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Bounded law of the iterated logarithm for discrepancies of permutations of lacunary sequences

By

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Abstract

For a sequence $\{n_k\}$ satisfying the Hadamard's gap condition, Philipp proved the bounded law of the iterated logarithm for discrepancies $\{n_k x\}$, and gave a concrete upperbound depending only on a constant in the gap condition. Recently Aistleitner gave much smaller constant by using martingale approximation. In this note, we give an almost optimal upper bound constant and prove that this bound is also valid for a permutation of a sequence satisfying the gap condition.

§ 1. Introduction

In this note, we will be concerned with the asymptotic behavior of discrepancies $D_N\{x_k\}$ and star discrepancies $D_N^*\{x_k\}$ of a sequence $\{x_k\}$, defined by

$$D_N\{x_k\} = \sup_{0\le a<b<1}\left|\frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[a,b)}(x_k)\right|;\ D_N^*\{x_k\} = \sup_{0\le a<1}\left|\frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{1}}_{[0,a)}(x_k)\right|,$$

where $\widetilde{\mathbf{1}}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(\langle x\rangle) - (b-a)$, $\mathbf{1}_{[a,b)}$ denotes the indicator function of $[a,b)$, and $\langle x\rangle$ denotes the fractional part of x.

Philipp [11, 12] assumed the Hadamard's gap condition

$$n_{k+1}/n_k \ge q > 1 \quad (k = 1, 2, \dots), \tag{1.1}$$

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and proved the bounded law of the iterated logarithm:

$$\frac{1}{4\sqrt{2}} \le \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} \le K_q = \frac{1}{\sqrt{2}}\Big(166 + \frac{664}{q^{1/2}-1}\Big). \quad \text{a.e.} \tag{1.2}$$

As Berkes-Philipp-Tichy [5] noted, the above result is permutation invariant, i.e., the inequalities remain valid if we change the order of $\{n_k\}$.

Aistleitner [2] gave a preciser estimate when $q \ge 2$:

$$\frac{1}{2} - \frac{8}{q^{1/4}} \le \varlimsup_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} \le \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} \le \frac{1}{2} + \frac{6}{q^{1/4}} \quad \text{a.e.} \tag{1.3}$$

Recently, the exact values of limsup became possible to calculate explicitly. When q is an odd integer greater than 2, we have

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{q^kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{q^kx\}}{\sqrt{2N\log\log N}} = \frac{1}{2}\sqrt{\frac{q+1}{q-1}} \quad \text{a.e.},$$

(Cf. [7]). When $q \to \infty$, we have

$$\frac{1}{2}\sqrt{\frac{q+1}{q-1}} = \frac{1}{2} + \frac{1}{2q} + o(q^{-1}) \tag{1.4}$$

and hence it is natural to expect that Aistleitner's upper bound estimate in (1.3) can be improved to $1/2 + O(1/q)$. Since Aistleitner used martingale approximation technique, which is hard to apply to the case when the sequence is permutated, it is not clear if the same estimate is valid for permutated sequences. We try to contribute to these points.

Now we are in a position to state our theorem.

Theorem 1.1. *Let $\{n_k\}$ be a sequence of real numbers (not necessarily integers nor positive) satisfying the gap condition*

$$n_1 \ne 0, \qquad |n_{k+1}/n_k| \ge q > 1 \quad (k = 1, 2, \dots). \tag{1.5}$$

Let ϖ be a permutation on $\mathbf{N}$, i.e., a bijection $\mathbf{N} \to \mathbf{N}$. Then for all countable dense set $S \subset [0, 1)$, we have

$$\begin{aligned}
\varlimsup_{N\to\infty} \frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} &= \sup_{S\ni a<b\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)}x)\Big| \\
&= \sup_{0\le a<b<1} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)}x)\Big|, \\
\varlimsup_{N\to\infty} \frac{ND_N^*\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} &= \sup_{a\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}(n_{\varpi(k)}x)\Big| \\
&= \sup_{0\le a<1} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}(n_{\varpi(k)}x)\Big|,
\end{aligned} \tag{1.6}$$

for almost every $x \in \mathbf{R}$. By denoting the d-th subsum of the Fourier series of $\widetilde{\mathbf{1}}_{[a,b)}$ by $\widetilde{\mathbf{1}}_{[a,b);d}$, we have

$$\begin{aligned}(1.7)\quad &\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)}x)\right| \\ &= \lim_{d\to\infty}\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)}x)\right|, \quad a.e. \quad x\in\mathbf{R}.\end{aligned}$$

By putting $\mathbf{N}_u = \{2^n + m2^{n-u} \mid n \geq u;\ 0 \leq m < 2^u\}$ for $u \in \mathbf{N}$, we have

$$\begin{aligned}(1.8)\quad &\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)}x)\right| \\ &= \lim_{u\to\infty}\overline{\lim_{\mathbf{N}_u\ni N\to\infty}} \frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)}x)\right|, \quad a.e. \quad x\in\mathbf{R}.\end{aligned}$$

As a byproduct of the proof of the above Theorem, we can prove the following.

Corollary 1.2. *Under the same conditions as those assumed in Proposition 1.1,*

$$\frac{1}{4\sqrt{2}} \leq \overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} \leq \overline{\lim_{N\to\infty}} \frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} \leq \left(\frac{1}{4} + \frac{1}{\sqrt{3}(q-1)}\right)^{1/2}, \ a.e.$$

This upper bound constant equals to $0.9095\ldots$ when $q = 2$, and is smaller than Philipp's constant $1050.898\ldots$ and Aistleitner's constant $5.545\ldots$. Because it asymptotically behaves like

$$\left(\frac{1}{4} + \frac{1}{\sqrt{3}(q-1)}\right)^{1/2} = \frac{1}{2} + \frac{1}{\sqrt{3}q} + o(q^{-1}), \quad (q\to\infty),$$

as comparered to (1.4), it gives an optimal estimate except for the multiple constant of $1/q$ term.

We improve the upperbound estimate by thoroughly using the exponential integrability technique which is invented around 1960's and 1970's.

§ 2. Exponential integrability

We follow the method of Philipp [11] and Takahashi [13] to give a refinement of exponential integrability results.

Denote the d-th subsum of the Fourier series of $\widetilde{\mathbf{1}}_{[a,b)}$ by $\widetilde{\mathbf{1}}_{[a,b);d}$ and the d-th Cesaro sum by $\mathcal{C}\widetilde{\mathbf{1}}_{[a,b);d}$. By $\|\widetilde{\mathbf{1}}_{[a,b)}\|_\infty \leq 1$, $\|\mathcal{C}\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty \leq \|\widetilde{\mathbf{1}}_{[a,b)}\|_\infty$, and $|\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j)| \leq 1/2|j|$, we have

$$\|\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty \le \|\mathcal{C}\widetilde{\mathbf{1}}_{[a,b);d}\|_\infty + \sum_{0<|j|\le d} \frac{|j|}{d} |\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j)| < 2.$$

Consequently we have $\|\widetilde{\mathbf{1}}_{[a,b)} - \widetilde{\mathbf{1}}_{[a,b);d}\|_\infty \le 3$.

We will be concerned with a real valued function f on $\mathbf{R}$ satisfying

$$\begin{aligned} &f(x+1) = f(x), \quad \int_0^1 f(x)\,dx = 0, \quad \|f\|_2^2 = \int_0^1 f^2(x)\,dx \le \frac{1}{4}, \quad \|f\|_\infty \le 3, \\ &|c_j| \le \frac{2}{\pi j}, \quad \text{where} \quad f(x) = \sum_{j=1}^\infty c_j \cos(2\pi j x + \gamma_j) \end{aligned} \tag{2.1}$$

It is easily verified that functions $\widetilde{\mathbf{1}}_{[a,b)}$, $\widetilde{\mathbf{1}}_{[a,b);d}$, and $\widetilde{\mathbf{1}}_{[a,b)} - \widetilde{\mathbf{1}}_{[a,b);d}$ satisfy the conditions (2.1).

By noting

$$\frac{1}{\pi}\int_{\mathbf{R}} \Big(\frac{\sin x}{x}\Big)^2 e^{2\pi\sqrt{-1}\,\lambda x}\,dx = \begin{cases} 1 - |\pi\lambda| & \pi|\lambda| \le 1, \\ 0 & \text{otherwise}, \end{cases}$$

we can define a probability measure μ on $(\mathbf{R}, \mathcal{B})$ by

$$\mu(dx) = \frac{1}{\pi}\Big(\frac{\sin x}{x}\Big)^2\,dx.$$

Clearly μ and the Lebesgue measure are mutually absolutely continuous.

We have an inequality

$$\Big|\int_{\mathbf{R}} \cos 2\pi(\alpha x + \gamma)\,\mu(dx)\Big| \le 1,$$

and the relation below: if $|\alpha| \ge 1/\pi$, then

$$\int_{\mathbf{R}} \cos 2\pi(\alpha x + \gamma)\,\mu(dx) = 0. \tag{2.2}$$

The target of this section is to prove the following:

Proposition 2.1. *Let $\{n_k\}$ be a sequence of real numbers satisfying (1.5). Suppose that*

$$|n_k| \ge 1, \qquad (k \in \mathbf{N}). \tag{2.3}$$

Let f be a function satisfying (2.1). For any $0 < \delta \le 1$ there exists $B_0 \ge 3$ depending only on q and δ such that for all integers $A \ge 0$ and $B \ge B_0$ with

$$\|f\|_2 \ge B^{-1/4}/4, \tag{2.4}$$

for all real number $R \geq 1$, *and for all permutation* ϖ *on* $\mathbf{N}$, *we have*

$$\mu\Big(\Big|\sum_{k=A+1}^{A+B} f(n_{\varpi(k)}x)\Big| \geq (1+2\delta)C_1R\|f\|_2^{1/4}(2B\log\log B)^{1/2}\Big) \tag{2.5}$$
$$\leq C_2\exp\big(-(1+\delta)\|f\|_2^{-1/2}R\log\log B\big) + C_3R^{-2}B^{-7/8}.$$

Here constants are given by

$$C_1^2 = 1 + \frac{4}{\sqrt{3}(q-1)}, \quad C_2 = 2\sup_{N\geq 3}\exp\big(24N^{-7/48}(\log\log N)^{1/2}\big), \quad \text{and} \quad C_3 = \frac{8}{\delta^2}.$$

If f *is a trigonometric polynomial satisfying (2.1) with degree* d, *for any* $0 < \delta \leq 1$ *there exists* $B_1 \geq 3$ *depending only on* q, δ *and* d *such that for all integers* $A \geq 0$ *and* $B \geq B_1$ *with (2.4) and for all permutation* ϖ *on* $\mathbf{N}$, *we have*

$$\mu\Big(\Big|\sum_{k=A+1}^{A+B} f(n_{\varpi(k)}x)\Big| \geq (1+\delta)\frac{C_1}{\sqrt{2}}\|f\|_2^{1/2}(2B\log\log B)^{1/2}\Big) \tag{2.6}$$
$$\leq C_2\exp\big(-(1+\delta)\log\log B\big).$$

To prove the above proposition, it is enough to prove the following lemma.

Lemma 2.2. *For any* $0 < \delta \leq 1$ *and* $q > 1$, *there exists an integer* $N_0 \geq 3$ *depending only on* δ *and* q *satisfying the following properties. If* f *satisfies (2.1),* $N \geq N_0$ *satisfies* $\|f\|_2 \geq N^{-1/4}/4$, *and a set of real numbers* $\{n_1, \dots, n_N\}$ *satisfies*

$$|n_k| \geq 1 \quad (k = 1, \dots, N) \quad \text{and} \quad |n_{k+1}/n_k| \geq q \quad (k = 1, \dots, N-1), \tag{2.7}$$

then

$$\mu\Big(\Big|\sum_{k=1}^{N} f(n_k x)\Big| \geq (1+2\delta)C_1R\|f\|_2^{1/4}(2N\log\log N)^{1/2}\Big) \tag{2.8}$$
$$\leq C_2\exp\big(-(1+\delta)\|f\|_2^{-1/2}R\log\log N\big) + C_3R^{-2}N^{-7/8}$$

holds.

For any $0 < \delta \leq 1$, $q > 1$, *and* $d \in \mathbf{N}$, *there exists an integer* $N_1 \geq 3$ *depending only on* q, δ *and* d *such that for any trigonometric polynomial* f *satisfying (2.1) with degree* d, *for any* $N \geq N_1$ *with* $\|f\|_2 \geq N^{-1/4}/4$, *and for any set of real numbers* $\{n_1, \dots, n_N\}$ *satisfying (2.7), we have*

$$\mu\Big(\Big|\sum_{k=1}^{N} f(n_k x)\Big| \geq (1+\delta)\frac{C_1}{\sqrt{2}}\|f\|_2^{1/2}(2N\log\log N)^{1/2}\Big) \tag{2.9}$$
$$\leq C_2\exp\big(-(1+\delta)\log\log N\big).$$

First we derive Proposition 2.1 from Lemma 2.2. Suppose that we are given a sequence $\{n_k\}$ of real numbers satisfying (1.5) and (2.3). Take $0 < \delta \le 1$ arbitrarily and take N_0 given in the lemma and denote it by B_0. Take a function f with (2.1), a permutation ϖ on $\mathbf{N}$, and $A \ge 0$ arbitrarily. Assume that $B \ge B_0$ and $\|f\|_2 \ge B^{-1/4}/4$ are satisfied. Since we have

$$\sum_{k=A+1}^{A+B} f(n_{\varpi(k)}x) = \sum_{j\in\varpi(\{A+1,\dots,A+B\})} f(n_j x)$$

and a set $\{n_j \mid j \in \varpi(\{A+1,\dots,A+B\})\}$ satisfies (2.7) with given q. Hence we have (2.8) for $\sum_{k=A+1}^{A+B} f(n_{\varpi(k)}x)$, which is identical with (2.5). The proof of (2.6) can be done in the same way from (2.9).

Now we prove Lemma 2.2. Denote the d-th subsum of the Fourier series of f by f_d. We prepare two lemmas.

Lemma 2.3. *We have*

$$\int_{\mathbf{R}} \Bigl(\sum_{k=1}^{N}(f-f_d)(n_k x)\Bigr)^2 \mu(dx) \le \frac{4q}{q-1}\frac{N}{d}, \qquad (N\in\mathbf{N}).$$

Proof. We prove by assuming that f is a trigonometric polynomial. The general case follows automatically by Fatou's lemma. By relation (2.2), we see

$$\begin{aligned}
&\int_{\mathbf{R}} \Bigl(\sum_{k=1}^{N}(f-f_d)(n_k x)\Bigr)^2 \mu(dx)\\
&\le \sum_{k,l:k\le l\le N}\ \sum_{i,j>d}\ \sum_{\varsigma=\pm1}\Bigl|c_ic_j\int_{\mathbf{R}}\cos(2\pi(in_k+\varsigma jn_l)+(\gamma_i+\varsigma\gamma_j))\,\mu(dx)\Bigr|\\
&\le \sum_{k,l:k\le l\le N}\ \sum_{i,j>d}\ \sum_{\varsigma=\pm1}|c_ic_j|\mathbf{1}\bigl(|in_k+\varsigma jn_l|<1/\pi\bigr).
\end{aligned}$$

We have $|n_l| \ge |n_k| \ge 1$, and hence at least one of $|in_k+jn_l|<1/\pi$ and $|in_k-jn_l|<1/\pi$ is false. Since we have $\bigl|i|n_k|-j|n_l|\bigr| \le \bigl|in_k+\varsigma jn_l\bigr|$, the condition $\bigl|in_k+\varsigma jn_l\bigr|<1/\pi$ implies $\bigl|i-j|n_l/n_k|\bigr| < 1/\pi|n_k| < 1/\pi$. Hence by denoting $[\,x\,]^* = [\,x+1/2\,]$, we have

$$\sum_{i,j>d}|c_ic_j|\sum_{\varsigma=\pm1}\mathbf{1}\bigl(|in_k+\varsigma jn_l|<1/\pi\bigr) \le \sum_{j>d}|c_{[j|n_l/n_k|]^*}c_j|.$$

By $|c_j| \le 1/j$ and $[j|n_l/n_k|]^* \ge j|n_l/n_k| - 1/2 \ge j|n_l/n_k|/2 \ge jq^{l-k}/2$, we have

$$\sum_{j>d}|c_{[j|n_l/n_k|]^*}c_j| \le 2q^{-(l-k)}\sum_{j>d}\frac{1}{j^2} \le 2q^{-(l-k)}\frac{1}{d}.$$

By taking summation for $k \le l \le N$, we have the conclusion. ∎

Take $0 < \delta \le 1$ arbitrarily and put $1 - \beta = 1/\sqrt{1+\delta}$. Clearly we have $0 < \beta < 1$. There exists an $x_\delta > 0$ such that

$$e^x \le 1 + x + \frac{\sqrt{1+\delta}}{2} x^2, \quad (|x| \le x_\delta).$$

Take $H \in \mathbf{N}$ large enough to satisfy

$$q^H > 3H^6/\beta, \quad 3/H^{1/2} \le x_\delta, \quad H^6 q^{-1} \ge 1, \tag{2.10}$$

and put

$$U_m(x) = \sum_{k=Hm+1}^{H(m+1)} f_{H^6}(n_k x).$$

Lemma 2.4. *If $a \in \mathbf{R}$ satisfies $2|a|H^{3/2} < 1$, then for all $P \in \mathbf{N}$ we have*

$$\int_{\mathbf{R}} \exp\Big(a \sum_{m=1}^{2P} U_m(x) \Big) \mu(dx) \le \exp\Big((1+\delta) \frac{C_1^2}{2} a^2 \|f\|_2 2HP \Big).$$

Proof. We assume $|a|H^{3/2} < 1$ and prove

$$\int_{\mathbf{R}} \exp\Big(a \sum_{m=1}^{P} U_{2m-w}(x) \Big) \mu(dx) \le \exp\Big(\frac{1+\delta}{2} \frac{C_1^2}{2} a^2 \|f\|_2 HP \Big) \tag{2.11}$$

for $w = 0, 1$. By assumption we have $|aU_m| \le 3H|a| < 3/H^{1/2} \le x_\delta$ and hence

$$\exp(aU_m) \le 1 + aU_m + \frac{\sqrt{1+\delta}}{2} a^2 U_m^2.$$

By defining

$$W_m(x) = \sum_{l=Hm+1}^{H(m+1)-1} \sum_{j=l+1}^{H(m+1)} \sum_{\varsigma=\pm 1} \sum_{\substack{1 \le r,s \le H^6: \\ |n_l r + \varsigma n_j s| < \beta |n_{Hm}|}} c_r c_s \cos\big(2\pi(n_l r + \varsigma n_j s)x + \gamma_r + \varsigma \gamma_s\big)$$

$$V_m = H \|f_{H^6}\|_2^2 = \frac{H}{2} \sum_{r=1}^{H^6} c_r^2,$$

we decompose U_m^2 in the following way:

$$\begin{aligned} U_m^2(x) = &\sum_{l=Hm+1}^{H(m+1)} f_{H^6}^2(n_l x) - V_m \\ &+ 2 \sum_{l=Hm+1}^{H(m+1)-1} \sum_{j=l+1}^{H(m+1)} f_{H^6}(n_l x) f_{H^6}(n_j x) - W_m(x) \\ &+ W_m(x) + V_m. \end{aligned}$$

We here prove that absolute values of frequencies in trigonometric polynomial expansion of

$$W_m^{\triangle} = \frac{\sqrt{1+\delta}}{2} a^2 (U_m^2 - W_m - V_m) + aU_m$$

belong to $[\beta|n_{Hm}|, 2\beta|n_{H(m+2)}|/3]$.

The frequencies of U_m are written as $n_l r$ where $Hm+1 \le l \le H(m+1)$ and $1 \le r \le H^6$, and hence we have

$$\beta|n_{Hm}| \le |n_{Hm}| \le |n_l r| \le H^6 |n_{H(m+1)}| \le H^6 q^{-H} |n_{H(m+2)}| \le \beta |n_{H(m+2)}|/3.$$

The frequencies of $f_{H^6}^2(n_l x) - \frac{1}{2}\sum_{j=1}^{H^6} c_j^2$ are written as $n_l(r+\varsigma s)$ where $Hm+1 \le l \le H(m+1)$, $r+\varsigma s \ne 0$, $1 \le r$, $s \le H^6$, and $\varsigma = \pm 1$, and hence we have

$$\beta|n_{Hm}| \le |n_{Hm}| \le |n_l(r+\varsigma s)| \le 2H^6 |n_{H(m+1)}| \le 2\beta |n_{H(m+2)}|/3.$$

By definition we have

$$\begin{aligned}
&2\sum_{l=Hm+1}^{H(m+1)-1} \sum_{j=l+1}^{H(m+1)} f_{H^6}(n_l x) f_{H^6}(n_j x) - W_m(x) \\
&= \sum_{l=Hm+1}^{H(m+1)-1} \sum_{j=l+1}^{H(m+1)} \sum_{\varsigma=\pm 1} \sum_{\substack{1 \le r,s \le H^6: \\ |n_l r + \varsigma n_j s| \ge \beta |n_{Hm}|}} c_r c_s \cos\bigl(2\pi(n_l r + \varsigma n_j s)x + \gamma_r + \varsigma \gamma_s\bigr).
\end{aligned}$$

Hence the frequencies $n_l r + \varsigma n_j s$ appearing here obey the following estimate.

$$\beta|n_{Hm}| \le |n_l r + \varsigma n_j s| \le 2H^6 |n_{H(m+1)}| \le 2\beta |n_{H(m+2)}|/3.$$

Take $0 < m_1 < m_2 < \cdots < m_t$ arbitrarily. Expand $W_{2m_i - w}^{\triangle}$ into trigonometric polynomial and denote a term by $\cos 2\pi(\alpha_i x + \Gamma_i)$. By $|n_{H(2m-w)}/n_{H(2(m-1)-w)}| \ge q^{2H} > 9/\beta^2 > 9$, we have

$$\begin{aligned}
&|\alpha_t \pm \alpha_{t-1} \pm \cdots \pm \alpha_1| \ge |\alpha_t| - |\alpha_{t-1}| - \cdots - |\alpha_1| \\
&\ge \beta |n_{H(2m_t - w)}| - (2\beta/3)(|n_{H(2m_{t-1}-w+2)}| + |n_{H(2m_{t-2}-w+2)}| + \cdots) \\
&\ge \beta |n_{H(2m_t - w)}| - (2\beta/3)|n_{H(2m_t - w)}|(1 + (1/9) + (1/9)^2 + \cdots) = \beta |n_{H(2m_t - w)}|/4 \\
&\ge \beta |n_H|/4 \ge q^{H-1}|n_1|\beta/4 \ge 3q^{-1}H^6/4 \ge 3/4 \ge 1/\pi.
\end{aligned}$$

Thanks to the relation (2.2), we have

$$\int_{\mathbf{R}} \prod_{i=1}^{t} \cos 2\pi(\alpha_i x + \Gamma_i)\, \mu(dx) = 0$$

and thereby we obtain the multiple orthogonality

$$\int_{\mathbf{R}} W_{2m_1}^{\triangle}(x) \dots W_{2m_t}^{\triangle}(x)\, \mu(dx) = 0.$$

Let us consider W_m. In the definition of W_m, for given l, j, r, s, the inequality $|n_l r + \varsigma n_j s| < \beta |n_{Hm}|$ can be valid at most one of the cases among $\varsigma = +1$ and $\varsigma = -1$, since $|n_j s| \geq |n_j| > |n_{Hm}|$. Since $|n_l r + \varsigma n_j s| < \beta |n_{Hm}|$ implies $|r - s|n_j/n_l|| < \beta$, for each s, there exists at most only one such r, and we have

$$|c_s c_r| \leq \frac{2}{\pi}|c_s| \frac{1}{s|n_j/n_l| - \beta} \leq \frac{2}{\pi}|c_s| \frac{1}{s|n_j/n_l|(1-\beta)} \leq \frac{2\sqrt{1+\delta}}{\pi}|c_s| \frac{1}{s} q^{-(j-l)}.$$

Here we used $\beta \leq \beta s|n_j/n_l|$. Hence by noting

$$\sum_{s=1}^{\infty} |c_s| \frac{1}{s} \leq \sqrt{2}\Big(\sum_{s=1}^{\infty} \frac{c_s^2}{2}\Big)^{1/2} \Big(\sum_{s=1}^{\infty} \frac{1}{s^2}\Big)^{1/2} = \|f\|_2 \frac{\pi}{\sqrt{3}},$$

we have

$$\sum_{\varsigma=\pm 1} \sum_{\substack{1 \leq r,s \leq H^6: \\ |n_l r + \varsigma n_j s| < \beta |n_{Hm}|}} |c_r c_s| \leq \frac{2\sqrt{1+\delta}}{\sqrt{3}} q^{-(j-l)} \|f\|_2. \tag{2.12}$$

By taking summation for l and j, we have

$$|W_m| \leq \sqrt{1+\delta} \frac{2}{\sqrt{3}(q-1)} \|f\|_2 H. \tag{2.13}$$

On the other hand, by

$$V_m \leq H \|f\|_2^2 \leq \sqrt{1+\delta} \|f\|_2 H/2, \tag{2.14}$$

we have

$$\begin{aligned} \exp(aU_m) &\leq 1 + aU_m + \frac{\sqrt{1+\delta}}{2} a^2 U_m^2 = 1 + \frac{\sqrt{1+\delta}}{2} a^2 (W_m + V_m) + W_m^{\triangle} \\ &\leq 1 + \frac{1+\delta}{2} \frac{C_1^2}{2} a^2 \|f\|_2 H + W_m^{\triangle}. \end{aligned}$$

By integrating this, we have

$$\begin{aligned} \int_{\mathbf{R}} \exp\Big(a \sum_{m=1}^{P} U_{2m-w}(x)\Big) \mu(dx) &\leq \int_{\mathbf{R}} \prod_{m=1}^{P} \Big(1 + \frac{1+\delta}{2} \frac{C_1^2}{2} a^2 \|f\|_2 H + W_{2m-w}^{\triangle}\Big) \mu(dx) \\ = \Big(1 + \frac{1+\delta}{2} \frac{C_1^2}{2} a^2 \|f\|_2 H\Big)^P &\leq \exp\Big(\frac{1+\delta}{2} \frac{C_1^2}{2} a^2 \|f\|_2 HP\Big). \end{aligned}$$

Let us now take $a \in \mathbf{R}$ satisfying $2|a|H^{3/2} < 1$. By applying Schwarz inequality and (2.11), we have

$$\begin{aligned} \int_{\mathbf{R}} \exp\Big(a \sum_{m=1}^{2P} U_m(x)\Big) \mu(dx) &\leq \prod_{w=0}^{1} \Big(\int_{\mathbf{R}} \exp\Big(2a \sum_{m=1}^{P} U_{2m-w}(x)\Big) \mu(dx)\Big)^{1/2} \\ &\leq \exp\Big((1+\delta) \frac{C_1^2}{2} a^2 \|f\|_2 2HP\Big), \end{aligned}$$

which completes the proof. □

Now we prove the first half of Lemma 2.2. Note that we are assuming $\|f\|_2 \geq N^{-1/4}/4$.

Take sufficiently large N_0 such that for all $N \geq N_0$,

$$C_1^{-1} 8 N^{-1/16} (\log\log N)^{1/2} < 1$$

holds and $H = [\, N^{1/6}\,]$ satisfies (2.10). Put

$$Q = C_1 \|f\|_2^{1/4} R (2N \log\log N)^{1/2} \geq C_1 R N^{7/16}.$$

We have

$$\begin{aligned}
&\mu\Bigl(x : \Bigl| \sum_{k=1}^{N} f(n_k x) \Bigr| \geq (1+2\delta) Q \Bigr) \\
&\leq \mu\Bigl(x : \Bigl| \sum_{k=1}^{N} f_{H^6}(n_k x) \Bigr| \geq (1+\delta) Q \Bigr) + \mu\Bigl(x : \Bigl| \sum_{k=1}^{N} (f - f_{H^6})(n_k x) \Bigr| \geq \delta Q \Bigr).
\end{aligned}$$

Thanks to the Chebyshev's inequality and by $H \geq (N/2)^{1/6}$, the second term is bounded by

$$\begin{aligned}
\frac{1}{(\delta Q)^2} \int_{\mathbf{R}} \Bigl(\sum_{k=1}^{N} (f - f_{H^6})(n_k x) \Bigr)^2 \mu(dx) &\leq \frac{4q}{q-1} \frac{N}{\delta^2 Q^2 H^6} \leq \frac{8q}{(q-1)\delta^2 C_1^2 N^{7/8} R^2} \\
&= \frac{\sqrt{3}(q-1) + \sqrt{3}}{\sqrt{3}(q-1) + 4} \frac{8}{\delta^2} \frac{1}{N^{7/8} R^2} \leq \frac{C_3}{N^{7/8} R^2}.
\end{aligned}$$

Put

$$a = C_1^{-1} \bigl(\|f\|_2^{-3/2} 2 N^{-1} \log\log N \bigr)^{1/2} \leq C_1^{-1} (16 N^{-5/8} \log\log N)^{1/2}.$$

We can verify the condition in Lemma 2.4 by $2|a| H^{3/2} \leq C_1^{-1} 8 N^{-1/16} (\log\log N)^{1/2} < 1$. By using the Chebyshev's inequality, we have the following bound for the first term.

$$e^{-(1+\delta)Qa} \int_{\mathbf{R}} \exp\Bigl(a \Bigl| \sum_{k=1}^{N} f_{H^6}(n_k x) \Bigr| \Bigr) \leq e^{-(1+\delta)Qa} \sum_{\varsigma = \pm 1} \int_{\mathbf{R}} \exp\Bigl(\varsigma a \sum_{k=1}^{N} f_{H^6}(n_k x) \Bigr) \mu(dx).$$

By taking P such that $2HP < N \leq 2H(P+1)$, we have

$$\begin{aligned}
a \Bigl| \sum_{k=1}^{N} f_{H^6}(n_k x) - \sum_{m=1}^{2P} U_m(x) \Bigr| &\leq 2Ha \|f_{H^6}\|_\infty \leq 6Ha \leq 24 C_1^{-1} N^{-7/48} (\log\log N)^{1/2} \\
&\leq 24 N^{-7/48} (\log\log N)^{1/2} \leq \log(C_2/2)
\end{aligned}$$

by $C_1 \geq 1$ and the definition of C_2. It implies

$$\int_{\mathbf{R}} \exp\Big(\varsigma a \sum_{k=1}^{N} f_{H^6}(n_k x)\Big)\,\mu(dx) \leq \frac{C_2}{2}\int_{\mathbf{R}} \exp\Big(\varsigma a \sum_{m=1}^{2P} U_m(x)\Big)\,\mu(dx)$$
$$\leq \frac{C_2}{2}\exp\Big((1+\delta)\frac{C_1^2}{2}a^2\|f\|_2 2HP\Big) \leq \frac{C_2}{2}\exp\Big((1+\delta)\frac{C_1^2}{2}a^2\|f\|_2 N\Big).$$

By noting

$$-(1+\delta)Qa + (1+\delta)\frac{C_1^2}{2}a^2\|f\|_2 N = -(1+\delta)\|f\|_2^{-1/2}(2R-1)\log\log N$$
$$\leq -(1+\delta)\|f\|_2^{-1/2}R\log\log N,$$

we see that the first term is bounded by $C_2\exp\Big(-(1+\delta)\|f\|_2^{-1/2}R\log\log N\Big)$. Combining these estimates, we complete the proof of the first half of Lemma 2.2.

Let us proceed to the proof of the last half of Lemma 2.2. Let f be a trigonometric polynomial with degree d. For arbitrary $0 < \delta \leq 1$ put $\check{\beta}$ by $1 - \check{\beta} = 1/\sqrt{1+\delta/3}$. Take K satisfying

$$q^K \geq 6d/\check{\beta}, \quad q^K \geq 1/\check{\beta}.$$

Take H sufficiently large to satisfy

$$q^H \geq 3, \quad K(1+6/\delta) \leq \delta H/6, \quad 3/H^{1/2} \leq x_{\delta/3}. \tag{2.15}$$

Put

$$\acute{U}_m(x) = \sum_{k=Hm+K+1}^{H(m+1)} f(n_k x) \quad \text{and} \quad \grave{U}_m(x) = \sum_{k=Hm+1}^{Hm+K} f(n_k x).$$

By assuming $2|a|H^{3/2} \leq 1$, we prove

$$\int_{\mathbf{R}} \exp\Big(a\sum_{m=0}^{P-1}\big(\acute{U}_m(x) + \grave{U}_{m+1}(x)\big)\Big)\,\mu(dx) \leq \exp\Big((1+\delta)\frac{C_1^2}{4}a^2\|f\|_2 HP\Big). \tag{2.16}$$

Let us assume $|\acute{a}|H^{3/2} \leq 1$ and $|\grave{a}|H^{3/2} \leq 1 + 6/\delta$. By (2.15) we have $|\acute{a}\acute{U}_m| \leq 3H|\acute{a}| \leq 3/H^{1/2} \leq x_{\delta/3}$ and $|\grave{a}\grave{U}_m| \leq 3K|\grave{a}| \leq 3K(1+6/\delta)/H^{3/2} \leq 3(\delta/6)/H^{1/2} \leq x_{\delta/3}$. Hence

$$\exp(\acute{a}\acute{U}_m) \leq 1 + \acute{a}\acute{U}_m + \frac{\sqrt{1+\delta/3}}{2}\acute{a}^2\acute{U}_m^2 \text{ and } \exp(\grave{a}\grave{U}_m) \leq 1 + \grave{a}\grave{U}_m + \frac{\sqrt{1+\delta/3}}{2}\grave{a}^2\grave{U}_m^2.$$

By defining

$$\acute{W}_m(x) = \sum_{l=Hm+K+1}^{H(m+1)-1} \sum_{j=l+1}^{H(m+1)} \sum_{\varsigma=\pm1} \sum_{\substack{1\le r,s\le d:\\ |n_l r+\varsigma n_j s|<\check{\beta}|n_{Hm+K+1}|}} c_r c_s \cos\big(2\pi(n_l r + \varsigma n_j s)x + \gamma_r + \varsigma\gamma_s\big),$$

$$\grave{W}_m(x) = \sum_{l=Hm+1}^{Hm+K-1} \sum_{j=l+1}^{Hm+K} \sum_{\varsigma=\pm1} \sum_{\substack{1\le r,s\le d:\\ |n_l r+\varsigma n_j s|<\check{\beta}|n_{Hm+1}|}} c_r c_s \cos\big(2\pi(n_l r + \varsigma n_j s)x + \gamma_r + \varsigma\gamma_s\big),$$

$$\acute{V}_m = \frac{H-K}{2}\sum_{r=1}^{d} c_r^2, \qquad \grave{V}_m = \frac{K}{2}\sum_{r=1}^{d} c_r^2,$$

we decompose $\acute{U}_m^2$ and $\grave{U}_m^2$ in the following ways:

$$\begin{aligned}\acute{U}_m^2(x) &= \Big(\sum_{l=Hm+K+1}^{H(m+1)} f^2(n_l x) - \acute{V}_m\Big) + \Big(2\sum_{l=Hm+K+1}^{H(m+1)-1} \sum_{j=l+1}^{H(m+1)} f(n_l x)f(n_j x) - \acute{W}_m(x)\Big)\\ &\quad + \acute{W}_m(x) + \acute{V}_m.\\ \grave{U}_m^2(x) &= \Big(\sum_{l=Hm+1}^{Hm+K} f^2(n_l x) - \grave{V}_m\Big) + \Big(2\sum_{l=Hm+1}^{Hm+K-1} \sum_{j=l+1}^{Hm+K} f(n_l x)f(n_j x) - \grave{W}_m(x)\Big)\\ &\quad + \grave{W}_m(x) + \grave{V}_m.\end{aligned}$$

We can prove in the same way as before that absolute values of frequencies of trigonometric polynomial expansion of

$$\acute{W}_m^{\triangle} = \frac{\sqrt{1+\delta/3}}{2}\acute{a}^2(\acute{U}_m^2 - \acute{W}_m - \acute{V}_m) + \acute{a}\acute{U}_m$$

belong to $[\,\check{\beta}|n_{Hm+K+1}|, 2d|n_{H(m+1)}|\,]$, and those of

$$\grave{W}_m^{\triangle} = \frac{\sqrt{1+\delta/3}}{2}\grave{a}^2(\grave{U}_m^2 - \grave{W}_m - \grave{V}_m) + \grave{a}\grave{U}_m$$

belong to $[\,\check{\beta}|n_{Hm+1}|, 2d|n_{Hm+K}|\,]$.

We prove the multiple orthogonality of $\acute{W}_{m_1}^{\triangle}, \dots, \acute{W}_{m_t}^{\triangle}$ for $0 \le m_1 < m_2 < \cdots < m_t$, and that of $\grave{W}_{m_1}^{\triangle}, \dots, \grave{W}_{m_t}^{\triangle}$ for $1 \le m_1 < m_2 < \cdots < m_t$.

Expand $\acute{W}_{m_i}^{\triangle}$ into trigonometric polynomial and denote a term by $\cos 2\pi(\acute{\alpha}_i x + \acute{\Gamma}_i)$. By $|n_{Hm+K+1}/n_{Hm}| \ge q^{K+1} > 6d/\check{\beta}$ and $|n_{H(m+1)}/n_{Hm}| \ge q^H \ge 3$, we have

$$\begin{aligned}|\acute{\alpha}_t \pm \acute{\alpha}_{t-1} \pm \cdots \pm \acute{\alpha}_1| &\ge \check{\beta}|n_{Hm_t+K+1}| - 2d(|n_{H(m_{t-1}+1)}| + |n_{H(m_{t-2}+1)}| + \cdots)\\ &\ge \check{\beta}|n_{Hm_t+K+1}| - 2d(|n_{Hm_t}| + |n_{H(m_t-1)}| + \cdots)\\ &\ge \check{\beta}|n_{Hm_t+K+1}| - 2d|n_{Hm_t}|(1 + 1/3 + 1/3^2 + \cdots)\\ &\ge \check{\beta}|n_{Hm_t+K+1}|/2 \ge \check{\beta}|n_{K+1}|/2 \ge q^K\check{\beta}/2 \ge 1/2 \ge 1/\pi.\end{aligned}$$

Hence we can prove multiple orthogonality in the same way as before.

Expand $\grave{W}^{\triangle}_{m_i}$ into trigonometric polynomial and denote a term by $\cos 2\pi(\grave{\alpha}_i x+\grave{\Gamma}_i)$. Because $7K \le K(6/\delta+1) \le H$, we have $H-K+1 \ge K$ and $|n_{Hm+1}/n_{H(m-1)+K}| \ge q^{H-K+1} > q^K \ge 6d/\check{\beta}$. By $|n_{H(m+1)+K}/n_{Hm+K}| \ge q^H \ge 3$, we have

$$\begin{aligned}|\grave{\alpha}_t \pm \grave{\alpha}_{t-1} \pm \cdots \pm \grave{\alpha}_1| &\ge \check{\beta}|n_{Hm_t+1}| - 2d(|n_{Hm_{t-1}+K}| + |n_{Hm_{t-2}+K}| + \cdots)\\ &\ge \check{\beta}|n_{Hm_t+1}| - 2d(|n_{H(m_t-1)+K}| + |n_{H(m_t-2)+K}| + \cdots)\\ &\ge \check{\beta}|n_{Hm_t+1}| - 2d|n_{H(m_t-1)+K}|(1+1/3+1/3^2+\cdots)\\ &\ge \check{\beta}|n_{Hm_t+1}|/2 \ge \check{\beta}|n_{H+1}|/2 \ge q^H\check{\beta}/2 \ge q^K\check{\beta}/2 \ge 1/2.\end{aligned}$$

Hence we can complete the proof of multiple orthogonality.

In the same way as before, we can verify

$$\begin{aligned}|\acute{W}_m| &\le \sqrt{1+\delta/3}\frac{2}{\sqrt{3}(q-1)}\|f\|_2 H, \quad \acute{V}_m \le \sqrt{1+\delta/3}\|f\|_2 H\frac{1}{2},\\ |\grave{W}_m| &\le \sqrt{1+\delta/3}\frac{2}{\sqrt{3}(q-1)}\|f\|_2 K, \quad \grave{V}_m \le \sqrt{1+\delta/3}\|f\|_2 K\frac{1}{2}.\end{aligned}$$

Hence in the same way as before, we have

$$\begin{aligned}\int_{\mathbf{R}} \exp\Big(\acute{a}\sum_{m=0}^{P-1}\acute{U}_m(x)\Big)\,\mu(dx) &\le \exp\Big(\frac{1+\delta/3}{2}\frac{C_1^2}{2}\acute{a}^2\|f\|_2 HP\Big),\\ \int_{\mathbf{R}} \exp\Big(\grave{a}\sum_{m=0}^{P-1}\grave{U}_{m+1}(x)\Big)\,\mu(dx) &\le \exp\Big(\frac{1+\delta/3}{2}\frac{C_1^2}{2}\grave{a}^2\|f\|_2 KP\Big).\end{aligned} \tag{2.17}$$

Put $\acute{\alpha} = 1+\delta/6$ and $\grave{\alpha} = 1+6/\delta$. By noting $\frac{1}{\acute{\alpha}}+\frac{1}{\grave{\alpha}} = 1$ and applying Hölder's inequality, we have

$$\begin{aligned}&\int_{\mathbf{R}} \exp\Big(a\sum_{m=0}^{P-1}\big(\acute{U}_m(x)+\grave{U}_{m+1}(x)\big)\Big)\,\mu(dx)\\ &\le \Big(\int_{\mathbf{R}} \exp\Big(\acute{\alpha}a\sum_{m=0}^{P-1}\acute{U}_m(x)\Big)\,\mu(dx)\Big)^{1/\acute{\alpha}}\Big(\int_{\mathbf{R}} \exp\Big(\grave{\alpha}a\sum_{m=0}^{P-1}\grave{U}_{m+1}(x)\Big)\,\mu(dx)\Big)^{1/\grave{\alpha}}\end{aligned}$$

By putting $\acute{a} = \acute{\alpha}a$ and $\grave{a} = \grave{\alpha}a$, we can verify by $2|a|H^{3/2} \le 1$ that $|\acute{a}|H^{3/2} \le \acute{\alpha}/2 \le 1$ and $|\grave{a}|H^{3/2} \le \grave{\alpha}/2 \le 1+6/\delta$, and hence we can apply (2.17) and have

$$\int_{\mathbf{R}} \exp\Big(a\sum_{m=0}^{P-1}\big(\acute{U}_m(x)+\grave{U}_{m+1}(x)\big)\Big)\,\mu(dx) \le \exp\Big(\frac{1+\delta/3}{2}\frac{C_1^2}{2}a^2\|f\|_2(\acute{\alpha}H+\grave{\alpha}K)P\Big).$$

Because of $\acute{\alpha}H+\grave{\alpha}K \le \acute{\alpha}H+\delta H/6 = (1+\delta/3)H$ and $(1+\delta/3)^2 \le 1+\delta$, we have

$$\int_{\mathbf{R}} \exp\Big(a\sum_{m=0}^{P-1}\big(\acute{U}_m(x)+\grave{U}_m(x)\big)\Big)\,\mu(dx) \le \exp\Big((1+\delta)\frac{C_1^2}{4}a^2\|f\|_2 HP\Big).$$

Let N_1 be sufficiently large such that $C_1^{-1}8N^{-1/8}(\log\log N)^{1/2} \le 1$ holds and $H = [N^{1/6}]$ satisfies (2.15) for all $N \ge N_1$. Note that we are assuming $\|f\|_2 \ge N^{-1/4}/4$. Put

$$a = C_1^{-1}\sqrt{2}\|f\|_2^{-1/2}(2N^{-1}\log\log N)^{1/2}$$

and take P such that $HP < N \le H(P+1)$. By noting $HP < HP + K \le H(P+1)$ and $7K \le H$, we have

$$\begin{aligned} a\Big|\sum_{k=1}^{N} f(n_k x) - \sum_{m=0}^{P-1}\big(\acute{U}_m(x) + \grave{U}_{m+1}(x)\big)\Big| &\le 3a(K+H) \le 6aH \\ \le 24C_1^{-1}N^{-5/24}(\log\log N)^{1/2} &\le 24N^{-5/24}(\log\log N)^{1/2} \le \log(C_2/2). \end{aligned}$$

Since we have $2aH^{3/2} \le C_1^{-1}8N^{-1/8}(\log\log N)^{1/2} \le 1$, the condition for (2.16) is verified. Hence we have

$$\int_{\mathbf{R}} \exp\Big(\pm a\sum_{k=1}^{N} f(n_k x)\Big)\,\mu(dx) \le \frac{C_2}{2}\exp\Big((1+\delta)\frac{C_1^2}{4}a^2\|f\|_2 N\Big).$$

By putting

$$Q = \frac{C_1}{\sqrt{2}}\|f\|_2^{1/2}(2N\log\log N)^{1/2},$$

we have

$$\begin{aligned} \mu\Big(\Big|\sum_{k=1}^{N} f(n_k x)\Big| \ge (1+\delta)Q\Big) &\le \exp(-(1+\delta)aQ)\sum_{\varsigma=\pm1}\int_{\mathbf{R}}\exp\Big(\varsigma a\sum_{k=1}^{N} f(n_k x)\Big)\,\mu(dx) \\ \le C_2\exp\Big(-(1+\delta)aQ + (1+\delta)\frac{C_1^2}{4}a^2\|f\|_2 N\Big) &= C_2\exp\Big(-(1+\delta)\log\log N\Big). \end{aligned}$$

§ 3. Exchange of sup and limsup

By following and modifying the method presented in [7, 9], Erdős-Gál [6], Gál-Gál [10], and Philipp [11], we prove the fundamental result for the exchange of order of sup and limsup appearing in the investigation of the asymptotics of discrepancies.

Although it is proved originally for lacunary series on the probability space $[0,1]$ equipped with the Borel field and the Lebesgue measure, we formulate it as a result on an abstract probability space. It makes it possible to apply the result for lacunary series on the probability space $(\mathbf{R}, \mathcal{B}, \mu)$, which is convenient when we consider non-integer sequences.

Proposition 3.1. *Suppose that a sequence $\{\xi_k\}$ of random variables satisfies the condition below: There exists $B_0 \geq 0$ and C_i $(i = 1, 2, 3)$ such that*

$$P\Big(\Big|\sum_{k=A+1}^{A+B} f(\xi_k)\Big| \geq C_1 R \|f\|_2^{1/4} (2B \log\log B)^{1/2}\Big) \leq C_2 \exp\Big(-2\|f\|_2^{-1/2} R \log\log B\Big) + C_3 R^{-2} B^{-3/4} \tag{3.1}$$

holds for any function f satisfying (2.1), for any $R \geq 1$, for any integer $A \geq 0$, and for any $B \geq B_0$ with

$$\|f\|_2 \geq B^{-1/4}/4. \tag{3.2}$$

Then for a countable dense set $S \subset [0, 1)$, we have

$$\begin{aligned}
\overline{\lim_{N\to\infty}} \frac{ND_N\{\xi_k\}}{\sqrt{2N\log\log N}} &= \sup_{S \ni a' < a \in S} \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a',a)}(\xi_k)\Big| \\
&= \sup_{0 \leq a' < a < 1} \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a',a)}(\xi_k)\Big|, \quad a.s., \\
\overline{\lim_{N\to\infty}} \frac{ND_N^*\{\xi_k\}}{\sqrt{2N\log\log N}} &= \sup_{a \in S} \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)}(\xi_k)\Big| \\
&= \sup_{0 \leq a < 1} \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)}(\xi_k)\Big|, \quad a.s.
\end{aligned} \tag{3.3}$$

Suppose that $f - f_d$ satisfies (2.1) for all $d \in \mathbf{N}$, where f_d is the d-th sub-sum of the Fourier series of f. Then

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} f(\xi_k)\Big| = \lim_{d\to\infty} \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} f_d(\xi_k)\Big|, \quad a.s., \tag{3.4}$$

For all f satisfying (2.1), we have

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} f(\xi_k)\Big| = \lim_{u\to\infty} \overline{\lim_{\mathbf{N}_u \ni N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \Big|\sum_{k=1}^{N} f(\xi_k)\Big|, \ a.s., \tag{3.5}$$

where $\mathbf{N}_u = \{2^n + m2^{n-u} \mid n \geq u;\ 0 \leq m < 2^u\}$.

Let us fix $L \in \mathbf{N}$ arbitrarily and take arbitrary $I = 0, \ldots, 2^L - 1$.

For $h \geq L$ and $\varepsilon(L+1), \ldots, \varepsilon(h) \in \{0, 1\}$, we denote by $\rho_{\varepsilon(L+1),\ldots,\varepsilon(h)}(x)$ the indicator function of the interval

$$\Big[2^{-L}I + \sum_{j=L+1}^{h-1} 2^{-j}\varepsilon(j), 2^{-L}I + \sum_{j=L+1}^{h} 2^{-j}\varepsilon(j)\Big),$$

and denote by $\sigma_{\varepsilon(L+1),\dots,\varepsilon(h)}(x)$ the indicator function of the interval

$$\Big[2^{-L}I+\sum_{j=L+1}^{h}2^{-j}\varepsilon(j),2^{-L}I+\sum_{j=L+1}^{h}2^{-j}\varepsilon(j)+2^{-h}\Big).$$

We define

$$\widetilde{\varphi}_{\varepsilon(L+1),\dots,\varepsilon(h)}(x)=\varphi_{\varepsilon(L+1),\dots,\varepsilon(h)}(\langle x\rangle)-2^{-h}\quad(\varphi=\rho,\sigma).$$

Clearly we see $\int_0^1\widetilde{\varphi}_{\varepsilon(L+1),\dots,\varepsilon(h)}(x)\,dx=0$ and $\widetilde{\varphi}_{\varepsilon(L+1),\dots,\varepsilon(h)}$ satisfies (2.1). Put

$$F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(A,B)(\omega)=\Big|\sum_{k=A+1}^{A+B}\widetilde{\varphi}_{\varepsilon(L+1),\dots,\varepsilon(h)}(\xi_k(\omega))\Big|,\qquad(\varphi=\rho,\sigma).$$

For $n\in\mathbf{N}$, $h\geq L$, $l\leq n$, $m<2^{n-l}$, $\varepsilon(L+1)$, …, $\varepsilon(h)\in\{0,1\}$, $\varphi=\rho$, σ, denote

$$\psi(N)=C_1(2N\log\log N)^{1/2},$$
$$G_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n)=\{F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(0,2^n)\geq 2^{-h/8}\psi(2^n)\},$$
$$H_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n,l,m)=\{F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(2^n+m2^l,2^{l-1})\geq 2^{-h/8}2^{(l-n-2)/9}\psi(2^n)\}.$$

We introduce the notation $H_B=[\,B/2\,]$.

Lemma 3.2. *For almost every ω, there exists an $n_0\in\mathbf{N}$ satisfying the following:*

1. *For all $n\geq n_0$, $h\in[\,L,n/2\,]$, $(\varepsilon(L+1),\dots,\varepsilon(h))\in\{0,1\}^{h-L}$, and $\varphi\in\{\rho,\sigma\}$, it holds that $\omega\notin G_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n)$;*
2. *For all $n\geq n_0$, $l\in[\,n/2,n\,]$, $h\in[\,L,l/2\,]$, $m\in[\,0,2^{n-l})$, $(\varepsilon(L+1),\dots,\varepsilon(h))\in\{0,1\}^{h-L}$, and $\varphi\in\{\rho,\sigma\}$, it holds that $\omega\notin H_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n,l,m)$.*

Proof. The assertion follows from the estimates below and application of the first Borel-Cantelli lemma.

$$\sum_{n=8}^{\infty}\sum_{\varphi=\rho,\sigma}\Big(\sum_{h=L+1}^{H_n}\sum_{(\varepsilon(L+1),\dots,\varepsilon(h))\in\{0,1\}^{h-L}}P\big(G_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n)\big)$$
$$+\sum_{l=H_n}^{n}\sum_{h=L+1}^{H_l}\sum_{m=1}^{2^{n-l}}\sum_{(\varepsilon(L+1),\dots,\varepsilon(h))\in\{0,1\}^{h-L}}P\big(H_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n,l,m)\big)\Big)<\infty. \tag{3.6}$$

By assuming $h=L+1$, …, H_n, we apply (3.1) with $A=0$, $B=2^n$, $R=1$ to have

$$\begin{aligned}&P\big(G_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n)\big)\\&\leq P\big(F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(0,2^n)\geq C_1\|\varepsilon(L+1),\dots,\varepsilon(h)\|_2^{1/4}(2\cdot 2^n\log\log 2^n)^{1/2}\big)\\&\leq C_2\exp\big(-2\|\varepsilon(L+1),\dots,\varepsilon(h)\|_2^{-1/2}\log\log 2^n\big)+C_3 2^{-3n/4}\\&\leq C_2\exp\big(-2\cdot 2^{h/4}\log\log 2^n\big)+C_3 2^{-3n/4}.\end{aligned}$$

Here the condition (3.2) is verified by $\|\widetilde{\varphi}_{\varepsilon(L+1),\dots,\varepsilon(h)}\|_2^2 = 2^{-h} - 2^{-2h} \geq 2^{-h-1} \geq \sqrt{B}/2$. By noting $-2 \cdot 2^{h/4} \leq -3/2 - 2^{h/4}/2$, the first term is estimated by $C_2 n^{-3/2} n^{-2^{h/4}/2} \leq C_2 n^{-3/2} 2^{-2^{h/4}/2}$. Therefore

$$\begin{aligned}\sum_{n=8}^{\infty} \sum_{\varphi=\rho,\sigma} \sum_{h=L+1}^{H_n} \sum_{(\varepsilon(L+1),\dots,\varepsilon(h))\in\{0,1\}^{h-L}} & P\big(G_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n)\big) \\ & \leq 2C_2 \sum_{n=8}^{\infty} n^{-3/2} \sum_{h=L+1}^{\infty} 2^{h-L} 2^{-2^{h/4}/2} + 2C_3 \sum_{n=8}^{\infty} \sum_{h=L+1}^{H_n} 2^{h-L} 2^{-3n/4}.\end{aligned} \tag{3.7}$$

While the first sum is clearly finite, the second sum is bounded by $2C_3 \sum_{n=8}^{\infty} 2^{n/2-L+1} 2^{-3n/4} < \infty$. Since we have

$$\begin{aligned} & P\big(H_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n,l,m)\big) \\ & \leq P\big(F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(2^n + m2^l, 2^{l-1}) \geq 2^{-h/8} C_1 2^{7(n-l)/18} (2 \cdot 2^{l-1} \log\log 2^{l-1})^{1/2}\big) \\ & \leq P\left(\begin{array}{l} F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(2^n + m2^l, 2^{l-1}) \\ \geq C_1 \|\widetilde{\varphi}_{\varepsilon(L+1),\dots,\varepsilon(h)}\|_2^{1/4} 2^{7(n-l)/18} (2 \cdot 2^{l-1} \log\log 2^{l-1})^{1/2} \end{array}\right),\end{aligned}$$

we apply the estimate (3.1) by putting $A = 2^n + m2^l$, $B = 2^{l-1}$, $R = 2^{7(n-l)/18}$. Here we can verify the condition (3.2) by $h \leq H_l$ and $\|\widetilde{\varphi}_{\varepsilon(L+1),\dots,\varepsilon(h)}\|_2^2 \geq 2^{-h-1} \geq 2^{-l/2-1} \geq \sqrt{B}/2$. Therefore we have

$$\begin{aligned} & P\big(H_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(n,l,m)\big) \\ & \quad \leq C_2 \exp\big(-2 \cdot 2^{h/4} 2^{7(n-l)/18} \log\log 2^{l-1}\big) + C_3 2^{7(l-n)/9} 2^{-3(l-1)/4}.\end{aligned}$$

By $-2 \cdot 2^{h/4} 2^{7(n-l)/18} \leq -4/3 - 2^{h/4}/3 - 2^{7(n-l)/18}/3$ and $l - 1 \geq n/2 - 2 \geq 2$, the summation of the first terms is bounded by

$$\begin{aligned}\sum_{n=8}^{\infty} \sum_{\varphi=\rho,\sigma} \sum_{l=H_n}^{n} \sum_{h=L+1}^{H_l} \sum_{m=1}^{2^{n-l}} \sum_{(\varepsilon(L+1),\dots,\varepsilon(h))\in\{0,1\}^{h-L}} & C_2 (n/2-2)^{-4/3} 2^{-2^{h/4}/3} 2^{-2^{7(n-l)/18}/3} \\ \leq 2C_1 \sum_{n=8}^{\infty} (n/2-2)^{-4/3} \sum_{l=H_n}^{n} 2^{n-l} 2^{-2^{7(n-l)/18}/3} & \sum_{h=L+1}^{\infty} 2^{h-L} 2^{-2^{h/4}/3} < \infty.\end{aligned}$$

The summation of the second term is estimated by

$$\begin{aligned}\sum_{n=8}^{\infty} \sum_{\varphi=\rho,\sigma} \sum_{l=H_n}^{n} \sum_{h=L+1}^{H_l} \sum_{m=1}^{2^{n-l}} & \sum_{(\varepsilon(L+1),\dots,\varepsilon(h))\in\{0,1\}^{h-L}} C_3 2^{7(l-n)/9} 2^{-3(l-1)/4} \\ & \leq 2C_3 \sum_{n=8}^{\infty} \sum_{l=H_n}^{n} \sum_{h=L+1}^{H_l} 2^{h-L} 2^{n-l} 2^{7(l-n)/9} 2^{-3(l-1)/4} \\ & \leq 2C_3 \sum_{n=8}^{\infty} \sum_{l=H_n}^{n} 2^{l/2-L+1} 2^{n-l} 2^{7(l-n)/9} 2^{-3(l-1)/4}\end{aligned}$$

which is less than a constant multiple of

$$\sum_{n=8}^{\infty} 2^{n/2-3n/4} \sum_{l=H_n}^{n} 2^{(l-n)/2} 2^{n-l} 2^{7(l-n)/9} 2^{-3(l-n)/4}$$
$$\ll \sum_{n=8}^{\infty} 2^{n/2-3n/4} \sum_{l=H_n}^{n} 2^{17(n-l)/36} \ll \sum_{n=8}^{\infty} 2^{n/2-3n/4+17n/72} = \sum_{n=8}^{\infty} 2^{-n/72} < \infty.$$

□

To show (3.3), we first prove

$$\overline{\lim_{N\to\infty}} \sup_{a<2^{-L}} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[I2^{-L}, I2^{-L}+a)}(\xi_k) \right| \le C_1 2^{9-L/8}, \quad \text{a.s.} \tag{3.8}$$

Assume $n \ge n_0$ and take N satisfying $2^n \le N < 2^{n+1}$. We express N by $N = 2^n + b_{n-1}2^{n-1} + \cdots + b_1 2 + b_0$, $(b_j = 0, 1)$. We set $b_n = 1$.

Defining $\Delta(A, B)$ by

$$\Delta(A, B) = \left| \sum_{k=A+1}^{A+B} \widetilde{\mathbf{1}}_{[2^{-L}I, 2^{-L}I+a)}(\xi_k) \right|,$$

we have the subadditivity $\Delta(A, B) \le \Delta(A, B') + \Delta(A + B', B - B')$ $(B' < B)$. By putting $m_l 2^l = b_{n-1}2^{n-1} + \cdots + b_l 2^l$ (note that $m_n = 0$), we have

$$\begin{aligned}\Delta(0, N) \le \Delta(0, 2^n) + \sum_{l=H_n+1}^{n} \Delta(2^n + b_{n-1}2^{n-1} + \cdots + b_l 2^l, b_{l-1}2^{l-1}\\ + \Delta(2^n + b_{n-1}2^{n-1} + \cdots + b_{H_n}2^{H_n}, b_{H_n-1}2^{H_n-1} + \cdots + b_0)\\ \le \Delta(0, 2^n) + \sum_{l=H_n+1}^{n} \Delta(2^n + 2^l m_l, 2^{l-1}) + 2^{H_n}.\end{aligned}$$

Expressing $a \in [0, 2^{-L})$ by $a = \sum_{j=L+1}^{\infty} 2^{-j}\varepsilon(j)$ $(\varepsilon(j) = 0, 1)$, we have

$$\sum_{j=L+1}^{H_l} 2^{-j}\varepsilon(j) \le a < \sum_{j=L+1}^{H_l} 2^{-j}\varepsilon(j) + 2^{-H_l} \tag{3.9}$$

and hence $\sum_{h=L+1}^{H_l} \rho_{\varepsilon(L+1),\ldots,\varepsilon(h)} \le \mathbf{1}_{[2^{-L}I, 2^{-L}I+a)} < \sum_{h=L+1}^{H_l} \rho_{\varepsilon(L+1),\ldots,\varepsilon(h)} + \sigma_{\varepsilon(L+1),\ldots,\varepsilon(H_l)}$. By subtracting (3.9), we have

$$\begin{aligned}\sum_{h=L+1}^{H_l} \widetilde{\rho}_{\varepsilon(L+1),\ldots,\varepsilon(h)}(x) - 2^{-H_l}\\ \le \widetilde{\mathbf{1}}_{[2^{-L}I, 2^{-L}I+a)}(x) < \sum_{h=L+1}^{H_l} \widetilde{\rho}_{\varepsilon(L+1),\ldots,\varepsilon(h)}(x) + \widetilde{\sigma}_{\varepsilon(L+1),\ldots,\varepsilon(H_l)}(x) + 2^{-H_l}.\end{aligned}$$

By substituting x by ξ_k and summing for $2^n + 2^l m_l < k \le 2^n + 2^l m_l + 2^{l-1}$, we have

$$\Delta(2^n + 2^l m_l, 2^{l-1}) \le \sum_{\varphi=\rho,\sigma} \sum_{h=L+1}^{H_l} F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(2^n + 2^l m_l, 2^{l-1}) + 2^{l/2},$$

where the last error term is produced by $2^{-H_l} 2^{l-1} \le 2^{l/2}$. Therefore we have

$$\begin{aligned}\Delta(0,N) \le \sum_{\varphi=\rho,\sigma} \Biggl(\sum_{h=L+1}^{H_n} & F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(0, 2^n) \\ & + \sum_{l=H_n}^{n} \Biggl\{ \sum_{h=L+1}^{H_l} F_{\varphi;\varepsilon(L+1),\dots,\varepsilon(h)}(2^n + 2^l m_l, 2^{l-1}) + 2^{l/2} \Biggr\} \Biggr) + 2^{n/2} \\ \le 2 \sum_{h=L+1}^{H_n} 2^{-h/8} \psi(2^n) & + 2 \sum_{l=H_n}^{n} \sum_{h=L+1}^{H_l} 2^{-h/8} 2^{(l-n-2)/9} \psi(2^n) + 2^{n/2} \\ \le 2^{-L/8} \psi(N) \frac{1}{1-2^{-1/8}} & \Bigl(1 + \frac{2^{-2/9}}{1-2^{-1/9}} \Bigr) + 4\sqrt{N},\end{aligned}$$

which implies (3.8).

Now we are in a position to prove (3.3). By denoting

$$J_{a',a}^{(N)} = \frac{1}{\sqrt{2N \log\log N}} \Bigl| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a',a)}(\xi_k) \Bigr|,$$

the first part of (3.3) is written as

$$\varlimsup_{N\to\infty} \sup_{0\le a'<a<1} J_{a',a}^{(N)} = \sup_{S\ni a'<a<S} \varlimsup_{N\to\infty} J_{a',a}^{(N)} = \sup_{0\le a'<a<1} \varlimsup_{N\to\infty} J_{a',a}^{(N)}.$$

Because of the trivial inequalities

$$\sup_{S\ni a'<a<S} \varlimsup_{N\to\infty} J_{a',a}^{(N)} \le \sup_{0\le a'<a<1} \varlimsup_{N\to\infty} J_{a',a}^{(N)} \le \varlimsup_{N\to\infty} \sup_{0\le a'<a<1} J_{a',a}^{(N)},$$

it is enough to prove

$$\varlimsup_{N\to\infty} \sup_{0\le a'<a<1} J_{a',a}^{(N)} \le \sup_{S\ni a'<a<S} \varlimsup_{N\to\infty} J_{a',a}^{(N)}.$$

For each $I'' = 0, \dots, 2^{-L} - 1$, we take $s(I'') \in S \cap [2^{-L} I'', 2^{-L}(I''+1))$. For arbitrary $0 \le a' < a < 1$, we take integers I' and $I < 2^{-L}$ such that $2^{-L} I' \le a' < 2^{-L}(I'+1)$ and $2^{-L} I \le a < 2^{-L}(I+1)$. We can easily verify

$$\mathbf{1}_{[a',a)} = -\mathbf{1}_{[2^{-L}I',a')} + \mathbf{1}_{[2^{-L}I',s(I'))} + \mathbf{1}_{[s(I'),s(I))} - \mathbf{1}_{[2^{-L}I,s(I))} + \mathbf{1}_{[2^{-L}I,a)},$$

and hence

$$\int_0^1 \mathbf{1}_{[a',a)} = -\int_0^1 \mathbf{1}_{[2^{-L}I',a')} + \int_0^1 \mathbf{1}_{[2^{-L}I',s(I'))} + \int_0^1 \mathbf{1}_{[s(I'),s(I))} - \int_0^1 \mathbf{1}_{[2^{-L}I,s(I))} + \int_0^1 \mathbf{1}_{[2^{-L}I,a)}.$$

Thanks to $\widetilde{\mathbf{1}}_{[a',a)} = \mathbf{1}_{[a',a)} - \int_0^1 \mathbf{1}_{[a',a)}$, we have

$$\widetilde{\mathbf{1}}_{[a',a)} = -\widetilde{\mathbf{1}}_{[2^{-L}I',a')} + \widetilde{\mathbf{1}}_{[2^{-L}I',s(I'))} + \widetilde{\mathbf{1}}_{[s(I'),s(I))} - \widetilde{\mathbf{1}}_{[2^{-L}I,s(I))} + \widetilde{\mathbf{1}}_{[2^{-L}I,a)},$$

and thereby

$$\begin{aligned} J^{(N)}_{a',a} &\le J^{(N)}_{s(I'),s(I)} + 4 \max_{I''<2^L} \sup_{a''<2^{-L}} J^{(N)}_{2^{-L}I'',2^{-L}I''+a''} \\ &\le \max_{I'<I<2^L} J^{(N)}_{s(I'),s(I)} + 4 \max_{I''<2^L} \sup_{a''<2^{-L}} J^{(N)}_{2^{-L}I'',2^{-L}I''+a''}. \end{aligned} \tag{3.10}$$

Hence $\sup_{0\le a'<a<1} J^{(N)}_{a',a}$ is bounded by (3.10), and by taking the limsup and by noting (3.8) we have

$$\begin{aligned} \varlimsup_{N\to\infty} \sup_{0\le a'<a<1} J^{(N)}_{a',a} &\le \max_{I'<I<2^L} \varlimsup_{N\to\infty} J^{(N)}_{s(I'),s(I)} + 4C_1 2^{9-L/8} \\ &\le \sup_{S\ni s'<s\in S} \varlimsup_{N\to\infty} J^{(N)}_{s',s} + 4C_1 2^{9-L/8}. \end{aligned}$$

By letting $L\to\infty$, we complete the proof of this part. The second part can be proved in the same way.

Secondly we prove (3.4). It is enough to prove the next lemma.

Lemma 3.3. *For any f satisfying (2.1) and any $\{\xi_k\}$ satisfying (3.1), we have*

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^N f(\xi_k)\right| \le C_4 \|f\|_2^{1/4}, \quad a.s., \tag{3.11}$$

where C_4 depends only on C_1.

Actually, by noting

$$\left|\sum_{k=1}^N f(\xi_k)\right| - \left|\sum_{k=1}^N (f_d-f)(\xi_k)\right| \le \left|\sum_{k=1}^N f_d(\xi_k)\right| \le \left|\sum_{k=1}^N f(\xi_k)\right| + \left|\sum_{k=1}^N (f_d-f)(\xi_k)\right|,$$

by taking the limsup with respect to N, and by applying (3.11) for $f_d - f$, we have

$$\begin{aligned} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^N f(\xi_k)\right| - C_4\|f-f_d\|_2^{1/4} &\le \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^N f_d(\xi_k)\right| \\ &\le \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^N f(\xi_k)\right| + C_4\|f-f_d\|_2^{1/4}. \end{aligned}$$

We have (3.4) by letting $d \to \infty$.

We prove (3.11) by assuming $\|f\|_2 > 0$, since everything is trivial when $\|f\|_2 = 0$ or $f = 0$ a.e. Denote

$$\Delta(A,B) = \Big|\sum_{k=A+1}^{A+B} f(\xi_k)\Big|.$$

By putting

$$\begin{aligned} G(n) &= \{\Delta(0,2^n) \geq \|f\|_2^{1/4}\psi(2^n)\} \\ H(n,l,m) &= \{\Delta(2^n+m2^l, 2^{l-1}) \geq 2^{(l-n)/9}\|f\|_2^{1/4}\psi(2^n)\}, \\ n_* &= \min\{n \geq 8 : \|f\|_2 \geq 2^{-n/8}/2\}, \end{aligned}$$

we can prove

$$\sum_{n=n_*}^{\infty}\Big(P(G_n) + \sum_{l=H_n}^{n}\sum_{m=1}^{2^{n-l}} P(H(n,l))\Big) < \infty.$$

Actually, by applying (3.1) for $R = 1$, $A = 0$, and $B = 2^n$, we have

$$P(G(n)) \leq C_2 \exp(-2\|f\|_2^{-1/2}\log\log 2^n) + C_3 2^{-3n/4} \leq C_2 n^{-2} + C_3 2^{-3n/4},$$

and we see that it is summable in n. Here (3.2) can be verified by $n \geq n_*$ and $\|f\|_2 \geq 2^{-n/8}/2 \geq 2^{-n/4}/4$.

By applying (3.1) for $R = 2^{(l-n)/9}2^{(n-l)/2+1/2} = \sqrt{2}\,2^{7(n-l)/18}$ and $B = 2^{l-1}$, we have

$$\begin{aligned} P(H(n,l)) &\leq P\big(\Delta(2^n+m2^l, 2^{l-1}) \geq 2^{(l-n)/9}2^{(n-l)/2+1/2}\|f\|_2^{1/4}(2\cdot 2^{l-1}\log\log 2^{l-1})^{1/2}\big) \\ &\leq C_2 \exp\Big(-2\|f\|_2^{-1/2}\cdot 2^{7(n-l)/18}\sqrt{2}\log\log 2^l\Big) + C_3 2^{7(l-n)/9}2^{-3l/4}. \end{aligned}$$

Here (3.2) can be verified by $n \geq n_*$, $l \geq n/2-1$, and $\|f\|_2 \geq 2^{-n/8}/2 \geq 2^{-l/4}/4$. By noting $\|f\|_2^{-1/2} \geq \sqrt{2}$, we have

$$\begin{aligned} &\exp\Big(-2\|f\|_2^{-1/2}\cdot 2^{7(n-l)/18}\sqrt{2}\log\log 2^l\Big) \leq \exp\Big(-4\cdot 2^{7(n-l)/18}\log\log 2^l\Big) \\ &\leq (l-1)^{-4\cdot 2^{7(n-l)/18}} \leq (n/2-1)^{-4\cdot 2^{7(n-l)/18}} \leq (n/2-1)^{-3}2^{-2^{7(n-l)/18}}, \end{aligned}$$

and hence we see that summation of these can be bounded by

$$\sum_{n=n_*}^{\infty}\sum_{l=H_n}^{n}\sum_{m=1}^{2^{n-l}}(n/2-1)^{-3}2^{-2^{7(n-l)/18}} < \sum_{n=n_*}^{\infty}(n/2-1)^{-3}\sum_{l=-\infty}^{n}2^{(n-l)-2^{7(n-l)/18}} < \infty,$$

$$\sum_{n=n_*}^{\infty}\sum_{l=H_n}^{n}\sum_{m=1}^{2^{n-l}}2^{-7n/9}2^{l/36} = \sum_{n=n_*}^{\infty}\sum_{l=H_n}^{n}2^{n-l}2^{-7n/9}2^{l/36} \ll \sum_{n=n_*}^{\infty}2^{-19n/72} < \infty.$$

After applying the first Borel-Cantelli lemma, we discuss in the same way as before. Let $n \geq n_0$ and take N satisfying $2^n \leq N < 2^{n+1}$. Expand N into $N = 2^n + b_{n-1}2^{n-1} + \cdots + b_1 2 + b_0$ $(b_j = 0, 1)$, and set m_l by $m_l 2^l = b_{n-1}2^{n-1} + \cdots + b_l 2^l$. We can prove

$$\begin{aligned}\Delta(0, N) &\leq \Delta(0, 2^n) + \sum_{l=H_n+1}^{n} \Delta(2^n + m_l 2^l, 2^{l-1}) + \sqrt{N} \\ &\leq \|f\|_2^{1/4}\psi(2^n)\Bigl(1 + \sum_{l=H_n+1}^{n} 2^{(l-n)/9}\Bigr) + \sqrt{N} \leq \frac{2}{1-2^{-1/9}}\|f\|_2^{1/4}\psi(N) + \sqrt{N}.\end{aligned}$$

Lastly we prove (3.5). For fixed u, and for n satisfying $n/2 \geq u$, by using the same notation as the last proof and putting N, n, m_l in the same way, we have

$$\begin{aligned}\Delta(0, N) &\leq \Delta(0, 2^n + m_{n-u}2^{n-u}) + \Delta(2^n + m_{n-u}2^{n-u}, N - (2^n + m_{n-u}2^{n-u})) \\ &\leq \Delta(0, 2^n + m_{n-u}2^{n-u}) + \sum_{l=H_n+1}^{n-u} \Delta(2^n + m_l 2^l, 2^{l-1}) + \sqrt{N} \\ &\leq \Delta(0, 2^n + m_{n-u}2^{n-u}) + \|f\|_2^{1/4}\psi(2^n) \sum_{l=H_n+1}^{n-u} 2^{(l-n)/9} + \sqrt{N} \\ &\leq \Delta(0, 2^n + m_{n-u}2^{n-u}) + \frac{2^{-u/9}}{1-2^{-1/9}}\|f\|_2^{1/4}\psi(N) + \sqrt{N}.\end{aligned}$$

Therefore we have

$$\varlimsup_{N\to\infty} \frac{\Delta(0, N)}{\sqrt{2N\log\log N}} \leq \varlimsup_{\mathbf{N}_u \ni N\to\infty} \frac{\Delta(0, N)}{\sqrt{2N\log\log N}} + C_1\frac{2^{-u/9}}{1-2^{-1/9}}\|f\|_2^{1/4},$$

and by letting $u \to \infty$, we have the '$\leq$' part of (3.5). Since the '$\geq$' part is trivial, the proof is over.

§ 4. Proof of the Main Theorem

Put $\xi_k(x) = n_{\varpi(k)}x$. We prove by assuming $|n_1| \geq 1$. The general case follows trivially, because there are at most finitely many k such that $|n_k| < 1$. By applying Proposition 2.1 for $\delta = 1$, we can verify the condition (3.1). Hence we can apply Proposition 3.1 and have the conclusion of Theorem 1.1.

§ 5. Hadamard gap sequences

Here we prove Corollary 1.2. By applying Koksma's inequality

$$\left|\sum_{k=1}^{N} f(x_k) - N\int_0^1 f(x)\,dx\right| \leq \mathrm{Var}(f)ND_N \tag{5.1}$$

to $f(x) = \cos 2\pi x$, by noting $\mathrm{Var}(f) = 4$, we have

$$\frac{1}{\sqrt{2}} = \overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^{N} \cos 2\pi n_{\varpi(k)} x\right| \le 4 \overline{\lim_{N\to\infty}} \frac{N D_N(n_{\varpi(k)})x}{\sqrt{2N\log\log N}} \quad \text{a.e.}$$

Here the left side equality is due to the law of the iterated logarithm for a permutation of lacunary trigonometric series by Aistleitner-Berkes-Tichy [4].

We prove

$$\overline{\lim_{\mathbf{N}_u \ni N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)} x)\right| \le (1+\delta)\frac{C_1}{\sqrt{2}} \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^{1/2}, \quad \text{a.e.} \tag{5.2}$$

for any $0 < \delta < 1$. By this together with (1.8) and $\|\widetilde{\mathbf{1}}_{[a,b);d}\|_2 \le 1/2$, we have

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)} x)\right| \le (1+\delta)\frac{C_1}{\sqrt{2}} \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^{1/2} \le (1+\delta)\frac{C_1}{2}, \quad \text{a.e.}$$

Since $0 < \delta < 1$ is arbitrary, we have

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)} x)\right| \le \frac{C_1}{2}, \quad \text{a.e.} \tag{5.3}$$

Therefore, by applying (1.7) and (1.6) in turn, we have the upper bound estimate part of Corollary 1.2.

The proof of (5.2) can be done in the following way. By Proposition 2.1 and the inequality (2.6), we have

$$\begin{aligned} &\mu\left(\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)} x)\right| \ge (1+\delta)\frac{C_1}{\sqrt{2}} \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^{1/2} (2N\log\log N)^{1/2}\right) \\ &\le C_2 \exp(-(1+\delta)\log\log N). \end{aligned}$$

Hence we have the following summability estimate, which proves (5.2) by Borel-Cantelli Lemma.

$$\begin{aligned} &\sum_{N\in\mathbf{N}_u} \mu\left(\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)} x)\right| \ge (1+\delta)\frac{C_1}{\sqrt{2}} \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^{1/2} (2N\log\log N)^{1/2}\right) \\ &\le C_2 \sum_{n=u}^{\infty} \sum_{m=0}^{2^u-1} \exp(-(1+\delta)\log\log(2^n + m2^{n-u})) \le C_2 2^u \sum_{n=u}^{\infty} \exp(-(1+\delta)\log\log 2^n) \\ &< \infty. \end{aligned}$$

Finally, we mention the following lemma, which may be convenient in some situation. It can be derived from Proposition 2.1 and Lemma 3.3.

Lemma 5.1. *For any $\{n_k\}$ satisfying (1.5), for any f satisfying (2.1), and for any permutation ϖ of $\mathbf{N}$, we have*

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^{N} f(n_{\varpi(k)}x)\right| \le C\|f\|_2^{1/4}, \quad a.e.,$$

where C depends only on q.

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References

[1] Aistleitner, C., On the law of the iterated logarithm for the discrepancy of $\langle n_k x\rangle$ with multidimensional indices. *Unif. Distrib. Theory,* **2** (2007), 89–104.

[2] Aistleitner, C., On the law of the iterated logarithm for the discrepancy of lacunary sequences, *Trans. Amer. Math. Soc.*, **362** (2010), 5967–5982.

[3] Aistleitner, C., Berkes, I. and Tichy, R., Lacunary sequences and permutations, *Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp,* Eds. Berkes, I., Bradley, R., Dehling, H., Peligrad, M., and Tichy, R., Kendrick press, 2010, pp. 35–49.

[4] Aistleitner, C., Berkes, I. and Tichy, R., *On the law of the iterated logarithm for permuted lacunary sequences*, Proc. Steklov Inst. Math., (to appear)

[5] Berkes, I., Philipp, W. and Tichy, R., Metric discrepancy results for sequence $\{n_k x\}$ and diophantine equations, *Diophantine Approximations*, Eds. Tichy R., Schlickewei H., Schmidt K., Festschrift for Wolfgang Schmidt, *Development in Mathematics* **17**, Springer, 2008, pp. 95–105.

[6] Erdős, P. and Gál, I., On the law of the iterated logarithm I, II, *Nederl. Akad. Wetensch. Proc. Ser. A*, **58** (*Indag. Math.***17**) (1955), 65–76, 77–84.

[7] Fukuyama, K., The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$, *Acta Math. Hungar.*, **118** (2008), 155–170.

[8] Fukuyama, K., The law of the iterated logarithm for the discrepancies of a permutation of $\{n_k x\}$, *Acta Math. Hungar.*, **123** (2009), 121–125.

[9] Fukuyama, K., A central limit theorem and a metric discrepancy result for sequence with bounded gaps, *Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp*, Eds. Berkes, I., Bradley, R., Dehling, H., Peligrad, M., and Tichy, R., Kendrick press, 2010, pp. 233–246.

[10] Gál, L. and Gál, S., The discrepancy of the sequence $\{2^n x\}$, *Indag. Math.*, **26** (1964), 129–143.

[11] Philipp, W., Limit theorems for lacunary series and uniform distribution mod 1, *Acta Arith.*, **26** (1975), 241–251.

[12] Philipp, W., A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables, *Ann. Probab.* **5** (1977), 319–350.

[13] Takahashi, S., An asymptotic property of a gap sequence, *Proc. Japan Acad.*, **38** (1962), 101–104.