\mu(i)} \le 2i^4 n_{i^+}$. Put

$$T_{a,b;d;i}(x) = \sum_{k\in\Delta_i} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x), \quad T'_{a,b;d;i}(x) = \sum_{k\in\Delta_i'} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x),$$

$$Y_{a,b;d;i} = E(T_{a,b;d;i} \mid \mathcal{F}_i) - E(T_{a,b;d;i} \mid \mathcal{F}_{i-1}).$$

Then $\{Y_{a,b;d;i}, \mathcal{F}_i\}$ forms a martingale difference sequence. Here let us prove

$$\|Y_{a,b;d;i} - T_{a,b;d;i}\|_\infty \le (\|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty + 2\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty)/i^3, \tag{3.10}$$

$$\|Y^2_{a,b;d;i} - T^2_{a,b;d;i}\|_\infty \le 3\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty(\|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty + 2\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty)/i^2, \tag{3.11}$$

$$\|Y^4_{a,b;d;i} - T^4_{a,b;d;i}\|_\infty \le 15\|\widetilde{\mathbf{1}}_{[a,b);d}\|^3_\infty(\|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty + 2\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty). \tag{3.12}$$

If $k \in \Delta_i$ and $x \in I = [\,L + j2^{-\mu(i)}, L + (j+1)2^{-\mu(i)}) \in \mathcal{F}_i$, then we have

$$\begin{aligned}
&|\widetilde{\mathbf{1}}_{[a,b);d}(n_k x) - E(\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,) \mid \mathcal{F}_i)| = \left| |I|^{-1} \int_I (\widetilde{\mathbf{1}}_{[a,b);d}(n_k x) - \widetilde{\mathbf{1}}_{[a,b);d}(n_k y))\,dy \right| \\
&\le \max_{y\in I} |\widetilde{\mathbf{1}}_{[a,b);d}(n_k x) - \widetilde{\mathbf{1}}_{[a,b);d}(n_k y))| \le \|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty n_k 2^{-\mu(i)} \le \|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty n_k/i^4 n_{i^+} \\
&\le \|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty/i^4.
\end{aligned}$$

Hence we obtain

$$|T_{a,b;d;i} - E(T_{a,b;d;i} \mid \mathcal{F}_i)| \le \|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty {}^\#\Delta_i/i^4 = \|\widetilde{\mathbf{1}}'_{[a,b);d}\|_\infty/i^3.$$

Take $J = [\,L + j2^{-\mu(i-1)}, L + (j+1)2^{-\mu(i-1)}) \in \mathcal{F}_{i-1}$. Then by Lemma 2.3, we have

$$\begin{aligned}|E(\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,) \mid \mathcal{F}_{i-1})| &= \left| |J|^{-1} \int_J \widetilde{\mathbf{1}}_{[a,b);d}(n_k y)\, dy \right| \le \|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty 2^{\mu(i-1)}/n_k \\ &\le \|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty 2(i-1)^4 n_{(i-1)^+}/n_{i^-} \le 2\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty/i^5.\end{aligned}$$

Thus $|E(T_{a,b;d;i} \mid \mathcal{F}_{i-1})| \le 2\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty {}^\#\Delta_i/i^5 = 2\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty/i^4$, and (3.10) is proved.

By $\|T_{a,b;d;i}\|_\infty \le i\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty$, we have

$$\|E(T_{a,b;d;i} \mid \mathcal{F}_i)\|_\infty,\ \|E(T_{a,b;d;i} \mid \mathcal{F}_{i-1})\|_\infty \le i\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty.$$

Hence we have

$$\begin{aligned}&\|Y_{a,b;d;i}\|_\infty \le 2i\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty, \quad \|Y_{a,b;d;i} + T_{a,b;d;i}\|_\infty \le 3i\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty, \\ &\|Y^2_{a,b;d;i} + T^2_{a,b;d;i}\|_\infty \le 5i^2\|\widetilde{\mathbf{1}}_{[a,b);d}\|^2_\infty.\end{aligned}$$

By applying these to $\|Y^2_{a,b;d;i} - T^2_{a,b;d;i}\|_\infty \le \|Y_{a,b;d;i} - T_{a,b;d;i}\|_\infty \|Y_{a,b;d;i} + T_{a,b;d;i}\|_\infty$ and $\|Y^4_{a,b;d;i} - T^4_{a,b;d;i}\|_\infty \le \|Y^2_{a,b;d;i} - T^2_{a,b;d;i}\|_\infty \|Y^2_{a,b;d;i} + T^2_{a,b;d;i}\|_\infty$, we have (3.11) and (3.12).

If we expand $T^2_{a,b;d;i}$ into trigonometric polynomial, the constant term is given by $v_{a,b;d;i} = \int_{\mathbf{R}} T^2_{a,b;d;i}(x)\mu_R(dx)$. Denote by $D > 0$ the infimum given in (3.1). The polynomial $T^2_{a,b;d;i} - v_{a,b;d;i}$ has at most $8(d+1)^2 i^2$ terms and the absolute values of frequencies are greater than Dn_{i^-}. Therefore, by Lemma 2.3 again, we have

$$|E(T^2_{a,b;d;i} - v_{a,b;d;i} \mid \mathcal{F}_{i-1})| \le 8(d+1)^2 i^2 (1/Dn_{i^-}) 2^{\mu(i-1)} = O(1/i^3).$$

Hence we have

$$\left\| \sum_{i=1}^M E(T^2_{a,b;d;i} \mid \mathcal{F}_{i-1}) - \beta_{a,b;d;M} \right\|_\infty = O(1), \tag{3.13}$$

where $\beta_{a,b;d;M} = \sum_{i=1}^M v_{a,b;d;i}$.

Since (3.11) implies $\big\|\sum_{i=1}^M \big(E(Y^2_{a,b;d;i} \mid \mathcal{F}_{i-1}) - E(T^2_{a,b;d;i} \mid \mathcal{F}_{i-1})\big)\big\|_\infty = O(1)$, we have

$$\left\| \sum_{i=1}^M E(Y^2_{a,b;d;i} \mid \mathcal{F}_{i-1}) - \beta_{a,b;d;M} \right\|_\infty = O(1). \tag{3.14}$$

Denote $l_M = M(M+1)/2$. By Lemma 2.2 we have

$$v_{a,b;d;i} \le v_{0,b-a;d;i} + iC_q \frac{\log d}{d}, \tag{3.15}$$

$$v_{a,b;d;i} \le C_q i, \tag{3.16}$$

$$\beta_{a,b;d;M} \le C_q l_M. \tag{3.17}$$

Now we use the following theorem by Monrad-Philipp [12] which is a version of Strassen's theorem [16].

Theorem 3.3. *Suppose that a square integrable martingale difference sequence* $\{\widehat{Y}_i, \widehat{\mathcal{F}}_i\}$ *satisfies*

$$\widehat{V}_M = \sum_{i=1}^{M} E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) \to \infty \quad a.s. \quad and \quad \sum_{i=1}^{\infty} E\big(\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2 \geq \psi(\widehat{V}_i)\}} / \psi(\widehat{V}_i)\big) < \infty$$

for some non-decreasing function ψ *with* $\psi(x) \to \infty$ $(x \to \infty)$ *such that* $\psi(x)(\log x)^{\alpha}/x$ *is non-increasing for some* $\alpha > 50$. *If there exists a uniformly distributed random variable* U *which is independent of* $\{\widehat{Y}_n\}$, *there exists a standard normal i.i.d.* $\{Z_i\}$ *such that*

$$\sum_{i\geq 1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i \leq t\}} = \sum_{i\leq t} Z_i + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big), \quad (t \to \infty) \quad a.s.$$

We prepare another probability space on which a uniform distributed random variable U and an i.i.d $\{\xi_k\}$ with $P(\xi_k = 1) = P(\xi_k = -1) = 1/2$ which is independent of U. Let $\mathcal{G}_i$ be a σ-field over this probability space which is generated by $\{\xi_k\}_{k\leq i^+}$. Put $\Xi_i = \sum_{k\in\Delta_i} \xi_k$.

We make a product of $[L, L+1)$ on which $\{Y_k\}$ is defined and this new probability space, and regard Y_k, U, and Ξ_k as random variables on this product probability space. Take $\varepsilon > 0$ arbitrarily and put

$$\widehat{Y}_{a,b;d;\varepsilon;i} = Y_{a,b;d;i} + \varepsilon\Xi_i, \quad \widehat{\mathcal{F}}_i = \mathcal{F}_i \otimes \mathcal{G}_i, \quad \widehat{V}_{a,b,d;\varepsilon;M} = \sum_{i=1}^{M} E(\widehat{Y}^2_{a,b,d;\varepsilon;i} \mid \widehat{\mathcal{F}}_{i-1}),$$
$$\widehat{v}_{a,b,d;\varepsilon;i} = v_{a,b;d;i} + \varepsilon^2 i, \quad \widehat{\beta}_{a,b,d;\varepsilon;M} = \beta_{a,b;d;M} + \varepsilon^2 l_M.$$

Clearly $\{\widehat{Y}_{a,b;d;\varepsilon;i}, \widehat{\mathcal{F}}_i\}$ is a martingale difference sequence.

By Lemma 2.4 and (3.12), we have

$$\|\widehat{Y}_{a,b;d;\varepsilon;i}\|_4 \leq \|Y_{a,b;d;i}\|_4 + \|\Xi_i\|_4 = \|T_{a,b;d;i}\|_4 + \|\Xi_i\|_4 + O(1) = O(i^{1/2}),$$

or

$$E\widehat{Y}^4_{a,b;d;\varepsilon;i} = O(i^2).$$

We have

$$E(\widehat{Y}^2_{a,b;d;\varepsilon;i} \mid \widehat{\mathcal{F}}_{i-1}) = E(Y^2_{a,b;d;i} \mid \mathcal{F}_{i-1}) + \varepsilon^2 i,$$

and hence

$$\widehat{V}_{a,b;d;\varepsilon;M} = \sum_{i=1}^{M} E(Y^2_{a,b;d;i} \mid \mathcal{F}_{i-1}) + \varepsilon^2 l_M \geq \varepsilon^2 l_M \to \infty.$$

We owe Aistleitner [2] this idea to prepare an independent rademacher i.i.d. to assure growth of $\widehat{V}_{a,b;d;\varepsilon;M}$. By (3.14), we have

$$\big\|\widehat{V}_{a,b;d;\varepsilon;M} - \widehat{\beta}_{a,b,d;\varepsilon;M}\big\|_\infty = O(1). \tag{3.18}$$

Hence by putting $\psi(x) = x/(\log x)^{51}$, we have

$$\sum_i E\Big(\frac{\widehat{Y}^2_{a,b;d;\varepsilon;i}\mathbf{1}_{\{\widehat{Y}^2_{a,b;d;\varepsilon;i}\geq\psi(\widehat{V}_{a,b;d;\varepsilon;i})\}}}{\psi(\widehat{V}_{a,b;d;\varepsilon;i})}\Big) \leq \sum_i \frac{E\widehat{Y}^4_{a,b;d;\varepsilon;i}}{\psi^2(\varepsilon^2 l_i)} = O\Big(\sum_i \frac{i^2(\log l_i)^{102}}{l_i^2}\Big) = O(1).$$

By (3.18) and $\widehat{V}_{a,b,d;\varepsilon;M} - \widehat{V}_{a,b,d;\varepsilon;M-1} \geq \varepsilon^2 M \to \infty$, we have $\widehat{V}_{a,b,d;\varepsilon;M-1} < \widehat{\beta}_{a,b,d;\varepsilon;M} < \widehat{V}_{a,b,d;\varepsilon;M+1}$ for large M. Hence $\widehat{V}_{a,b;d;\varepsilon;i} \leq \widehat{\beta}_{a,b;d;\varepsilon;M}$ is equivalent to $i \leq M-1$ or $i \leq M$. By $\|\widehat{Y}_{a,b;d;\varepsilon;i}\|_\infty = O(i)$ we have

$$\sum_{i\geq 1}\widehat{Y}_{a,b;d;\varepsilon;i}\mathbf{1}_{\{\widehat{V}_{a,b;d;\varepsilon;i}\leq\widehat{\beta}_{a,b;d;\varepsilon;M}\}} = \sum_{k=1}^M \widehat{Y}_{a,b;d;\varepsilon;k} + O(M) = \sum_{k=1}^M \widehat{Y}_{a,b;d;\varepsilon;k} + o(\phi(l_M)),$$

where $\phi(x) = \sqrt{x\log\log x}$. By (3.17) we have $\widehat{\beta}_{a,b;d;\varepsilon;M} = O(l_M)$. By applying Theorem 3.3 and putting $t = \widehat{\beta}_{a,b;d;\varepsilon;M}$, we have

$$\sum_{k=1}^M \widehat{Y}_{a,b;d;\varepsilon;k} = \sum_{i\geq 1}\widehat{Y}_{a,b;d;\varepsilon;i}\mathbf{1}_{\{\widehat{V}_{a,b;d;\varepsilon;i}\leq\widehat{\beta}_{a,b;d;\varepsilon;M}\}} + o(\phi(l_M)) = \sum_{i\leq\widehat{\beta}_{a,b;d;\varepsilon;M}} Z_i + o(\phi(l_M)), \quad \text{a.s.}$$

and

$$\sum_{k=1}^M \widehat{Y}_{0,b-a;d;\varepsilon;k} = \sum_{i\leq\widehat{\beta}_{0,b-a;d;\varepsilon;M}} Z'_i + o(\phi(l_M)), \quad \text{a.s.}$$

where $\{Z'_i\}$ is another standard normal i.i.d. By (3.15), (3.16), and (3.17), we have $\widehat{v}_{a,b;d;\varepsilon;i} \leq \widehat{v}_{0,b-a;d;\varepsilon;i} + iC_q(\log d)/d$, $\varepsilon^2 i \leq \widehat{v}_{a,b;d;\varepsilon;i} \leq (C_q+\varepsilon^2)i$, $\varepsilon^2 l_M \leq \widehat{\beta}_{a,b;d;\varepsilon;M} \leq (C_q+\varepsilon^2)l_M$. By applying Lemma 2.5, we have

$$\varlimsup_{M\to\infty}\frac{1}{\phi(l_M)}\Big|\sum_{i\leq\widehat{\beta}_{a,b;d;\varepsilon;M}} Z_i\Big| \leq \varlimsup_{M\to\infty}\frac{1}{\phi(l_M)}\Big|\sum_{i\leq\widehat{\beta}_{0,b-a;d;\varepsilon;M}} Z'_i\Big| + \sqrt{C_q\frac{\log d}{d}},$$

where both limsups are constants. Hence there exists a constant $C(a,b;d;\varepsilon)$ such that

$$\varlimsup_{M\to\infty}\frac{1}{\phi(l_M)}\Big|\sum_{k=1}^M \widehat{Y}_{a,b;d;\varepsilon;k}\Big| = \varlimsup_{M\to\infty}\frac{1}{\phi(l_M)}\Big|\sum_{i\leq\widehat{\beta}_{a,b;d;\varepsilon;M}} Z_i\Big| = C(a,b;d;\varepsilon), \quad \text{a.s.} \tag{3.19}$$

satisfying an inequality below:

$$C(a,b;d;\varepsilon) \leq C(0,b-a;d;\varepsilon) + \sqrt{C_q\frac{\log d}{d}}. \tag{3.20}$$

Clearly we have

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k=1}^{M} \varepsilon \Xi_k\Big| = \varepsilon, \quad \text{a.s.}$$

Hence dividing

$$\Big|\sum_{k=1}^{M} Y_{a,b;d;k}\Big| - \Big|\sum_{k=1}^{M} \varepsilon\Xi_k\Big| \le \Big|\sum_{k=1}^{M} \widehat{Y}_{a,b;d;\varepsilon;k}\Big| \le \Big|\sum_{k=1}^{M} Y_{a,b;d;k}\Big| + \Big|\sum_{k=1}^{M} \varepsilon\Xi_k\Big|$$

by $\phi(l_M)$ and taking limsup, we have

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k=1}^{M} Y_{a,b;d;k}\Big| - \varepsilon \le C(a,b;d;\varepsilon) \le \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k=1}^{M} Y_{a,b;d;k}\Big| + \varepsilon, \quad \text{a.s.}$$

By letting $\varepsilon \to 0$ we have

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k=1}^{M} Y_{a,b;d;k}\Big| = \lim_{\varepsilon\to 0} C(a,b;d;\varepsilon) := C(a,b;d), \quad \text{a.s.}$$

By (3.20), we have (3.3). By noting (3.10), we have

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k=1}^{M} T_{a,b;d;k}\Big| = \varlimsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k=1}^{M} Y_{a,b;d;k}\Big| = C(a,b;d), \quad \text{a.s.} \tag{3.21}$$

Since this conclusion is valid over the product probability space with probability 1, by applying Fubini's theorem we see that it is valid on $[L, L+1)$, a.e.

We here apply (1.2) and Koksma's inequality by noting $\sum_{i=1}^{M}[1+9\log_q i] = O(M\log M)$, we have

$$\Big|\sum_{k=1}^{M} T'_{a,b;d;k}\Big| = O\big(\sqrt{M\log M \log\log(M\log M)}\big) = o(\sqrt{l_M}),$$

and thereby

$$\varlimsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k=1}^{M} T'_{a,b;d;k}\Big| = 0, \quad \text{a.e.}$$

This together with $M^+ = l_M + \sum_{i=1}^{M}[1+9\log_q i] \sim l_M$ and (3.21) implies

$$\varlimsup_{M\to\infty} \frac{1}{\phi(M^+)}\Big|\sum_{i=1}^{M} \sum_{k\in\Delta_i'\cup\Delta_i} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\Big| = C(a,b;d), \quad \text{a.e.}$$

Moreover we have $\sum_{k\in\Delta_M'\cup\Delta_M} \|\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\|_\infty = o(\phi(M^+))$, and hence

$$\varlimsup_{N\to\infty} \frac{1}{\phi(N)}\Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\Big| = C(a,b;d), \quad \text{a.e.}$$

Here we prove (3.4).

We prove

$$\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T_{a,b;d;i}(x)\Bigr)^2 \mu_R(x) = \beta_{a,b;d;M} + O(1), \tag{3.22}$$

$$\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T'_{a,b;d;i}(x)\Bigr)^2 \mu_R(x) = o(l_M). \tag{3.23}$$

If $j \in \Delta_i$ and $k \in \Delta_{i+1}$, then $n_j/n_k \le 1/q^{9\log_q i} = i^{-9}$. Thus we have

$$\Bigl|\int_{\mathbf{R}} T_{a,b;d;i}(x) T_{a,b;d;i+1}(x) \mu_R(dx)\Bigr| \le {}^{\#}\Delta_i {}^{\#}\Delta_{i+1} O(i^{-9}) = O(i^{-7})$$

by (2.6). If $j \in \Delta_i$ and $k \in \Delta_{i+l}$ $(l \ge 2)$, we have $k - j \ge {}^{\#}\Delta_{i+1} + \cdots + {}^{\#}\Delta_{i+l-1} \ge ((i+1) + (i+l-1))/2$, and $n_j/n_k \le 1/q^{i/2+(i+l)/2}$. Thereby

$$\Bigl|\int_{\mathbf{R}} T_{a,b;d;i}(x) T_{a,b;d;i+l}(x) \mu_R(dx)\Bigr| \le \frac{i(i+l)}{q^{i/2+(i+l)/2}}.$$

Hence we have

$$\sum_{i=1}^{\infty}\sum_{l=1}^{\infty} \Bigl|\int_{\mathbf{R}} T_{a,b;d;i}(x) T_{a,b;d;i+l}(x) \mu_R(dx)\Bigr| < \infty,$$

which implies

$$\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T_{a,b;d;i}(x)\Bigr)^2 \mu_R(x) = \sum_{i=1}^{M} \int_{\mathbf{R}} T^2_{a,b;d;i}(x)\, \mu_R(x) + O(1) = \beta_{a,b;d;M} + O(1).$$

If $j \in \Delta'_i$ and $k \in \Delta'_{i+l}$ $(l \ge 1)$, then $k - j \ge {}^{\#}\Delta_i + \cdots + {}^{\#}\Delta_{i+l-1} \ge (i + (i+l-1))/2$. In the same way as before, we have

$$\sum_{i=1}^{\infty}\sum_{l=1}^{\infty} \Bigl|\int_{\mathbf{R}} T'_{a,b;d;i}(x) T'_{a,b;d;i+l}(x) \mu_R(dx)\Bigr| \le O\Bigl(\sum_{i=1}^{\infty}\sum_{l=1}^{\infty} \frac{\log_q i \log_q(i+l)}{q^{i/2+(i+l-1)/2}}\Bigr) = O(1),$$

and

$$\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T'_{a,b;d;i}(x)\Bigr)^2 \mu_R(x) = \sum_{i=1}^{M} \int_{\mathbf{R}} T'^2_{a,b;d;i}(x)\, \mu_R(x) + O(1) = O\bigl(M(\log_q M)^2\bigr) = o(l_M).$$

By

$$\Bigl(\frac{1}{l_M}\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T_{a,b;d;i}(x)\Bigr)^2 \mu_R(x)\Bigr)^{1/2} - \Bigl(\frac{1}{l_M}\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T'_{a,b;d;i}(x)\Bigr)^2 \mu_R(x)\Bigr)^{1/2}$$

$$\le \Bigl(\frac{1}{l_M}\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T_{a,b;d;i}(x) + T'_{a,b;d;i}(x)\Bigr)^2 \mu_R(x)\Bigr)^{1/2}$$

$$\le \Bigl(\frac{1}{l_M}\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T_{a,b;d;i}(x)\Bigr)^2 \mu_R(x)\Bigr)^{1/2} + \Bigl(\frac{1}{l_M}\int_{\mathbf{R}} \Bigl(\sum_{i=1}^{M} T'_{a,b;d;i}(x)\Bigr)^2 \mu_R(x)\Bigr)^{1/2},$$

we have

$$\int_{\mathbf{R}}\Big(\sum_{i=1}^{M} T_{a,b;d;i}(x)+T'_{a,b;d;i}(x)\Big)^2\,\mu_R(x)=\beta_{a,b;d;M}+o(l_M)=\widehat{\beta}_{a,b;d;\varepsilon;M}-\varepsilon^2 l_M+o(l_M).$$

By definition of $\underline{\sigma}^2_{a,b;d}$ and $\overline{\sigma}^2_{a,b;d}$, we have

$$\underline{\sigma}^2_{a,b;d}l_M+o(l_M)\le\int_{\mathbf{R}}\Big(\sum_{i=1}^{M} T_{a,b;d;i}(x)+T'_{a,b;d;i}(x)\Big)^2\,\mu_R(x)\le\overline{\sigma}^2_{a,b;d}l_M+o(l_M),$$

and hence we have

$$(\underline{\sigma}^2_{a,b;d}+\varepsilon^2)l_M+o(l_M)\le\widehat{\beta}_{a,b;d;\varepsilon;M}\le(\overline{\sigma}^2_{a,b;d}+\varepsilon^2)l_M+o(l_M).$$

By (3.19), we have

$$(\underline{\sigma}^2_{a,b;d}+\varepsilon^2)^{1/2}\le C(a,b;d;\varepsilon)\le(\overline{\sigma}^2_{a,b;d}+\varepsilon^2)^{1/2},$$

we have (3.4) by letting $\varepsilon\downarrow 0$.

Now we assume (3.1) for all $d\ge 3$. We apply the fundamental proposition below:

Proposition 3.4 ([10]). *Let $\{n_k\}$ be a sequence of real numbers satisfying*

$$n_1\ne 0,\qquad |n_{k+1}/n_k|\ge q>1\quad(k=1,2,\dots), \tag{3.24}$$

and ϖ be a permutation of $\mathbf{N}$, i.e., a bijection $\mathbf{N}\to\mathbf{N}$. Then for any dense countable set $S\subset[0,1)$, we have

$$\begin{aligned}\varlimsup_{N\to\infty}\frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}}&=\sup_{S\ni a<b\in S}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)}x)\Big|\\&=\sup_{0\le a<b<1}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)}x)\Big|,\\\varlimsup_{N\to\infty}\frac{ND^*_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}}&=\sup_{a\in S}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}(n_{\varpi(k)}x)\Big|\\&=\sup_{0\le a<1}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}(n_{\varpi(k)}x)\Big|,\end{aligned}\tag{3.25}$$

for almost every $x\in\mathbf{R}$. If we denote the d-th subsum of the Fourier series of $\widetilde{\mathbf{1}}_{[a,b)}$ by $\widetilde{\mathbf{1}}_{[a,b);d}$, we have

$$\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)}x)\Big|=\lim_{d\to\infty}\varlimsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)}x)\Big|$$

for almost every $x\in\mathbf{R}$.

By applying the proposition for $\varpi(k) = k$, we have

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \sup_{0\le a<1} \lim_{d\to\infty} C(0,a;d), \quad \text{a.e.},$$

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \sup_{0\le a<b<1} \lim_{d\to\infty} C(a,b;d), \quad \text{a.e.}$$

Clearly we have

$$\sup_{0\le a<1} \lim_{d\to\infty} C(0,a;d) \le \sup_{0\le a<b<1} \lim_{d\to\infty} C(a,b;d),$$

and by (3.3) we have

$$\lim_{d\to\infty} C(a,b;d) \le \lim_{d\to\infty} C(0,b-a;d),$$

and hence we have

$$\sup_{0\le a<1} \lim_{d\to\infty} C(0,a;d) = \sup_{0\le a<b<1} \lim_{d\to\infty} C(a,b;d) := \Sigma,$$

which shows (3.5). By (3.4), we have

$$\varlimsup_{d\to\infty} \underline{\sigma}_{a,b;d} \le \lim_{d\to\infty} C(a,b;d) \le \varliminf_{d\to\infty} \overline{\sigma}_{a,b;d},$$

which implies (3.6). The formula (3.8) is clear from this inequality and (3.7).

§ 4. Random ratio

We prove Theorem 1.2. By the condition (1.6), we can verify Hadamard's gap condition (1.1) for $\{n_k\}$. We can also verify that the condition (3.9) holds. Actually, all n_{j+1}/n_j belongs to the set $\{b/a \mid a \in A, b \in B\}$. By (1.6), we see that A is a finite set, and for fixed $a \in A$, the number of b satisfying $b/a \in [\,q,Q\,]$ is finite. Hence $\{b/a \in [\,q,Q\,] \mid a \in A, b \in B\}$ is proved to be finite set and the condition (3.9) is verified.

By applying Lemma 3.2 and Proposition 3.1, we see that there exists a Σ such that (3.5) holds.

We put

$$\sigma^2(f;\mathcal{L}(X_1,Y_1)) = E\left(\int_0^1 f^2(x)\,dx + 2\sum_{k=1}^{\infty}\int_0^1 f(X_1\dots X_k x) f(Y_1\dots Y_k x)\,dx\right)$$

and verify that it is well defined for $f = \widetilde{\mathbf{1}}_{[a,b);d}$ and $f = \widetilde{\mathbf{1}}_{[a,b)}$. We first note

$$\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\widetilde{\mathbf{1}}_{[a,b);d}(n_{k+l}x)\mu_R(dx) = \int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}((n_{k+l}/n_k)x)\mu_R(dx)$$

$$= \int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}\left(\frac{Y_{k+1}\dots Y_{k+l}}{X_{k+1}\dots X_{k+l}}x\right)\mu_R(dx)$$

$$= \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_{k+1}\dots Y_{k+l}x)\,dx.$$

By applying the inequality (2.6) to the above formula, we have

$$\left|\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_{k+1}\dots Y_{k+l}x)\,dx\right| \le 2\frac{n_k}{n_{k+l}} \le \frac{2}{q^l}. \tag{4.1}$$

Putting $k = 0$ and by noting that we naturally have $n_0 = 1$, we see that the series appearing in the definition of $\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\mathcal{L}(X_1,Y_1))$ is absolutely convergent and $\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\mathcal{L}(X_1,Y_1))$ is well defined. By noting (2.1), $|\widetilde{V}(x,\xi,y,\eta)| \le 1$, and $X_1\dots X_k \ge q^k$, we see that the series appearing in the definition of $\sigma^2(\widetilde{\mathbf{1}}_{[a,b)};\mathcal{L}(X_1,Y_1))$ is also absolutely convergent and $\sigma^2(\widetilde{\mathbf{1}}_{[a,b)};\mathcal{L}(X_1,Y_1))$ is well defined.

Since $\widetilde{\mathbf{1}}_{[a,b);d}$ converges to $\widetilde{\mathbf{1}}_{[a,b)}$ in L^2-sense, we have

$$\begin{aligned}\lim_{d\to\infty}&\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_{k+1}\dots Y_{k+l}x)\,dx\\ &= \int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b)}(Y_{k+1}\dots Y_{k+l}x)\,dx.\end{aligned}$$

Because each term is bounded by $1/q^l$ which is summable in l, by applying Lebesgue's convergence theorem for series, we have

$$\begin{aligned}\lim_{d\to\infty}&\sum_{k=1}^{\infty}\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_{k+1}\dots Y_{k+l}x)\,dx\\ &= \sum_{k=1}^{\infty}\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b)}(Y_{k+1}\dots Y_{k+l}x)\,dx.\end{aligned}$$

Since the absolute value of the above series is bounded by $\sum_k 2/q^k < \infty$, by taking the expectation and by applying bounded convergence theorem, we have

$$\lim_{d\to\infty}\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\mathcal{L}(X_1,Y_1)) = \sigma^2(\widetilde{\mathbf{1}}_{[a,b)};\mathcal{L}(X_1,Y_1)). \tag{4.2}$$

Hence we have verified the second formula of (3.7) for $\sigma^2_{a,b;d} = \sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\mathcal{L}(X_1,Y_1))$ and $\sigma^2_{a,b} = \sigma^2(\widetilde{\mathbf{1}}_{[a,b)};\mathcal{L}(X_1,Y_1))$. Now we verify that the first formula of (3.7) holds with probability one. By applying the above formula, we have

$$\begin{aligned}&\frac{1}{N}\int_{\mathbf{R}}\Big(\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\Big)^2\mu_R(dx)\\ &= \frac{1}{N}\sum_{k=1}^{N}\int_{\mathbf{R}}\widetilde{\mathbf{1}}^2_{[a,b);d}(n_kx)\mu_R(dx) + \frac{2}{N}\sum_{l=1}^{N-1}\sum_{k=1}^{N-l}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(n_kx)\widetilde{\mathbf{1}}_{[a,b);d}(n_{k+l}x)\mu_R(dx)\\ &= \int_0^1\widetilde{\mathbf{1}}^2_{[a,b);d}(x)\,dx + 2\sum_{l=1}^{N-1}\frac{1}{N}\sum_{k=1}^{N-l}\int_0^1\widetilde{\mathbf{1}}_{[a,b);d}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_{k+1}\dots Y_{k+l}x)\,dx\\ &\to \int_0^1\widetilde{\mathbf{1}}^2_{[a,b);d}(x)\,dx + 2\sum_{l=1}^{\infty}E\int_0^1\widetilde{\mathbf{1}}_{[a,b);d}(X_1\dots X_lx)\widetilde{\mathbf{1}}_{[a,b);d}(Y_1\dots Y_lx)\,dx.\end{aligned}$$

The last limiting procedure is verified in the following way. Firstly, by the law of large numbers, we have

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N-l}\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_{k+1}\dots Y_{k+l}x)\,dx \\ \to E\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(X_1\dots X_l x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_1\dots Y_l x)\,dx, \quad \text{a.s.}\end{aligned}$$

since the summands form a sequence of l-dependent identically distributed sequence of random variables. Secondly, by (4.1), we have

$$\left|\frac{1}{N}\sum_{k=1}^{N-l}\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(X_{k+1}\dots X_{k+l}x)\widetilde{\mathbf{1}}_{[a,b);d}(Y_{k+1}\dots Y_{k+l}x)\,dx\right| \le \frac{2}{q^l},$$

where the right hand side is summable in l. Hence by Lebesgue's convergence theorem for series, we can verify the last limiting procedure.

Since $\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\mathcal{L}(X_1,Y_1))$ depends only on a, b, d, and $\mathcal{L}(X_1,Y_1)$, by (3.7) and (3.8) we see that Σ depends only on $\mathcal{L}(X_1,Y_1)$.

Finally we prove (1.9) by assuming that A and B consist of odd numbers. Suppose that P and Q are positive odd integers. In this case we have $\langle P/2\rangle = \langle Q/2\rangle = 1/2$. By (2.2), we have

$$\widetilde{V}(\langle Pa\rangle,\langle Pb\rangle,\langle Qa\rangle,\langle Qb\rangle) \le \frac{1}{4}$$

and see that the equality holds if $a=0$ and $b=1/2$.

By (2.1) we see that $\sigma^2(\widetilde{\mathbf{1}}_{[a,b)};\mathcal{L}(X_1,Y_1))$ takes its maximum value when $a=0$ and $b=1/2$. Hence we have

$$\Sigma^2 = \sigma^2(\widetilde{\mathbf{1}}_{[0,1/2)};\mathcal{L}(X_1,Y_1)) = E\left(\frac{1}{4}+\sum_{k=1}^{\infty}\frac{2}{4X_1\dots X_kY_1\dots Y_k}\right) = \frac{1}{4}+\sum_{k=1}^{\infty}\frac{1}{2\Upsilon^k_{\mathcal{L}(X_1,Y_1)}},$$

which shows (1.9).

§ 5. Periodic ratio

We prove Theorem 1.3 by applying Proposition 3.1. In this case the condition (3.9) is clearly satisfied, and then by Lemma 3.2, the condition (3.1) can be verified for all $d \ge 3$. Since the sequence clearly satisfies Hadamard's gap condition (1.1), we can verify that (3.5) holds for some constant Σ. We evaluate $\sigma_{a,b;d}$ and $\sigma_{a,b}$ in (3.7). Denote $\theta_* = \theta_1\dots\theta_\tau$,

$$g_d(x) = \widetilde{\mathbf{1}}_{[a,b);d}(x) + \widetilde{\mathbf{1}}_{[a,b);d}(\theta_1 x) + \widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\theta_2 x) + \cdots + \widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{\tau-1}x),$$

and

$$\sigma^2(f;\theta_1,\dots,\theta_\tau)$$
$$= -\int_{\mathbf{R}} f^2(x)\,\mu_R(dx) + 2\sum_{l=0}^{\infty}\left(\begin{array}{l}\displaystyle\int_{\mathbf{R}} f(x)f(\theta_*^l x)\,\mu_R(dx)\\ \displaystyle+\frac{1}{\tau}\sum_{1\le j<k\le\tau}\Big(\int_{\mathbf{R}} f(x)f(\theta_*^l\theta_j\dots\theta_{k-1}x)\,\mu_R(dx)\\ \displaystyle\qquad+\int_{\mathbf{R}} f(x)f(\theta_*^l\theta_1\dots\theta_{j-1}\theta_k\dots\theta_\tau x)\,\mu_R(dx)\Big)\end{array}\right),$$

where we obey the convention $\theta_j\dots\theta_{k-1}=1$ if $j=k$. Since we have

$$\begin{aligned}\int_{\mathbf{R}} g_d^2(x)\,\mu_R(dx) &= \sum_{1\le j=k\le\tau}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{j-1}x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{k-1}x)\,\mu_R(dx)\\ &\quad+2\sum_{1\le j<k\le\tau}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{j-1}x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{k-1}x)\,\mu_R(dx)\\ &=\tau\int_{\mathbf{R}}\widetilde{\mathbf{1}}^2_{[a,b);d}(x)\,\mu_R(dx)\\ &\quad+2\sum_{1\le j<k\le\tau}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_j\dots\theta_{k-1}x)\,\mu_R(dx),\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbf{R}} g_d(x)g_d(\theta_*^l x)\,\mu_R(dx)\\ &=\sum_{1\le j=k\le\tau}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{j-1}x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_*^l\theta_1\dots\theta_{k-1}x)\,\mu_R(dx)\\ &\quad+\sum_{1\le j<k\le\tau}\Big(\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{j-1}x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_*^l\theta_1\dots\theta_{k-1}x)\,\mu_R(dx)\\ &\qquad+\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_1\dots\theta_{k-1}x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_*^l\theta_1\dots\theta_{j-1}x)\,\mu_R(dx)\Big)\\ &=\tau\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_*^l x)\,\mu_R(dx)\\ &\quad+\sum_{1\le j<k\le\tau}\Big(\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_*^l\theta_j\dots\theta_{k-1}x)\,\mu_R(dx)\\ &\qquad+\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_*^{l-1}\theta_1\dots\theta_{j-1}\theta_k\dots\theta_\tau x)\,\mu_R(dx)\Big),\end{aligned}$$

we can verify

$$\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\theta_1,\dots,\theta_\tau)=\frac{1}{\tau}\Big(\int_{\mathbf{R}} g_d^2(x)\,\mu_R(dx)+2\sum_{l=1}^{\infty}\int_{\mathbf{R}} g_d(x)g_d(\theta_*^l x)\,\mu_R(dx)\Big). \tag{5.1}$$

By applying (2.6), we have

$$\left|\int_{\mathbf{R}} g_d(x)g_d(\theta_*^l x)\,\mu_R(dx)\right| \leq \frac{2\tau^2}{\theta_*^{l-1}},$$

we see that the series converges absolutely and $\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\theta_1,\ldots,\theta_\tau)$ is well defined. We also apply Lebesgue's convergence theorem when $d\to\infty$ to have

$$\begin{aligned}(5.2)\qquad &\lim_{d\to\infty}\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\theta_1,\ldots,\theta_\tau)\\ &= \frac{1}{\tau}\biggl(\lim_{d\to\infty}\int_{\mathbf{R}} g_d^2(x)\,\mu_R(dx) + 2\sum_{l=1}^{\infty}\lim_{d\to\infty}\int_{\mathbf{R}} g_d(x)g_d(\theta_*^l x)\,\mu_R(dx)\biggr).\end{aligned}$$

We can verify (3.7) for $\sigma^2_{a,b;d} = \sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\theta_1,\ldots,\theta_\tau)$ as below:

$$\begin{aligned}\frac{1}{N\tau}\int_{\mathbf{R}}\biggl(\sum_{k=1}^{N\tau}\widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\biggr)^2\mu_R(dx) &= \frac{1}{N\tau}\int_{\mathbf{R}}\biggl(\sum_{k=1}^{N} g_d(\theta_*^{k-1}x)\biggr)^2\mu_R(dx)\\ &= \int_{\mathbf{R}} g_d^2(x)\,\mu_R(dx) + \sum_{l=1}^{N-1}\frac{N-l}{N}\int_{\mathbf{R}} g_d(x)g_d(\theta_*^l x)\,\mu_R(dx)\\ &\to \sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\theta_1,\ldots,\theta_\tau).\end{aligned}$$

By denoting

$$H(\Theta) = \lim_{d\to\infty}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(x)\widetilde{\mathbf{1}}_{[a,b);d}(\Theta x)\,\mu_R(dx),$$

we have

$$\begin{aligned}\sigma^2_{a,b} &= \lim_{d\to\infty}\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d};\theta_1,\ldots,\theta_\tau)\\ &= -H(1) + 2\sum_{l=0}^{\infty}\biggl(H(\theta_*^l) + \frac{1}{\tau}\sum_{1\leq j<k\leq\tau}\bigl(H(\theta_*^l\theta_j\ldots\theta_{k-1}) + H(\theta_*^l\theta_1\ldots\theta_{j-1}\theta_k\ldots\theta_\tau)\bigr)\biggr).\end{aligned}$$

When $\tau = 2$, we have

$$\sigma^2_{a,b} = -H(1) + 2\sum_{l=0}^{\infty}\biggl(H(\theta_*^l) + \frac{1}{2}\bigl(H(\theta_*^l\theta_1) + H(\theta_*^l\theta_2)\bigr)\biggr),$$

and we see that it is invariant under substitution of θ_1 and θ_2. It proves (1.11).

When $\tau = 3$, we have

$$\begin{aligned}\sigma^2_{a,b} = &-H(1) + 2\sum_{l=0}^{\infty}\biggl(H(\theta_*^l)\\ &+ \frac{H(\theta_*^l\theta_1) + H(\theta_*^l\theta_2) + H(\theta_*^l\theta_3) + H(\theta_*^l\theta_1\theta_2) + H(\theta_*^l\theta_2\theta_3) + H(\theta_*^l\theta_3\theta_1)}{3}\biggr),\end{aligned}$$

and we see that it is invariant under permutation among θ_1, θ_2, and θ_3. It proves (1.12).

We consider the case when $\theta_j = p_j/q_j$ for odd $p_j \in B$ and $q_j \in A$. Put $s_j = p_j q_j$, $s_* = s_1 \dots s_\tau$, $p_* = p_1 \dots p_\tau$, and $q_* = q_1 \dots q_\tau$. In this case

$$\begin{aligned} H(\theta_*^l \theta_j \dots \theta_{k-1}) &= \lim_{d\to\infty} \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(p_*^l p_j \dots p_{k-1} x) \widetilde{\mathbf{1}}_{[a,b);d}(q_*^l q_j \dots q_{k-1} x)\, dx \\ &= \int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(p_*^l p_j \dots p_{k-1} x) \widetilde{\mathbf{1}}_{[a,b)}(q_*^l q_j \dots q_{k-1} x)\, dx \\ &= \frac{\widetilde{V}(\langle p_*^l p_j \dots p_{k-1} a\rangle, \langle p_*^l p_j \dots p_{k-1} b\rangle, \langle q_*^l q_j \dots q_{k-1} a\rangle, \langle q_*^l q_j \dots q_{k-1} b\rangle)}{s_*^l s_j \dots s_{k-1}} \\ &\le \frac{1}{4 s_*^l s_j \dots s_{k-1}}, \end{aligned}$$

where equality holds when $a = 0$ and $b = 1/2$. Similarly we have

$$\begin{aligned} H(\theta_*^l \theta_1 \dots \theta_{j-1} \theta_k \dots \theta_\tau) &= \frac{\widetilde{V}\begin{pmatrix} \langle p_*^l p_1 \dots p_{j-1} p_k \dots p_\tau a\rangle, \langle p_*^l p_1 \dots p_{j-1} p_k \dots p_\tau b\rangle, \\ \langle q_*^l q_1 \dots q_{j-1} q_k \dots q_\tau a\rangle, \langle q_*^l q_1 \dots q_{j-1} q_k \dots q_\tau b\rangle \end{pmatrix}}{s_*^l s_1 \dots s_{j-1} s_k \dots s_\tau} \\ &\le \frac{1}{4 s_*^l s_1 \dots s_{j-1} s_k \dots s_\tau}, \end{aligned}$$

where equality holds when $a = 0$ and $b = 1/2$. Hence we see that $\sigma_{a,b}$ takes its maximum at $a = 0$ and $a = 1/2$ and its maximum equals to

$$\begin{aligned} &\frac{1}{4}\Bigl(-1 + 2\sum_{l=0}^\infty \frac{1}{s_*^l} + \frac{2}{\tau}\sum_{l=0}^\infty \sum_{1\le j<k\le\tau} \Bigl(\frac{1}{s_*^l s_j \dots s_{k-1}} + \frac{1}{s_*^l s_1 \dots s_{j-1} s_k \dots s_\tau}\Bigr)\Bigr) \\ &= \frac{1}{4(s_*-1)}\Bigl(1 + s_* + \frac{2}{\tau} s_* \sum_{1\le j<k\le\tau} \Bigl(\frac{1}{s_j \dots s_{k-1}} + \frac{1}{s_1 \dots s_{j-1} s_k \dots s_\tau}\Bigr)\Bigr) \\ &= \frac{1}{4(s_*-1)}\Bigl(1 + s_* + \frac{2}{\tau} \sum_{1\le j<k\le\tau} \Bigl(s_j \dots s_{k-1} + s_1 \dots s_{j-1} s_k \dots s_\tau\Bigr)\Bigr). \end{aligned}$$

§ 6. Union of geometric progressions

We prove Theorem 1.4 by assuming $\theta_1 < \dots < \theta_\tau$. First we recall an outline of the proof of Theorem 1.1. For a function of bounded variation over the unit interval with period 1 satisfying $\int_0^1 f = 0$, we define

$$\sigma^2(f,\theta) = \begin{cases} \int_0^1 f^2(y)\, dy & \text{if } \theta \text{ satisfies (1.4)}, \\ \int_0^1 f^2(y)\, dy + 2\sum_{k=1}^\infty \int_0^1 f(p^k y) f(q^k y)\, dy & \text{if } \theta \text{ is given by (1.5)}. \end{cases}$$

We have

$$\lim_{N\to\infty}\frac{1}{N}\int_{\mathbf{R}}\Big(\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(\theta^k x)\Big)^2\mu_R(dx)=\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d},\theta), \tag{6.1}$$

$$\lim_{d\to\infty}\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta)=\sigma(\widetilde{\mathbf{1}}_{[a,b)},\theta),\quad \text{uniformly in } a \text{ and } b \tag{6.2}$$

$$\Sigma_\theta=\sup_{0\le a<b<1}\sigma(\widetilde{\mathbf{1}}_{[a,b)},\theta)=\sigma(\widetilde{\mathbf{1}}_{[0,1/2)},\theta), \tag{6.3}$$

when θ satisfies (1.4) or is given by (1.5) with odd p and q. Details are given in section 2 of [8]. See also [6].

Fix a positive integer d arbitrarily. Let $1\le i,\ j\le\tau$ and $1\le\nu,\ \nu'\le d$, and denote

$$M_{i;j,\nu,\nu'}=\{k\in\mathbf{N}\mid|\log_{\theta_j}(\nu\theta_i^k/\nu'\theta_j^l)|\ge 1/2\tau d^3 \text{ for all } l\in\mathbf{Z}\}.$$

The condition defining $M_{i;j,\nu,\nu'}$ is equivalent to

$$\langle k\log_{\theta_j}\theta_i+(\log_{\theta_j}\nu-\log_{\theta_j}\nu')\rangle^*\ge 1/2\tau d^3$$

where $\langle x\rangle^*$ denotes the distance between x and the nearest integer of x. Since $\log_{\theta_j}\theta_i$ is irrational, we see that

$$^\#\big(M_{i;j,\nu,\nu'}\cap[1,N]\big)\sim(1-1/\tau d^3)N,\quad (N\to\infty).$$

Denote

$$M_i=\bigcap_{j\ne i}\bigcap_{1\le\nu,\nu'\le d}M_{i;j,\nu,\nu'}\quad\text{and}\quad R_i=M_i^c.$$

We divide the sequence $\{\theta_i^k\}_{k\in\mathbf{N}}$ into the main part $\{\theta_i^k\}_{k\in M_i}$ and the remainder part $\{\theta_i^k\}_{k\in R_i}$. Let $\{n_k^\circ\}$ be an arrangement in increasing order of

$$\bigcup_{i=1}^{\tau}\{\theta_i^k\}_{k\in M_i}.$$

Denote the sequence $\{\theta_i^j\}_{j\in R_i}$ simply by $\{n_k^{(i)}\}_{k\in\mathbf{N}}$.

By noting $\log_{\theta_j}x<\log_{\theta_1}x$ $(x>0)$, by definition we have $\log_{\theta_1}(\nu n_k^\circ/\nu' n_l^\circ)\ge 1/2\tau d^3$ if $k,\ l\in\mathbf{N}$, $1\le\nu,\nu'\le d$ and $\nu n_k^\circ/\nu' n_l^\circ>1$. If we consider the special case $\nu=\nu'=1$, we have $n_k^\circ/n_l^\circ\ge\theta_1^{1/2\tau d^3}$ if $n_k^\circ/n_l^\circ>1$. It implies that the sequence $\{n_k^\circ\}$ satisfies Hadamard's gap condition with $q=\theta_1^{1/2\tau d^3}$. Moreover we have

$$\nu n_k^\circ-\nu' n_l^\circ\ge(\theta_1^{1/2\tau d^3}-1)\nu' n_l^\circ\ge(\theta_1^{1/2\tau d^3}-1)n_{l\wedge k}^\circ$$

if $\nu n_k^\circ>\nu' n_l^\circ$. It verifies the condition (3.1) for $\{n_k^\circ\}$. Hence we can apply the first half of Proposition 3.1 and see that there exists a real number $C(a,b;d)$ such that

$$\varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k^\circ x)\Big|=C(a,b;d),\quad \text{a.e.} \tag{6.4}$$

Denote

$$M_i(t) = \{k \in M_i \mid \theta_i^k \le t\}, \quad R_i(t) = \{k \in R_i \mid \theta_i^k \le t\},$$
$$N_i^\circ(t) = {}^\# M_i(t), \quad N^{(i)}(t) = {}^\# R_i(t), \quad N_i(t) = {}^\#\{k \in \mathbf{N} \mid \theta_i^k \le t\} \sim \log t / \log \theta_i.$$

We clearly have

$$\varliminf_{N\to\infty} \frac{{}^\#\big(M_i \cap [\,1, N\,]\big)}{N} \ge (I_d^\circ)^{-2} \quad \text{and} \quad \varlimsup_{N\to\infty} \frac{{}^\#\big(R_i \cap [\,1, N\,]\big)}{N} \le 1/d,$$

or

$$\varliminf_{t\to\infty} \frac{N_i^\circ(t)}{N_i(t)} \ge (I_d^\circ)^{-2} \quad \text{and} \quad \varlimsup_{t\to\infty} \frac{N^{(i)}(t)}{N_i(t)} \le 1/d,$$

where $(I_d^\circ)^2$ denotes $1/(1-1/d)$. By denoting

$$w_i = \frac{1/\log\theta_i}{1/\log\theta_1 + \cdots + 1/\log\theta_\tau}$$
$$N(t) = N_1(t) + \cdots + N_\tau(t), \quad N^\circ(t) = N_1^\circ(t) + \cdots + N_\tau^\circ(t),$$

we have

$$\begin{aligned} 1 \ge \varliminf_{t\to\infty} \frac{\phi(N^\circ(t))}{\phi(N(t))} &= \varliminf_{t\to\infty} \Big(\frac{N^\circ(t)}{N(t)}\Big)^{1/2} \ge \Big(1 - \frac{1}{d}\Big)^{1/2}, \\ \varlimsup_{t\to\infty} \frac{\phi(N^{(i)}(t))}{\phi(N(t))} &= \varlimsup_{t\to\infty} \Big(\frac{N^{(i)}(t)}{N(t)}\Big)^{1/2} \le \Big(\frac{w_i}{d}\Big)^{1/2}. \end{aligned} \tag{6.5}$$

If $i \ne j$, $k \in M_i$ and $l \in M_j$, we have $\nu\theta_i^k \ne \nu'\theta_j^l$ for $1 \le \nu, \nu' \le d$, and

$$\int_{\mathbf{R}} \widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_j^l x)\mu_R(dx) = 0.$$

Hence we have

$$\int_{\mathbf{R}} \Big(\sum_{i=1}^{\tau} \sum_{k\in M_i(t)} \widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2 \mu_R(dx) = \sum_{i=1}^{\tau} \int_{\mathbf{R}} \Big(\sum_{k\in M_i(t)} \widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2 \mu_R(dx). \tag{6.6}$$

By noting $w_i(I_d^\circ)^{-2} \le \varliminf_{t\to\infty} N_i^\circ(t)/N^\circ(t) \le \varlimsup_{t\to\infty} N_i^\circ(t)/N^\circ(t) \le w_i$, we have

$$\begin{aligned} &\varlimsup_{N\to\infty} \frac{1}{N} \int_{\mathbf{R}} \Big(\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k^\circ x)\Big)^2 \mu_R(dx) \\ &= \varlimsup_{t\to\infty} \frac{1}{N^\circ(t)} \int_{\mathbf{R}} \Big(\sum_{i=1}^{\tau} \sum_{k\in M_i(t)} \widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2 \mu_R(dx) \\ &\le \sum_{i=1}^{\tau} w_i \varlimsup_{t\to\infty} \frac{1}{N_i^\circ(t)} \int_{\mathbf{R}} \Big(\sum_{k\in M_i(t)} \widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2 \mu_R(dx), \end{aligned}$$

and similarly, we have

$$\begin{aligned}&\varliminf_{N\to\infty}\frac{1}{N}\int_{\mathbf{R}}\Big(\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k^{\circ}x)\Big)^2\mu_R(dx)\\&\geq\sum_{i=1}^{\tau}w_i(I_d^{\circ})^{-2}\varliminf_{t\to\infty}\frac{1}{N_i^{\circ}(t)}\int_{\mathbf{R}}\Big(\sum_{k\in M_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx).\end{aligned}$$

By denoting $R_i(t)$ by $\{m(1)<m(2)<\cdots<m(N^{(i)}(t))\}$, we have

$$\begin{aligned}&\int_{\mathbf{R}}\Big(\sum_{k\in R_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)\\&=\sum_{k=1}^{N^{(i)}(t)}\int_{\mathbf{R}}\widetilde{\mathbf{1}}^2_{[a,b);d}(\theta_i^{m(k)}x)\mu_R(dx)\\&\qquad+2\sum_{l=1}^{N^{(i)}(t)-1}\sum_{k=1}^{N^{(i)}(t)-l}\int_{\mathbf{R}}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^{m(k)}x)\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^{m(k+l)}x)\mu_R(dx)\\&\leq 2N^{(i)}(t)+4\sum_{l=1}^{N^{(i)}(t)-1}\sum_{k=1}^{N^{(i)}(t)-l}\theta_i^{m(k)-m(k+l)}\leq 2\frac{\theta_1+1}{\theta_1-1}N^{(i)}(t),\end{aligned}$$

where we applied (2.6) and $m(k)-m(k+l)\leq -l$. Hence we have

$$\begin{aligned}\varlimsup_{t\to\infty}\frac{1}{N_i^{\circ}(t)}\int_{\mathbf{R}}\Big(\sum_{k\in R_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)&\leq\frac{\theta_1+1}{\theta_1-1}\varlimsup_{t\to\infty}2\frac{N^{(i)}(t)}{N_i^{\circ}(t)}\\&\leq\frac{\theta_1+1}{\theta_1-1}\frac{2/d}{1-1/d}=:(I_d^{\bullet})^2.\end{aligned}$$

By (6.1), we have

$$\begin{aligned}&\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)\leq\varliminf_{t\to\infty}\frac{1}{N_i^{\circ}(t)}\int_{\mathbf{R}}\Big(\sum_{k=1}^{N_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)\\&\leq\varlimsup_{t\to\infty}\frac{1}{N_i^{\circ}(t)}\int_{\mathbf{R}}\Big(\sum_{k=1}^{N_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)\leq(I_d^{\circ})^2\sigma^2(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i).\end{aligned}\tag{6.7}$$

By applying these estimates to

$$\begin{aligned}&\left|\Big(\int_{\mathbf{R}}\Big(\sum_{k=1}^{N_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)\Big)^{1/2}-\Big(\int_{\mathbf{R}}\Big(\sum_{k\in M_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)\Big)^{1/2}\right|\\&\qquad\leq\int_{\mathbf{R}}\Big(\sum_{k\in R_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)\Big)^{1/2},\end{aligned}$$

we have

$$\big(\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)-I_d^\bullet\big)^2 \le \varliminf_{t\to\infty}\frac{1}{N_i^\circ(t)}\int_{\mathbf{R}}\Big(\sum_{k\in M_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx)$$
$$\le \varlimsup_{t\to\infty}\frac{1}{N_i^\circ(t)}\int_{\mathbf{R}}\Big(\sum_{k\in M_i(t)}\widetilde{\mathbf{1}}_{[a,b);d}(\theta_i^k x)\Big)^2\mu_R(dx) \le \big(I_d^\circ\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)+I_d^\bullet\big)^2.$$

Hence

$$\sum_{i=1}^{\tau}w_i\Big(\frac{\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)-I_d^\bullet}{I_d^\circ}\Big)^2 \le \varliminf_{N\to\infty}\frac{1}{N}\int_{\mathbf{R}}\Big(\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k^\circ x)\Big)^2\mu_R(dx)$$
$$\le \varlimsup_{N\to\infty}\frac{1}{N}\int_{\mathbf{R}}\Big(\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k^\circ x)\Big)^2\mu_R(dx) \le \sum_{i=1}^{\tau}w_i\big(I_d^\circ\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)+I_d^\bullet\big)^2.$$

By Proposition 3.1, we have

$$\bigg(\sum_{i=1}^{\tau}w_i\Big(\frac{\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)-I_d^\bullet}{I_d^\circ}\Big)^2\bigg)^{1/2} \le C(a,b;d) \le \bigg(\sum_{i=1}^{\tau}w_i\big(I_d^\circ\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)+I_d^\bullet\big)^2\bigg)^{1/2}.$$

Now we recall the following inequality. For any sequence $\{m_k\}$ satisfying Hadamard's gap condition (1.1), there exists a constant C depending only on q in (1.1) such that

$$\varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}(\widetilde{\mathbf{1}}_{[a,b)}-\widetilde{\mathbf{1}}_{[a,b);d})(m_k x)\Big| \le C\|\widetilde{\mathbf{1}}_{[a,b)}-\widetilde{\mathbf{1}}_{[a,b);d}\|_2^{1/4}, \quad \text{a.e.}$$

This inequality can be proved by method of Takahashi [17], a detailed proof can be found in [8] (Lemma 4.1). By applying this and $\|\widetilde{\mathbf{1}}_{[a,b)}-\widetilde{\mathbf{1}}_{[a,b);d}\|_2 \le 1/d^{1/2}$ to a triangle inequality relation

$$\varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k^\circ x)\Big| - \varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}(\widetilde{\mathbf{1}}_{[a,b)}-\widetilde{\mathbf{1}}_{[a,b);d})(n_k^\circ x)\Big|$$
$$\le \varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b)}(n_k^\circ x)\Big|$$
$$\le \varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k^\circ x)\Big| + \varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}(\widetilde{\mathbf{1}}_{[a,b)}-\widetilde{\mathbf{1}}_{[a,b);d})(n_k^\circ x)\Big|,$$

we have

$$C(a,b;d)-\frac{C}{d^{1/8}} \le \varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b)}(n_k^\circ x)\Big| \le C(a,b;d)+\frac{C}{d^{1/8}}.$$

By combining these, we have

$$\Big(\sum_{i=1}^{\tau} w_i\Big(\frac{\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)-I_d^\bullet}{I_d^\circ}\Big)^2\Big)^{1/2}-\frac{C}{d^{1/8}}$$
$$\le \varlimsup_{N\to\infty}\frac{1}{\phi(N)}\Big|\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b)}(n_k^\circ x)\Big| \le \Big(\sum_{i=1}^{\tau} w_i\big(I_d^\circ\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)+I_d^\bullet\big)^2\Big)^{1/2}+\frac{C}{d^{1/8}}.$$

By applying Proposition 3.4, we have

$$\sup_{0\le a<b<1}\Big(\sum_{i=1}^{\tau} w_i\Big(\frac{\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)-I_d^\bullet}{I_d^\circ}\Big)^2\Big)^{1/2}-\frac{C}{d^{1/8}}$$
$$\le \varlimsup_{N\to\infty}\frac{ND_N\{n_k^\circ x\}}{\phi(N)} \le \sup_{0\le a<b<1}\Big(\sum_{i=1}^{\tau} w_i\big(I_d^\circ\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)+I_d^\bullet\big)^2\Big)^{1/2}+\frac{C}{d^{1/8}}. \tag{6.8}$$

On the other hand, since each $\{n_k^{(i)}\}$ is a subsequence of $\{\theta_i^k\}$, it satisfies Hadamard's gap condition (1.1) with $q=\theta_1$. Hence by (1.2), we have

$$\varlimsup_{N\to\infty}\frac{ND_N\{n_k^{(i)}x\}}{\phi(N)}\le K_{\theta_1},\quad \text{a.e.} \tag{6.9}$$

By

$$\Big|\sum_{k=1}^{N(t)}\widetilde{\mathbf{1}}_{[a,b)}(n_k x)\Big| \le \Big|\sum_{k=1}^{N^\circ(t)}\widetilde{\mathbf{1}}_{[a,b)}(n_k^\circ x)\Big| + \sum_{i=1}^{\tau}\Big|\sum_{k=1}^{N^{(i)}(t)}\widetilde{\mathbf{1}}_{[a,b)}(n_k^{(i)} x)\Big|,$$

we have

$$\varlimsup_{t\to\infty}\frac{1}{\phi(N(t))}\sup_{0\le a<b<1}\Big|\sum_{k=1}^{N(t)}\widetilde{\mathbf{1}}_{[a,b)}(n_k x)\Big| \le \varlimsup_{t\to\infty}\frac{1}{\phi(N(t))}\sup_{0\le a<b<1}\Big|\sum_{k=1}^{N^\circ(t)}\widetilde{\mathbf{1}}_{[a,b)}(n_k^\circ x)\Big|$$
$$+\sum_{i=1}^{\tau}\varlimsup_{t\to\infty}\frac{1}{\phi(N(t))}\sup_{0\le a<b<1}\Big|\sum_{k=1}^{N^{(i)}(t)}\widetilde{\mathbf{1}}_{[a,b)}(n_k^{(i)} x)\Big|,$$

and hence by applying (6.5), (6.8), and (6.9), we have

$$\varlimsup_{N\to\infty}\frac{ND_N\{n_k x\}}{\phi(N)}$$
$$\le \varlimsup_{t\to\infty}\frac{\phi(N^\circ(t))}{\phi(N(t))}\varlimsup_{N\to\infty}\frac{ND_N\{n_k^\circ x\}}{\phi(N)} + \sum_{i=1}^{\tau}\varlimsup_{t\to\infty}\frac{\phi(N^{(i)}(t))}{\phi(N(t))}\varlimsup_{N\to\infty}\frac{ND_N\{n_k^{(i)} x\}}{\phi(N)}$$
$$\le \sup_{0\le a<b<1}\Big(\sum_{i=1}^{\tau} w_i\big(I_d^\circ\sigma(\widetilde{\mathbf{1}}_{[a,b);d},\theta_i)+I_d^\bullet\big)^2\Big)^{1/2}+\frac{C}{d^{1/8}}+\sum_{i=1}^{\tau}\Big(\frac{w_i}{d}\Big)^{1/2}K_{\theta_1}.$$

By letting $d \to \infty$, and by noting (6.2), (6.3), $I_d^\bullet \to 0$, and $I_d^\circ \to 1$, we have

$$\overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x)}{\phi(N)} \le \sup_{0\le a<b<1} \Big(\sum_{i=1}^{\tau} w_i \sigma^2(\widetilde{\mathbf{1}}_{[a,b)}, \theta_i) \Big)^{1/2} = \Big(\sum_{i=1}^{\tau} w_i \Sigma_{\theta_i}^2 \Big)^{1/2}.$$

By

$$\Big| \sum_{k=1}^{N^\circ(t)} \widetilde{\mathbf{1}}_{[a,b)}(n_k^\circ x) \Big| \le \Big| \sum_{k=1}^{N(t)} \widetilde{\mathbf{1}}_{[a,b)}(n_k x) \Big| + \sum_{i=1}^{\tau} \Big| \sum_{k=1}^{N^{(i)}(t)} \widetilde{\mathbf{1}}_{[a,b)}(n_k^{(i)} x) \Big|,$$

we can derive in the same way

$$\begin{aligned} \underline{\lim_{t\to\infty}} \frac{\phi(N^\circ(t))}{\phi(N(t))} \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k^\circ x\}}{\phi(N)} \le{}& \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x\}}{\phi(N)} \\ &+ \sum_{i=1}^{\tau} \overline{\lim_{t\to\infty}} \frac{\phi(N^{(i)}(t))}{\phi(N(t))} \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k^{(i)} x\}}{\phi(N)}, \end{aligned}$$

or

$$\begin{aligned} &\overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x\}}{\phi(N)} \\ &\ge \underline{\lim_{t\to\infty}} \frac{\phi(N^\circ(t))}{\phi(N(t))} \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k^\circ x\}}{\phi(N)} - \sum_{i=1}^{\tau} \overline{\lim_{t\to\infty}} \frac{\phi(N^{(i)}(t))}{\phi(N(t))} \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k^{(i)} x\}}{\phi(N)} \\ &\ge \frac{1}{I_d^\circ} \sup_{0\le a<b<1} \Big(\sum_{i=1}^{\tau} w_i \Big(\frac{\sigma(\widetilde{\mathbf{1}}_{[a,b);d}, \theta_i) - I_d^\bullet}{I_d^\circ} \Big)^2 \Big)^{1/2} - \frac{C}{d^{1/8}} - \sum_{i=1}^{\tau} \Big(\frac{w_i}{d} \Big)^{1/2} K_{\theta_1}. \end{aligned}$$

By letting $d \to \infty$, we have

$$\overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x\}}{\phi(N)} \ge \sup_{0\le a<b<1} \Big(\sum_{i=1}^{\tau} w_i \sigma^2(\widetilde{\mathbf{1}}_{[a,b)}, \theta_i) \Big)^{1/2} = \Big(\sum_{i=1}^{\tau} w_i \Sigma_{\theta_i}^2 \Big)^{1/2},$$

which proves the conclusion.

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