RIMS Kôkyûroku Bessatsu
B29 (2012), 41–64

Metric discrepancy results for geometric progressions and variations

By

Katusi FUKUYAMA*

Abstract

In the first section, we make a brief survey of studies on metric discrepancy results for geometric progressions and variations. After announcing a few new results, we show the law of the iterated logarithm for discrepancies of geometric progressions including the case when the common ratio is negative.

§ 1. Introduction

We say that a sequence $\{x_k\}$ is uniformly distributed modulo 1 if

$$\lim_{N\to\infty}\frac{1}{N}{}^{\#}\{k\le N\mid\langle x_k\rangle\in[\,a,b)\}=b-a \tag{1.1}$$

holds for every $[\,a,b)\subset[\,0,1)$. Here $\langle x\rangle$ denotes the fractional part $x-[\,x\,]$ of x. This definition is to be equivalent to the condition that

$$\lim_{N\to\infty}\frac{1}{N}{}^{\#}\{k\le N\mid\langle x_k\rangle\in[\,0,a)\}=a \tag{1.2}$$

holds for all $0\le a<1$. We can also see that the convergence (1.2) is uniform in a, and the convergence (1.1) is uniform in $[\,a,b)$.

We introduce the notion of discrepancy which measures the speed of the uniform convergence by means of supremum norm. We define the discrepancy D_N and the star

Received 25 Feb. 2011, Revised 12 Dec. 2011, Accepted 29 Feb. 2012
2000 Mathematics Subject Classification(s): 11K38, 42A55, 60F15
Key Words: uniform distribution, discrepancy, lacunary sequence, law of the iterated logarithm
The author is supported by KAKENHI 19204008 and 21340027.
*Department of Mathematics, Kobe University, Rokko, Kobe, 657-8501, Japan.
e-mail: `fukuyama@math.kobe-u.ac.jp`

discrepancy D_N^* of a sequence $\{x_k\}$ by

$$\begin{aligned} D_N\{x_k\} &= \sup_{0\le a<b<1}\left|\frac{1}{N}\#\{k\le N \mid \langle x_k\rangle \in [\,a,b)\} - (b-a)\right| \\ &= \sup_{0\le a<b<1}\left|\frac{1}{N}\sum_{k=1}^N \widetilde{\mathbf{1}}_{[a,b)}(x_k)\right|; \\ D_N^*\{x_k\} &= \sup_{0\le a<1}\left|\frac{1}{N}\#\{k\le N \mid \langle x_k\rangle \in [\,0,a)\} - a\right| \\ &= \sup_{0\le a<1}\left|\frac{1}{N}\sum_{k=1}^N \widetilde{\mathbf{1}}_{[0,a)}(x_k)\right|, \end{aligned}$$

where $\widetilde{\mathbf{1}}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(\langle x\rangle) - (b-a)$, and $\mathbf{1}_{[a,b)}$ denotes the indicator function of $[a,b)$.

The most well known and investigated sequences which are uniformly distributed are arithmetic progressions $\{k\alpha\}$ with irrational common difference $\alpha \notin \mathbf{Q}$. Kronecker [39] proved that the fractional parts are dense in unit interval, and Weyl [51], Sierpiński [48], and Bohl [20] proved independently that they are uniformly distributed modulo 1.

Weyl also succeeded in giving metric results. He proved for every strictly increasing sequence $\{n_k\}$ of integers, $\{n_k x\}$ is uniformly distributed modulo 1 for almost every x with respect to the Lebesgue measure. But we cannot say that it holds for every x. Actually, $\{k!x\}$ is uniformly distributed modulo 1 for almost every x, but the fractional parts of $\{k!e\}$ converge to 0 and cannot be distributed uniformly. In this case the set of x for which the sequence is not uniformly distributed is very large and has Hausdorff dimension 1.

Since $\{kx\}$ is uniformly distributed a.e., we have $D_N^*\{kx\} \to 0$ a.e. As to the speed of convergence, we have two major results. One is proved by Khintchine [38] that

$$ND_N^*\{kx\} = O\left((\log N) g(\log\log N)\right), \quad \text{a.e. } x$$

if and only if an increasing function g satisfies $\sum 1/g(n) < \infty$. Kesten [37] showed the following result and proved that the speed of convergence is determined in the sense of convergence in measure: For any $\varepsilon > 0$, it holds that

$$\lim_{N\to\infty} \mathrm{Leb}\left\{x \in [\,0,1\,] \;\middle|\; \left|\frac{ND_N^*\{kx\}}{\log N \log\log N} - \frac{2}{\pi^2}\right| > \varepsilon\right\} = 0.$$

For a sequence $\{n_k\}$ satisfying Hadamard's gap condition

$$n_{k+1}/n_k \ge q > 1 \quad (k = 1, 2, \dots), \tag{1.3}$$

Erdős-Gál conjectured that

$$ND_N\{n_k x\} = O((N\log\log N)^{1/2}), \quad \text{a.e.}$$

By applying methods due to Erdős-Gál [23], Gál-Gál [36], and Takahashi [47], Philipp [43] proved the next theorem and solved the conjecture.

Theorem 1.1. *If a sequence $\{n_k\}$ of integers satisfies Hadamard's gap condition (1.3), then*

$$\frac{1}{4\sqrt{2}} \leq \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} \leq C < \infty, \quad a.e. \tag{1.4}$$

Here $C = (166 + 664/(q^{1/2}-1))/\sqrt{2}$.

It is proved by the argument of exponential integrability, i.e., by a method of real analysis.

Philipp [44] proved it again by the method of martingale approximations, and removed the assumption that n_k are integers. This method became the main stream of the investigation. But, as Aistleitner-Berkes-Tichy [11] pointed out, it is not appropriate for the study of the permutation of $\{n_k\}$, and there still remains a merit of the method of exponential integrability.

Philipp [41] also proved that

$$\overline{\lim_{N\to\infty}} \frac{ND_N\{2^k x\}}{\sqrt{2N\log\log N}}$$

equals to a constant a.e., but did not evaluate the explicit value of this constant.

Dhompongsa [22] assumed a very strong gap condition

$$\log(n_{k+1}/n_k)/\log\log k \to \infty \quad \text{as} \quad k \to \infty$$

and approximated the empirical process by the Kiefer process. As a corollary of this we can derive

$$\overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad \text{a.e.}$$

This value $\frac{1}{2}$ is the same as that appears in Chung-Smirnov theorem [21, 49], and it shows that the sequence is extremely nearly independent. The gap condition was weakened to

$$n_{k+1}/n_k \to \infty \quad \text{as} \quad k \to \infty,$$

and the same law of the iterated logarithm was proved in [26].

It is very natural to ask if Hadamard's gap condition is really necessary to have the bounded law of the iterated logarithm (1.4), since the gap condition can be replaced by weaker one to have the central limit theorem for lacunary trigonometric series.

It was prove by Berkes-Philipp [16] that it is impossible to replace the gap condition by weaker ones.

Theorem 1.2 ([16]). *For any $0 < \varepsilon_k \downarrow 0$, there exists a sequence $\{n_k\}$ satisfying $n_{k+1}/n_k \geq 1 + \varepsilon_k$ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \infty, \quad a.e.$$

The next is a recent result which gives explicit values appearing in the law of the iterated logarithm for discrepancies of geometric progressions $\{\theta^k x\}$.

Theorem 1.3 ([25, 27]). *For any $\theta > 1$, there exists a real number Σ_θ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{\theta^k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta, \quad a.e. \tag{1.5}$$

We have

$$\Sigma_\theta = 1/2$$

if and only if θ satisfies

$$\theta^r \notin \mathbf{Q} \quad (r \in \mathbf{N}). \tag{1.6}$$

In other cases, θ can be written uniquely by

$$\theta = \sqrt[r]{p/q}, \quad r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\},\ p, q \in \mathbf{N}, \quad \gcd(p,q) = 1. \tag{1.7}$$

In this case Σ_θ does not depend on r and satisfies

$$1/2 < \Sigma_\theta \leq \sqrt{(pq+1)/(pq-1)}/2.$$

Moreover, we can evaluate it in the following cases:

$$\Sigma_\theta = \begin{cases} \sqrt{(pq+1)/(pq-1)}/2, & \textit{if } p \textit{ and } q \textit{ are both odd;} \\ \sqrt{(p+1)/(p-1)}/2, & \textit{especially if } p \textit{ is odd and } q = 1; \\ \sqrt{(p+1)p(p-2)/(p-1)^3}/2, & \textit{if } p \geq 4 \textit{ is even and } q = 1; \\ \sqrt{42}/9, & \textit{if } p = 2 \textit{ and } q = 1; \\ \sqrt{22}/9, & \textit{if } p = 5 \textit{ and } q = 2. \end{cases}$$

If we regard Σ_θ as a function of $\theta > 1$, it is discontinuous at every θ which is a power root of a rational number, and is continuous elsewhere. We can also say that the maximum value of Σ_θ is $\sqrt{42}/9$ which is taken at the points $\theta = \sqrt[r]{2}$. In other words, the geometric progression with ratio 2 is furthest from the uniform distribution.

We can evaluate in a way that $\Sigma_3 = \sqrt{2}/2 = 0.707\ldots$, $\Sigma_9 = \sqrt{5}/4 = 0.559\ldots$, $\Sigma_{27} = \sqrt{14/13}/2 = 0.518\ldots$, and see that the subsequences $\{9^k x\}$, $\{27^k x\}$, $\ldots$ of $\{3^k x\}$ seem to approach to the uniform distribution. Actually we can prove the following result showing that any subsequence of the geometric progression is closer to the uniform distribution.

Theorem 1.4 ([30]). *Let $\mathcal{N}$ be the collection of all strictly increasing sequences of positive integers. For any $\theta > 1$ and $\{m(k)\} \in \mathcal{N}$, there exists a real number $\Sigma_{\theta,\{m(k)\}} \geq 1/2$ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{\theta^{m(k)}x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{\theta^{m(k)}x\}}{\sqrt{2N\log\log N}} = \Sigma_{\theta,\{m(k)\}}, \quad a.e.$$

We have

$$\left\{ \Sigma_{\theta,\{m(k)\}} \,\middle|\, \{m(k)\} \in \mathcal{N} \right\} = [\,1/2, \Sigma_\theta\,].$$

As to the question if every positive number can be a constant appearing in the law of the iterated logarithm for discrepancies of some sequence $\{n_k\}$, we have the following two results.

Theorem 1.5 ([35]). *For any $\Sigma \geq 1/2$, there exists a sequence $\{n_k\}$ of positive integers satisfying Hadamard's gap condition and*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \Sigma, \quad a.e.$$

Theorem 1.6 ([27]). *For any $0 < \Sigma < 1/2$, there exists a sequence $\{n_k\}$ of positive integers such that $1 \leq n_{k+1} - n_k \leq \lceil (1+4\Sigma^2)^2/(1-4\Sigma^2)^2 \rceil$ and*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \Sigma, \quad a.e.$$

Above results are proved by means of technique of randomization. As to the randomly generated sequences, we have some results by Weber [50]. One of these gives a metric discrepancy result for sequences defined by sum of random variables. For sequences defined by product of random integers, we can show the following:

Theorem 1.7 ([34]). *Let A and B be subsets of $\{1, 2, \dots\}$ satisfying the condition below: there exist $c > 1$ and $q > 1$ such that*

$$b/a \geq q \quad \textit{and} \quad \gcd(a,b) = 1 \quad \textit{for all} \quad a \in A \quad \textit{and} \quad b \in B. \tag{1.8}$$

Let $\{(X_k, Y_k)\}$ be an $A \times B$-valued i.i.d. For a sequence $\{n_k\}$ of integers given by

$$n_k = \prod_{j=1}^{k} \frac{Y_j}{X_j}, \tag{1.9}$$

there exists a constant $\Sigma_{\mathcal{L}(X_1,Y_1)}$ *depending only on the law* $\mathcal{L}(X_1,Y_1)$ *of the random variable* (X_1,Y_1) *such that*

$$P\Big(\varlimsup_{N\to\infty}\frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}}=\varlimsup_{N\to\infty}\frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}}=\Sigma_{\mathcal{L}(X_1,Y_1)},\quad a.e.\ x.\Big)=1. \tag{1.10}$$

When A *and* B *both consist of odd numbers, we have*

$$\Sigma_{\mathcal{L}(X_1,Y_1)}=\frac{1}{2}\sqrt{\frac{\Upsilon_{\mathcal{L}(X_1,Y_1)}+1}{\Upsilon_{\mathcal{L}(X_1,Y_1)}-1}},\quad \textit{where}\quad \Upsilon_{\mathcal{L}(X_1,Y_1)}=\Big(E\Big(\frac{1}{X_1Y_1}\Big)\Big)^{-1}. \tag{1.11}$$

We can investigate the following variation of geometric progressions.

Example 1.8. Let $\{n_k\}$ be the arrangement in increasing order of the union of $\{15^k\}$ and $\{15^k\cdot 3\}$. Then

$$\varlimsup_{N\to\infty}\frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}}=\varlimsup_{N\to\infty}\frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}}=\sqrt{\frac{3}{7}},\quad \text{a.e.} \tag{1.12}$$

For an arrangement in increasing order of the union of $\{15^k\}$ and $\{15^k\cdot 5\}$, we have the same result.

We can explain the reason why these two sequences obey the same limiting behavior by the fact that these two sequences are transformed to the other if we replace x by $3x$ or $5x$ and omit the first few terms. We have the same situation for $\{\theta^kx\}\cup\{\theta^kAx\}$ and $\{\theta^kx\}\cup\{\theta^kBx\}$ if we take positive A and B with $\theta=AB$.

This example can be regarded as a special case of the result below.

Theorem 1.9 ([34]). *For* $\theta_1,\ \dots,\ \theta_\tau>1$, *we define a sequence* $\{n_k\}$ *by*

$$n_0=1,\quad n_{k+1}=\theta_{j+1}n_k\ \textit{if}\ k=j \bmod \tau\ \textit{and}\ j=0,\dots,\tau-1.$$

Then there exists a constant $\Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{periodic}}$ *such that*

$$\varlimsup_{N\to\infty}\frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}}=\varlimsup_{N\to\infty}\frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}}=\Sigma_{\theta_1,\dots,\theta_\tau;\mathrm{periodic}},\quad a.e. \tag{1.13}$$

We have permutation invariance among generators when $\tau=2,\ 3$, *i.e.,*

$$\Sigma_{\theta_1,\theta_2;\mathrm{periodic}}=\Sigma_{\theta_2,\theta_1;\mathrm{periodic}}, \tag{1.14}$$

and

$$\begin{aligned}&\Sigma_{\theta_1,\theta_2,\theta_3;\mathrm{periodic}}=\Sigma_{\theta_2,\theta_3,\theta_1;\mathrm{periodic}}=\Sigma_{\theta_3,\theta_1,\theta_2;\mathrm{periodic}}\\&=\Sigma_{\theta_1,\theta_3,\theta_2;\mathrm{periodic}}=\Sigma_{\theta_3,\theta_2,\theta_1;\mathrm{periodic}}=\Sigma_{\theta_2,\theta_1,\theta_3;\mathrm{periodic}}.\end{aligned} \tag{1.15}$$

Let A and B be sets of positive integers satisfying (1.8). If A and B both consist of odd numbers, and if $p_j \in B$ and $q_j \in A$ $(j = 1, \dots, \tau)$, then

$$\Sigma_{p_1/q_1,\dots,p_\tau/q_\tau;\text{periodic}}$$
$$= \frac{1}{2}\sqrt{\frac{1}{(s_1 \dots s_\tau - 1)}\Big(1 + s_1 \dots s_\tau + \frac{2}{\tau}\sum_{1\le j<k\le\tau}\big(s_j \dots s_{k-1} + s_1 \dots s_{j-1}s_k \dots s_\tau\big)\Big)},$$

where $s_j = p_j q_j$.

The other interesting example is a union of geometric progressions. Although it does not satisfy Hadamard's gap condition anymore, we still have a metric discrepancy result below:

Theorem 1.10 ([34]). *Suppose that $\theta_1, \dots, \theta_\tau > 1$ are given and that geometric progressions $\{\theta_1^k\}, \dots, \{\theta_\tau^k\}$ are mutually disjoint from each other, i.e.,*

$$\log\theta_i/\log\theta_j \notin \mathbf{Q}, \quad (i \ne j). \tag{1.16}$$

Let $\{n_k\}$ be the arrangement in increasing order of $\{\theta_1^k\} \cup \dots \cup \{\theta_\tau^k\}$. Then there exists a real number $\Sigma_{\theta_1,\dots,\theta_\tau;\text{union}}$ such that

$$\varlimsup_{N\to\infty}\frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty}\frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma_{\theta_1,\dots,\theta_\tau;\text{union}}, \quad a.e.$$

If each θ_j satisfies (1.6) or given by (1.7) with odd p and q, then

$$\Sigma_{\theta_1,\dots,\theta_\tau;\text{union}} = \sqrt{\Big(\frac{\Sigma_{\theta_1}^2}{\log\theta_1} + \dots + \frac{\Sigma_{\theta_\tau}^2}{\log\theta_\tau}\Big) \Big/ \Big(\frac{1}{\log\theta_1} + \dots + \frac{1}{\log\theta_\tau}\Big)},$$

where $\Sigma_{\theta_1}, \dots, \Sigma_{\theta_\tau}$ are defined by (1.5).

We here introduce the Hardy-Littlewood-Pólya sequences which are extensions of geometric progressions and have been investigated well. When relatively prime positive integers $q_1, \dots, q_\tau$ are given, a Hardy-Littlewood-Pólya sequence is the arrangement in increasing order of the semigroup

$$\{q_1^{i_1} \dots q_\tau^{i_\tau} \mid i_1, \dots, i_\tau \in \{0, 1, 2, \dots\}\}.$$

Although these sequences do not satisfy Hadamard's gap condition, Philipp [45] proved the bounded law of the iterated logarithm (1.4) and solved Baker's conjecture. In this case upper bound constant C depends on τ, the number of generators.

We here present a recently proved preciser result.

Theorem 1.11 ([32]). *Let $\{n_k\}$ be the Hardy-Littlewood-Pólya sequence generated by $q_1, \ldots, q_\tau$. There exists a real number $\Sigma_{q_1,\ldots,q_\tau;\mathrm{HLP}}$ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \Sigma_{q_1,\ldots,q_\tau;\mathrm{HLP}}, \quad a.e.$$

If $q_1, \ldots, q_\tau$ are odd numbers, we have

$$\Sigma_{q_1,\ldots,q_\tau;\mathrm{HLP}} = \frac{1}{2}\biggl(\prod_{i=1}^{\tau} \frac{q_i+1}{q_i-1}\biggr)^{1/2}.$$

In case when $\tau \geq 2$ and q_1 is even, we have an estimate

$$\frac{1}{2}\biggl(\prod_{i=2}^{\tau} \frac{q_i+1}{q_i-1}\biggr)^{1/2} \leq \Sigma_{q_1,\ldots,q_\tau;\mathrm{HLP}} \leq \frac{1}{2}\biggl(\prod_{i=1}^{\tau} \frac{q_i+1}{q_i-1}\biggr)^{1/2}.$$

Aistleitner gave an almost optimal condition to have the Chung-Smirnov type result for a sequence satisfying Hadamard's gap condition.

Theorem 1.12 (Aistleitner [2]). *For positive integers N and d, and for non-negative integer ν, we denote*

$$L(N,d,\nu) = {}^{\#}\Bigl(\{(a,b,k,l) \in [1,d]^2 \times [1,N]^2 \mid an_k - bn_l = \nu\} \setminus \{(a,a,k,k) \mid a,k \in \mathbf{N}\}\Bigr).$$

If

$$\sup_{\nu\geq 0} L(N,d,\nu) = O\Bigl(\frac{N}{(\log N)^{1+\varepsilon}}\Bigr) \quad (N\to\infty)$$

holds for any d, we have

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad a.e.$$

Under a gap condition which is weaker than the Hadamard's, Aistleitner [6] also gave a stronger Diophantus condition and succeeded in proving the same conclusion.

As to the question if there exists very slowly diverging sequence which obeys the Chung-Smirnov type limiting behavior, we have the following result.

Theorem 1.13 ([29]). *For any sequence $\{G(k)\}$ of real numbers satisfying $1 \leq G(k) \uparrow \infty$, there exists a sequence $\{n_k\}$ of integers satisfying $1 \leq n_{k+1} - n_k \leq G(k)$ and*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad a.e.$$

So far we have concerned with the results in which both limsups concerning the discrepancies and the star discrepancies equal to some constant for almost every x, and it was open if there exists a case in which they are not constant. As to this problem, we have the following three answers.

Theorem 1.14 ([28]). *Put*

$$\Sigma^2(x) = \begin{cases} 4x, & 0 \le x \le 1/4; \\ 1, & 1/4 \le x \le 1/2; \\ \Sigma^2(1-x), & 1/2 \le x < 1. \end{cases}$$

There exists a sequence $\{n_k\}$ of positive integers such that $1 \le n_{k+1} - n_k \le 5$ and

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma(x), \quad a.e.$$

Theorem 1.15 (Aistleitner [3]). *For the sequence defined by $n_{2k-1} = 2^{k^2}$ and $n_{2k} = 2^{k^2+1} - 1$, $(k = 1, 2, \dots)$. we have*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma(x), \qquad \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \frac{3}{4\sqrt{2}}, \quad a.e.,$$

where

$$\Sigma^2(x) = \begin{cases} 9/32, & 0 \le x \le 3/8; \\ (4x(1-x)-x)/2, & 3/8 \le x \le 1/2; \\ \Sigma^2(1-x), & 1/2 \le x < 1. \end{cases}$$

Theorem 1.16 (Aistleitner [7]). *For the sequence*

$$n_k = \begin{cases} 2^{k^2}, & k = 1 \mod 4; \\ 2^{(k-1)^2+1} - 1, & k = 2 \mod 4; \\ 2^{k^2+k}, & k = 3 \mod 4; \\ 2^{(k-1)^2+(k-1)+1} - 2, & k = 0 \mod 4, \end{cases}$$

we have

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma(x), \qquad a.e.,$$

where

$$\Sigma^2(x) = \begin{cases} 9/32, & 0 \le x \le 3/8; \\ (4x(1-x)-x)/2, & 3/8 \le x \le 7/16; \\ 49/128 - x/4, & 7/16 \le x \le 1/2; \\ \Sigma^2(1-x), & 1/2 \le x < 1. \end{cases}$$

The above two examples by Aistleitner can be regarded as modifications of subsequences of Erdős-Fortet sequence $\{2^k - 1\}$. As to the asymptotic behavior of this sequence itself, we have the following result.

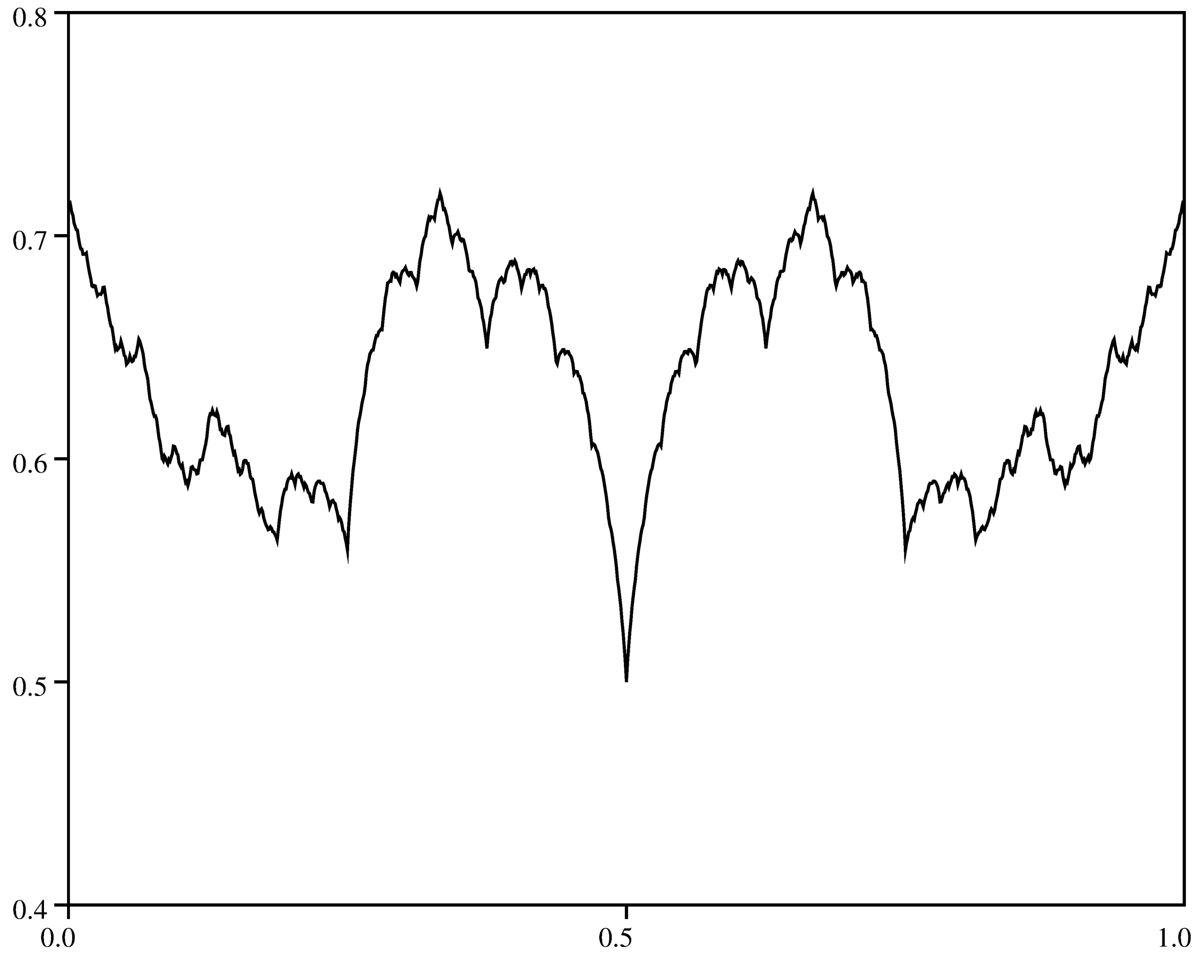

Figure 1. Graph of $\Sigma_2^*(x)$.

Theorem 1.17 ([31]). *For any $\theta > 1$, we have*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{(\theta^k - 1)x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta \quad \textit{and} \quad \varlimsup_{N\to\infty} \frac{ND_N^*\{(\theta^k - 1)x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta^*(x), \qquad \textit{a.e.}$$

Here Σ_θ is a constant defined by (1.5) and $\Sigma_\theta^*(x)$ is a continuous function on the torus.

Especially if θ satisfies (1.6), then we have $\Sigma_\theta^*(x) = 1/2 = \Sigma_\theta$.

If θ is given by (1.7) and satisfies one of the following conditions, then $\Sigma_\theta^*(x)$ is not a constant, and $\Sigma_\theta^*(x) < \Sigma_\theta$ holds except for finitely many x:

(i) both of p and q are odd;

(ii) $q = 1$;

(iii) $p = 5$ and $q = 2$.

We can see an irregular nature of the function $\Sigma_\theta^*(x)$ by the graph of $\Sigma_2^*(x)$ in Figure 1.

By recalling the definition of discrepancy, we have

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \sup_{0\le a<b<1} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x) \right|.$$

To determine the exact value, it is always difficult to calculate the limsup of sup with respect to $a < b$. If we abbreviate taking sup for $a < b$, calculating limsup correspond to the law of the iterated logarithm in probability theory, which have been investigated very well. Since the calculation of

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x) \right|$$

is rather classical, it is very natural to expect to have the change of order of these limiting procedure in the following way:

$$\begin{aligned} &\varlimsup_{N\to\infty} \sup_{0\le a<b<1} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x) \right| \\ &= \sup_{0\le a<b<1} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x) \right|. \end{aligned}$$

Almost all results so far we have explained was proved by showing this exchange individually in each cases. Actually, it is valid for every sequence of positive numbers satisfying Hadamard's gap condition. ([27]).

We can also prove the following version of this result, which removes the assumption of positivity of n_k and includes permutations of sequences.

Proposition 1.18 ([33]). *Let $\{n_k\}$ be a sequence of real numbers satisfying*

$$n_1 \neq 0, \qquad |n_{k+1}/n_k| \ge q > 1 \quad (k = 1, 2, \dots), \tag{1.17}$$

and ϖ be a permutation of $\mathbf{N}$, i.e., a bijection $\mathbf{N} \to \mathbf{N}$. Then for any dense countable set $S \subset [0, 1)$, we have

$$\begin{aligned} \varlimsup_{N\to\infty} \frac{ND_N\{n_{\varpi(k)} x\}}{\sqrt{2N\log\log N}} &= \sup_{S\ni a<b\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)} x) \right| \\ &= \sup_{0\le a<b<1} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)} x) \right|, \\ \varlimsup_{N\to\infty} \frac{ND^*_N\{n_{\varpi(k)} x\}}{\sqrt{2N\log\log N}} &= \sup_{a\in S} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)}(n_{\varpi(k)} x) \right| \\ &= \sup_{0\le a<1} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)}(n_{\varpi(k)} x) \right|, \end{aligned} \tag{1.18}$$

for almost every $x \in \mathbf{R}$. If we denote the d-th subsum of the Fourier series of $\widetilde{\mathbf{1}}_{[a,b)}$ by $\widetilde{\mathbf{1}}_{[a,b);d}$, we have

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \Big| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_{\varpi(k)}x) \Big| = \lim_{d\to\infty} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \Big| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_{\varpi(k)}x) \Big|$$

for almost every $x \in \mathbf{R}$.

Usually, a result of this kind is that for a.e. $x \in [0,1]$. In case when all n_k are integers, then the limsup clearly has period 1, and the result is known to be valid for a.e. $x \in \mathbf{R}$. But when n_k is not necessarily integers, this point is not trivial.

Let us go back to a classical result, the bounded law of the iterated logarithm (1.4) by Philipp. As to this result, Aistleitner [2] recently proved the following result and gave a preciser estimate in the case $q \geq 2$.

Theorem 1.19. *If a sequence $\{n_k\}$ of integers satisfies Hadamard's gap condition (1.3) with $q \geq 2$, then*

$$\frac{1}{2} - \frac{8}{q^{1/4}} \leq \varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} \leq \frac{1}{2} + \frac{6}{q^{1/4}}, \quad a.e. \tag{1.19}$$

Berkes-Philipp-Tichy [18] stated that the bounded law of the iterated logarithm (1.4) is invariant if we permute the sequence $\{n_k\}$. Since Aistleitner proved (1.19) by using martingale approximation technique, it is not clear if it is permutation invariant or not. As to this point we can prove the following by applying the above proposition.

Corollary 1.20 ([33]). *If a sequence $\{n_k\}$ of real numbers satisfies (1.17), then for any permutation ϖ of $\mathbf{N}$,*

$$\frac{1}{4\sqrt{2}} \leq \varlimsup_{N\to\infty} \frac{ND_N^*\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} \leq \varlimsup_{N\to\infty} \frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} \leq \Big(\frac{1}{4} + \frac{1}{\sqrt{3}(q-1)}\Big)^{1/2}, \quad a.e.$$

We here emphasize that we succeeded in taking very small upper bound constant. This constant equals to $0.9095\ldots$ when $q = 2$, and is much smaller than Philipp's constant $1050.898\ldots$ and Aistleitner's constant $5.545\ldots$. Moreover we have

$$\Big(\frac{1}{4} + \frac{1}{\sqrt{3}(q-1)}\Big)^{1/2} = \frac{1}{2} + \frac{1}{\sqrt{3}q} + O\Big(\frac{1}{q^2}\Big), \quad (q \to \infty).$$

If we consider a geometric progression $n_k = q^k$ for odd q, Theorem 1.3 implies that

$$\Sigma_q = \frac{1}{2} + \frac{1}{2q} + O\Big(\frac{1}{q^2}\Big), \quad (q \to \infty).$$

Therefore we gave an optimal estimate beside of multiple constant of $1/q$.

By the above extension to allow negative n_k, we can prove a result for geometric progression $\{\theta^k x\}$ whose ratio θ is less than -1.

Theorem 1.21. *For $\theta < -1$, there exist constants Σ_θ and Σ_θ^* such that*

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta^*, \quad \textit{and} \quad \varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta, \quad \textit{a.e.} \tag{1.20}$$

We prove (1.20) by assuming $|\theta| > 1$ in Section 2. The constants Σ_θ and Σ_θ^* are evaluated in the following way except for countable many θ.

Theorem 1.22. *Suppose that $|\theta| > 1$. When θ satisfies (1.6), then*

$$\Sigma_\theta^* = \Sigma_\theta = 1/2.$$

When $\theta < -1$ does not satisfy (1.6), we express θ in the following way:

$$\theta = -\sqrt[r]{p/q} \quad \textit{where} \quad r = \min\{n \in \mathbf{N} \mid \theta^n \in \mathbf{Q}\},\ p, q \in \mathbf{N},\ \textit{and}\ \gcd(p,q) = 1. \tag{1.21}$$

When θ is given by (1.21), then we have

$$\frac{1}{2} < \Sigma_\theta \le \frac{1}{2}\sqrt{\frac{pq+1}{pq-1}}. \tag{1.22}$$

And moreover, if r is even, then

$$\Sigma_\theta^* = \Sigma_\theta = \Sigma_{|\theta|}. \tag{1.23}$$

We have

$$\Sigma_\theta = \Sigma_{|\theta|}. \tag{1.24}$$

in the following cases: (1) r, p, and q are odd; (2) r is odd, $p \ge 4$ is even, and $q = 1$; (3) r is odd, $p = 5$, and $q = 2$.
If r and p are odd, and $q = 1$, then we have

$$\Sigma_\theta^* = \frac{1}{2}\sqrt{\frac{p(p^3+2p^2-p+2)}{(p-1)(p+1)^3}}. \tag{1.25}$$

It is bigger than $1/2$ if $p = 3$, and less than $1/2$ otherwise.

If r is odd and pq is even, then we have

$$\Sigma_\theta^* = \frac{1}{2}. \tag{1.26}$$

If r, p, and $q \ge 3$ are odd, then we have

$$\Sigma_\theta^* < \frac{1}{2}. \tag{1.27}$$

We will prove this theorem in a separate paper. We can summarize as below:

Corollary 1.23. *Suppose that $\theta < -1$. We have $\Sigma_\theta^* \neq \Sigma_\theta$ if and only if θ is given by (1.21) with odd r.*

Recently Aistleitner-Berkes-Tichy [9, 10, 11, 12, 13] investigated in details about asymptotic behavior of permutated sequences. We here state some related results.

By using the argument in [26], we have the following:

Theorem 1.24. *Let $\{n_k\}$ be a sequence of real numbers. If there exists a subsequence $\{n_{k_j}\}$ such that*

$$0 < \overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_{k_j}x\}}{\sqrt{2N\log\log N}}, \quad a.e.,$$

then there exists a permutation ϖ of $\mathbf{N}$ such that

$$\overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_{k_j}x\}}{\sqrt{2N\log\log N}}, \quad a.e., \text{ and}$$
$$\overline{\lim_{N\to\infty}} \frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N\{n_{k_j}x\}}{\sqrt{2N\log\log N}}, \quad a.e.$$

By applying this, we can deduce the results below:

Corollary 1.25 ([30]). *Suppose that $\theta > 1$. For any $\Sigma \in [1/2, \Sigma_\theta]$, there exists a permutation ϖ of $\mathbf{N}$ such that*

$$\overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \Sigma, \quad a.e.$$

Corollary 1.26 ([26]). *If a sequence $\{n_k\}$ of real numbers is not bounded from above, there exists a permutation ϖ of $\mathbf{N}$ such that*

$$\overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N\{n_{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad a.e.$$

For any $a = 2, 3, \ldots$, there exists a permutation ϖ of $\mathbf{N}$ such that

$$\overline{\lim_{N\to\infty}} \frac{ND_N^*\{2^{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N\{2^{\varpi(k)}x\}}{\sqrt{2N\log\log N}} = \frac{1}{2}\sqrt{\frac{(2^a+1)2^a(2^a-2)}{(2^a-1)^3}}, \quad a.e.$$

Corollary 1.27. *For any positive number Σ, there exists a permutation ϖ of $\mathbf{N}$ such that*

$$\overline{\lim_{N\to\infty}} \frac{ND_N^*\{\varpi(k)x\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N\{\varpi(k)x\}}{\sqrt{2N\log\log N}} = \Sigma \quad a.e.$$

The last corollary can be derived from Theorems 1.5 and 1.6.

§ 2. Geometric progressions

We assume $|\theta| > 1$ and prove (1.20). We denote $\varsigma = 1$ if θ is given by (1.7), and $\varsigma = (-1)^r$ if θ is given by (1.21).

For a function f of bounded variation over the unit interval with period 1 satisfying $\int_0^1 f = 0$, we define

$$\sigma^2(f,\theta) = \begin{cases} \int_0^1 f^2(y)\,dy & \text{if } \theta \text{ satisfies (1.6)},\\ \int_0^1 f^2(y)\,dy + 2\sum_{k=1}^\infty \int_0^1 f((\varsigma p)^k y) f(q^k y)\,dy & \text{if } \theta \text{ is given by (1.7) or (1.21)}.\end{cases}$$

We prove the next proposition which is an extension of the result given in [25].

Proposition 2.1. *For a function f of bounded variation with period 1 satisfying $\int_0^1 f = 0$, $\sigma^2(f,\theta)$ is well defined and we have*

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^N f(\theta^k x) \right| = \sigma(f,\theta), \quad a.e.,$$

and

$$\overline{\lim_{N\to\infty}} \frac{N D_N^*\{\theta^k x\}}{\sqrt{2N\log\log N}} = \sup_{0\le a<1} \sigma(\widetilde{\mathbf{1}}_{[0,a)},\theta), \quad a.e.,$$
$$\overline{\lim_{N\to\infty}} \frac{N D_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \sup_{0\le a<b<1} \sigma(\widetilde{\mathbf{1}}_{[a,b)},\theta), \quad a.e.$$

First we prove that $\sigma^2(f,\theta)$ is well defined. When θ satisfies (1.6), it is trivial. We consider the case when θ is given by (1.7) or (1.21).

Because f is of bounded variation, we have a constant C_f such that $|\widehat{f}(n)| \le C_f/|n|$. Note that $\widehat{f}(0) = 0$. Hence we have

$$\left| \int_0^1 f((\varsigma p)^k y) f(q^k y)\,dy \right| \le \sum_{l\ne 0} \left| \widehat{f}(lq^k)\widehat{f}(-l(\varsigma p)^k) \right| \le \sum_{l\ne 0} \frac{C_f^2}{l^2 p^k q^k} \le \frac{C_f^2 \pi^2}{3p^k q^k},$$

which is summable in k. Therefore the series defining $\sigma^2(f,\theta)$ is absolutely convergent and $\sigma^2(f,\theta)$ is well defined.

Denote by f_d the d-th subsum of the Fourier series of f. If we prove both of

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^N f_d(\theta^k x) \right| = \sigma(f_d,\theta), \quad \text{a.e.}, \tag{2.1}$$

$$\lim_{d\to\infty} \sigma(f_d,\theta) = \sigma(f,\theta), \tag{2.2}$$

then by putting $f = \widetilde{\mathbf{1}}_{[a,b)}$ and applying Proposition 1.18, we have the conclusions of Proposition 2.1.

Here we prove (2.2). Since $f_d \to f$ in L^2, we have

$$\int_0^1 f_d((\varsigma p)^k y) f_d(q^k y)\, dy \to \int_0^1 f((\varsigma p)^k y) f(q^k y)\, dy.$$

In the same way as above, we can verify

$$\left| \int_0^1 f_d((\varsigma p)^k y) f_d(q^k y)\, dy \right| \le \frac{C_f^2 \pi^2}{3 p^k q^k}.$$

Since the right hand side is summable in k and independent of d, we have (2.2) by regarding the series appearing in the definition of $\sigma^2(f_d, \theta)$ as an integral and by applying the dominated convergence theorem.

To prove (2.1), we prepare some lemmas.

For a bounded measurable function g, we introduce *a mean value* $\int_{\mathbf{R}} g(x)\, \mu_R(dx)$ by

$$\int_{\mathbf{R}} g(x)\, \mu_R(dx) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(x)\, dx$$

if the right hand side limit exists.

For a trigonometric polynomial g with period 1 satisfying $\int_0^1 g = 0$, we have

$$\int_{\mathbf{R}} g(\Theta x) g(x)\, \mu_R(dx) = 0$$

if $\Theta \notin \mathbf{Q}$, and

$$\int_{\mathbf{R}} g((P/Q)x) g(x)\, \mu_R(dx) = \int_{\mathbf{R}} g(Px) g(Qx)\, \mu_R(dx) = \int_0^1 g(Px) g(Qx)\, dx$$

if P and Q are non-zero integers.

We prove the next key lemma.

Lemma 2.2. *For a trigonometric polynomial with period 1 satisfying $\int_0^1 g = 0$, we have*

$$\int_{\mathbf{R}} \Big(\sum_{k=M+1}^{M+N} g(\theta^k x) \Big)^2 \mu_R(dx) \sim N \sigma^2(g, \theta). \tag{2.3}$$

Proof. By changing variable $\theta^{M+1}x$ by x, we see that the integral equals to $\int_{\mathbf{R}} \big(\sum_{k=0}^{N-1} g(\theta^k x) \big)^2 \mu_R(dx)$.

First assume that θ is given by (1.7) or (1.21). We can easily see that it is enough to show the behavior of $\int_{\mathbf{R}}\bigl(\sum_{k=0}^{Nr-1} g(\theta^k x)\bigr)^2 \mu_R(dx)$. By noting

$$\int_{\mathbf{R}} g(\theta^{ri+j}x)g(\theta^{ri'+j'}x)\,\mu_R(dx) = 0, \qquad (0 \le j < j' \le r-1),$$

we have

$$\begin{aligned}\int_{\mathbf{R}}\Bigl(\sum_{k=0}^{Nr-1} g(\theta^k x)\Bigr)^2 \mu_R(dx) &= \sum_{j=0}^{r-1}\int_{\mathbf{R}}\Bigl(\sum_{i=0}^{N-1} g(\theta^{ri+j}x)\Bigr)^2 \mu_R(dx)\\ &= r\int_{\mathbf{R}}\Bigl(\sum_{i=0}^{N-1} g(\theta^{ri}x)\Bigr)^2 \mu_R(dx),\end{aligned}$$

where the last equality is proved by changing $\theta^j x$ by x. By noting

$$\int_{\mathbf{R}} g(\theta^{ri}x)g(\theta^{ri'}x)\,\mu_R(dx) = \int_0^1 g((\varsigma p)^{i'-i}x)g(q^{i'-i}x)\,dx \quad (i \le i'),$$

we have

$$\begin{aligned}\int_{\mathbf{R}}\Bigl(\sum_{k=0}^{Nr-1} g(\theta^k x)\Bigr)^2 \mu_R(dx) &= Nr\int_0^1 g^2(x)\,dx + 2r\sum_{k=1}^{N-1}(N-k)\int_0^1 g((\varsigma p)^k x)g(q^k x)\,dx\\ &\sim Nr\sigma^2(g,\theta).\end{aligned}$$

On the other hand, when θ satisfies (1.6), we have $\int_{\mathbf{R}} g(\theta^k x)g(\theta^{k'}x)\,\mu_R(dx) = 0$ if $k \ne k'$. Hence we have $\int_{\mathbf{R}} g^2(\theta^k x)\,\mu_R(dx) = \int_0^1 g^2(x)\,dx$ and thereby

$$\int_{\mathbf{R}}\Bigl(\sum_{k=M+1}^{M+N} g(\theta^k x)\Bigr)^2 \mu_R(dx) = N\int_0^1 g^2(x)\,dx.$$

$\square$

The next two lemmas are used in martingale approximation procedure, which is one of the main part of the proof.

Lemma 2.3. *Let g be a bounded measurable function with period 1 satisfying $\int_0^1 g = 0$. For all $a < b$ and $\lambda > 0$, we have*

$$\left|\int_a^b g(\lambda x)\,dx\right| \le \frac{\|g\|_\infty}{\lambda}.$$

Proof. By changing variables $y = \lambda x$, the integral is written by

$$\frac{1}{\lambda}\int_{\lambda a}^{\lambda b} g(y)\,dy = \frac{1}{\lambda}\sum_{k=0}^{[\lambda(b-a)]-1}\int_{\lambda a+k}^{\lambda a+k+1} g(y)\,dy + \frac{1}{\lambda}\int_{\lambda a+[\lambda(b-a)]}^{\lambda b} g(y)\,dy.$$

Since we have $|\int_{\lambda a+[\lambda(b-a)]}^{\lambda b} g(y)\,dy| \le (\lambda(b-a)-[\lambda(b-a)])\|g\|_\infty \le \|g\|_\infty$, we can complete the proof. $\square$

Lemma 2.4. *Suppose that a trigonometric polynomial g with degree d and period 1 satisfies $\int_0^1 g = 0$. There exists a constant C_q depending only on q such that*

$$\int_A^{A+1}\Big(\sum_{k=M+1}^{M+N} g(\lambda_k x)\Big)^4 dx \le C_q\Big(\sum_{|\nu|\le d}|\widehat{g}(\nu)|\Big)^4 N^2$$

holds for any integer A and for every sequence $\{\lambda_k\}$ satisfying $|\lambda_1|\ge 1$ and the generalized Hadamard's gap condition $|\lambda_{k+1}/\lambda_k|\ge q>1$.

Proof. It is enough to prove for $A=0$. The general case can be proved by trivial modification. By the triangle inequality of L^4-norm, the left hand side is less than

$$\Big(\sum_{|\nu|\le d}|\widehat{g}(\nu)|\Big)^4\int_0^1\Big(\sum_{k=M+1}^{M+N}\exp(2\pi\sqrt{-1}\lambda_k\nu x)\Big)^4 dx.$$

When $\lambda_k\ge 0$, we already proved (Lemma 1 (1) of [24]) that

$$\int_0^1\Big(\sum_{j=1}^{\infty}(c_j\cos 2\pi\lambda_j x + d_j\sin 2\pi\lambda_j x)\Big)^4 dx \le C_q\Big(\sum_{j=1}^{\infty}(c_j^2+d_j^2)\Big)^2. \tag{2.4}$$

holds for a constant C_q depending only on q. When $\lambda_k\ge 0$ is not assumed, by noting $c_j\cos 2\pi\lambda_j x + d_j\sin 2\pi\lambda_j x = c_j\cos 2\pi|\lambda_j|x + (\pm d_j)\sin 2\pi|\lambda_j|x$, we see that (2.4) is valid under the generalized Hadamard's gap condition above.

By combining these, we have the conclusion. $\square$

We take an arbitrary integer A and prove (1.20) on $[A, A+1)$. Here we adopt the method of martingale approximation, which is a simplification of the proof given by Aistleitner [2] and originated with Berkes [14] and Philipp [42].

We simply denote f_d by g. We put $n_k=\theta^k$ and note $|n_{k+1}/n_k|\ge|\theta|$. We divide $\mathbf{N}$ into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \dots$ satisfying $^\#\Delta_i' = [1+9\log_{|\theta|} i]$ and $^\#\Delta_i = i$. By putting $i^- = \min\Delta_i$ and $i^+ = \max\Delta_i$, we have

$$|n_{i^-}/n_{(i-1)^+}| \ge |\theta|^{9\log_{|\theta|} i} = i^9.$$

We denote $\mu(i) = [\log_2 i^4|n_{i^+}|]+1$ and introduce a σ-field $\mathcal{F}_i$ on $[A, A+1)$ defined by

$$\mathcal{F}_i = \sigma\{[A+j2^{-\mu(i)}, A+(j+1)2^{-\mu(i)}) \mid j=0,\dots,2^{\mu(i)}-1\}.$$

We here note $i^4|n_{i^+}| \le 2^{\mu(i)} \le 2i^4|n_{i^+}|$. Set

$$T_i(x) = \sum_{k\in\Delta_i} g(n_k x),\quad T_i'(x) = \sum_{k\in\Delta_i'} g(n_k x),\quad Y_i = E(T_i\mid\mathcal{F}_i) - E(T_i\mid\mathcal{F}_{i-1}).$$

We see that $\{Y_i, \mathcal{F}_i\}$ forms a martingale difference sequence. We prove

$$\|Y_i - T_i\|_\infty \le (\|g'\|_\infty + 2\|g\|_\infty)/i^3, \tag{2.5}$$

$$\|Y_i^2 - T_i^2\|_\infty \le 3\|g\|_\infty(\|g'\|_\infty + 2\|g\|_\infty)/i^2, \tag{2.6}$$

$$\|Y_i^4 - T_i^4\|_\infty \le 15\|g\|_\infty^3(\|g'\|_\infty + 2\|g\|_\infty). \tag{2.7}$$

For $k \in \Delta_i$ and $x \in I = [\,A + j2^{-\mu(i)}, A + (j+1)2^{-\mu(i)}) \in \mathcal{F}_i$, we have

$$\begin{aligned}|g(n_k x) - E(g(n_k\,\cdot\,) \mid \mathcal{F}_i)| &= \left| |I|^{-1} \int_I (g(n_k x) - g(n_k y))\, dy \right| \le \max_{y \in I} |g(n_k x) - g(n_k y))| \\ &\le \|g'\|_\infty |n_k| 2^{-\mu(i)} \le \|g'\|_\infty |n_k| / |n_{i^+}| i^4 \le \|g'\|_\infty / i^4,\end{aligned}$$

and hence we have $|T_i - E(T_i \mid \mathcal{F}_i)| \le \|g'\|_\infty {}^\#\Delta_i / i^4 = \|g'\|_\infty / i^3$.

On $J = [\,A + j2^{-\mu(i-1)}, A + (j+1)2^{-\mu(i-1)}) \in \mathcal{F}_{i-1}$, by Lemma 2.3 we have

$$\begin{aligned}|E(g(n_k\,\cdot\,) \mid \mathcal{F}_{i-1})| &= \left| |J|^{-1} \int_J g(n_k y)\, dy \right| \le \|g\|_\infty 2^{\mu(i-1)} / |n_k| \\ &\le \|g\|_\infty 2(i-1)^4 |n_{(i-1)^+} / n_{i^-}| \le 2\|g\|_\infty / i^5,\end{aligned}$$

and hence we have $|E(T_i \mid \mathcal{F}_{i-1})| \le 2\|g\|_\infty {}^\#\Delta_i / i^5 = 2\|g\|_\infty / i^4$, which shows (2.5).

By $\|T_i\|_\infty \le i\|g\|_\infty$, we have $\|E(T_i \mid \mathcal{F}_i)\|_\infty$, $\|E(T_i \mid \mathcal{F}_{i-1})\|_\infty \le i\|g\|_\infty$, which imply $\|Y_i\|_\infty \le 2i\|g\|_\infty$ and $\|Y_i + T_i\|_\infty \le 3i\|g\|_\infty$. Similarly we have $\|Y_i^2 + T_i^2\|_\infty \le 5i^2\|g\|_\infty^2$. By applying these to $\|Y_i^2 - T_i^2\|_\infty \le \|Y_i - T_i\|_\infty \|Y_i + T_i\|_\infty$ and $\|Y_i^4 - T_i^4\|_\infty \le \|Y_i^2 - T_i^2\|_\infty \|Y_i^2 + T_i^2\|_\infty$, we have (2.6) and (2.7).

Put $C = \min\{\langle \log_{|\theta|} \nu - \log_{|\theta|} \nu' \rangle^* \mid \nu, \nu' = 1, \dots, d, \log_{|\theta|} \nu - \log_{|\theta|} \nu' \notin \mathbf{Z}\} \in (0,1)$, where $\langle x \rangle^* = \min_{n \in \mathbf{Z}} |x - n|$. By denoting $D = |\theta|^C - 1 > 0$, we prove

$$|\theta^k \nu + \theta^l \nu'| \ge D|\theta|^L \quad \text{if} \quad k, l \ge L,\ |\nu|, |\nu'| \le d,\ \theta^k \nu + \theta^l \nu' \ne 0. \tag{2.8}$$

Since the assertion is true if $\theta^k \nu$ and $\theta^l \nu'$ are both positive or negative, we assume that one is positive and the other is negative. By $\theta^k \nu + \theta^l \nu' \ne 0$, we have $|\theta^k \nu| \ne |\theta^l \nu'|$. In case when $\log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'| \notin \mathbf{Z}$, we have

$$\begin{aligned}\left|\log_{|\theta|} |\theta^k \nu| - \log_{|\theta|} |\theta^l \nu'|\right| &= \left|(k - l) + (\log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'|)\right| \\ &\ge \langle \log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'| \rangle^* \ge C.\end{aligned}$$

If $\log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'| \in \mathbf{Z}$, by $|\theta^k \nu| \ne |\theta^l \nu'|$ we see that $\log_{|\theta|} |\theta^k \nu| - \log_{|\theta|} |\theta^l \nu'|$ is a non-zero integer and $\left|\log_{|\theta|} |\theta^k \nu| - \log_{|\theta|} |\theta^l \nu'|\right| \ge 1 \ge C$. Therefore we have $|\theta^k \nu / \theta^l \nu'| \ge |\theta|^C$ when $|\theta^k \nu| > |\theta^l \nu'|$, and $|\theta^l \nu' / \theta^k \nu| \ge |\theta|^C$ when $|\theta^k \nu| \le |\theta^l \nu'|$. From $|\theta^k \nu / \theta^l \nu'| \ge |\theta|^C$ we can derive $|\theta^k \nu + \theta^l \nu'| = |\theta^k \nu| - |\theta^l \nu'| \ge (|\theta|^C - 1)|\theta^l \nu'| \ge D\theta^L$, and from $|\theta^l \nu' / \theta^k \nu| \ge |\theta|^C$ we can derive $|\theta^k \nu + \theta^l \nu'| = |\theta^l \nu'| - |\theta^k \nu| \ge (|\theta|^C - 1)|\theta^k \nu| \ge D|\theta|^L$.

If we expand T_i^2 into trigonometric polynomial, the constant term equals to $v_i = \int_{\mathbf{R}} T_i^2(x)\mu_R(dx)$. The trigonometric polynomial expansion of $T_i^2 - v_i$ has at most $8(d+1)^2 i^2$ terms, and the absolute value of frequency of each term is greater than $D|n_{i-}|$. Hence by Lemma 2.3, we have $|E(T_i^2 - v_i \mid \mathcal{F}_{i-1})| \le 8(d+1)^2 i^2 (1/D|n_{i-}|) 2^{\mu(i-1)} = O(1/i^3)$. By putting $\beta_M = \sum_{i=1}^M v_i$, we have

$$\Big\| \sum_{i=1}^M E(T_i^2 \mid \mathcal{F}_{i-1}) - \beta_M \Big\|_\infty = O(1). \tag{2.9}$$

Denote $V_M = \sum_{i=1}^M E(Y_i^2 \mid \mathcal{F}_{i-1})$. By (2.6), we see

$$\Big\| \sum_{i=1}^M \big(E(Y_i^2 \mid \mathcal{F}_{i-1}) - E(T_i^2 \mid \mathcal{F}_{i-1}) \big) \Big\|_\infty = O(1)$$

and

$$\big\| V_M - \beta_M \big\|_\infty = O(1). \tag{2.10}$$

Denote $l_M = M(M+1)/2$. By (2.3) we have $v_i \sim i\sigma^2(g,\theta)$ and

$$\beta_M \sim l_M \sigma^2(g,\theta). \tag{2.11}$$

Here we use the following theorem by Monrad-Philipp [40], which is a variation of Strassen's theorem [46].

Theorem 2.5. *Suppose that a square integrable martingale difference sequence* $\{\widehat{Y}_i, \widehat{\mathcal{F}}_i\}$ *satisfies*

$$\widehat{V}_M = \sum_{i=1}^M E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) \to \infty \quad a.s. \quad and \quad \sum_{i=1}^\infty E\big(\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2 \ge \psi(\widehat{V}_i)\}} / \psi(\widehat{V}_i)\big) < \infty$$

for some non-decreasing function ψ *with* $\psi(x) \to \infty$ $(x \to \infty)$ *such that* $\psi(x)(\log x)^\alpha / x$ *is non-increasing for some* $\alpha > 50$. *If there exists a uniformly distributed random variable* U *which is independent of* $\{\widehat{Y}_n\}$, *there exists a standard normal i.i.d.* $\{Z_i\}$ *such that*

$$\sum_{i \ge 1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i \le t\}} = \sum_{i \le t} Z_i + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big), \quad (t \to \infty) \quad a.s.$$

We prepare another probability space on which a uniform distributed random variable U and an i.i.d $\{\xi_k\}$ with $P(\xi_k = 1) = P(\xi_k = -1) = 1/2$ which is independent of

U. Let $\mathcal{G}_i$ be a σ-field over this probability space which is generated by $\{\xi_k\}_{k\le i^+}$. Put $\Xi_i = \sum_{k\in\Delta_i}\xi_k$.

We make a product of $[A, A+1)$ on which $\{Y_k\}$ is defined and this new probability space, and regard Y_k, U, and Ξ_k as random variables on this product probability space. Take $\varepsilon > 0$ arbitrarily and put

$$\widehat{Y}_i = Y_i + \varepsilon\Xi_i, \quad \widehat{\mathcal{F}}_i = \mathcal{F}_i \otimes \mathcal{G}_i, \quad \widehat{\beta}_M = \beta_M + \varepsilon^2 l_M.$$

Clearly $\{\widehat{Y}_i, \widehat{\mathcal{F}}_i\}$ is a martingale difference sequence.

By Lemma 2.4 and (2.7), we have $\|\widehat{Y}_i\|_4 \le \|Y_i\|_4 + \|\Xi_i\|_4 = \|T_i\|_4 + \|\Xi_i\|_4 + O(1) = O(i^{1/2})$ or $E\widehat{Y}_i^4 = O(i^2)$. We have $E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) = E(Y_i^2 \mid \mathcal{F}_{i-1}) + \varepsilon^2 i$ and hence $\widehat{V}_M = V_M + \varepsilon^2 l_M \ge \varepsilon^2 l_M$. We owe Aistleitner [5] this idea to prepare an independent rademacher i.i.d. to assure the growth of $\widehat{V}_M$. By (2.10), we have

$$\big\|\widehat{V}_M - \widehat{\beta}_M\big\|_\infty = O(1). \tag{2.12}$$

Hence by putting $\psi(x) = x/(\log x)^{51}$, we have

$$\sum_i E\big(\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2\ge\psi(\widehat{V}_i)\}}/\psi(\widehat{V}_i)\big) \le \sum_i \frac{E\widehat{Y}_i^4}{\psi^2(\varepsilon^2 l_i)} = O\Big(\sum_i \frac{i^2(\log l_i)^{102}}{l_i^2}\Big) = O(1).$$

By (2.12) and $\widehat{V}_M - \widehat{V}_{M-1} \ge \varepsilon^2 M \to \infty$, we have $\widehat{V}_{M-1} < \widehat{\beta}_M < \widehat{V}_{M+1}$ for large M. Hence $\widehat{V}_i \le \widehat{\beta}_M$ is equivalent to $i \le M-1$ or $i \le M$. By $\|\widehat{Y}_i\|_\infty = O(i)$ we have

$$\sum_{i\ge1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i\le\widehat{\beta}_M\}} = \sum_{k=1}^M \widehat{Y}_k + O(M) = \sum_{k=1}^M \widehat{Y}_k + o(\phi(l_M)),$$

where $\phi(x) = \sqrt{2x\log\log x}$. By (2.11) we have $\widehat{\beta}_M = O(l_M)$. By applying Theorem 2.5 and putting $t = \widehat{\beta}_M$, we have

$$\sum_k^M \widehat{Y}_k = \sum_{i\ge1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i\le\widehat{\beta}_M\}} + o(\phi(l_M)) = \sum_{i\le\widehat{\beta}_M} Z_i + o(\phi(l_M)), \quad \text{a.s.}$$

By noting $\widehat{\beta}_M \sim (\sigma^2(g,\theta) + \varepsilon^2) l_M$, we have

$$\varlimsup_{M\to\infty} \phi^{-1}(l_M)\Big|\sum_{k=1}^M \widehat{Y}_k\Big| = \varlimsup_{M\to\infty} \phi^{-1}(l_M)\Big|\sum_{i\le\widehat{\beta}_M} Z_i\Big| = (\sigma^2(g,\theta) + \varepsilon^2)^{1/2}, \quad \text{a.s.}$$

By noting

$$\varlimsup_{M\to\infty} \phi^{-1}(l_M)\Big|\sum_{k=1}^M \varepsilon\Xi_k\Big| = \varepsilon, \quad \text{a.s.},$$

dividing $\left|\sum_{k=1}^M \widehat{Y}_k\right| - \left|\sum_{k=1}^M \varepsilon\Xi_k\right| \le \left|\sum_{k=1}^M Y_k\right| \le \left|\sum_{k=1}^M \widehat{Y}_k\right| + \left|\sum_{k=1}^M \varepsilon\Xi_k\right|$ by $\phi(l_M)$ and taking limsup, we have

$$(\sigma^2(g,\theta)+\varepsilon^2)^{1/2} - \varepsilon \le \varlimsup_{M\to\infty} \phi^{-1}(l_M)\left|\sum_{k=1}^M Y_k\right| \le (\sigma^2(g,\theta)+\varepsilon^2)^{1/2} + \varepsilon, \quad \text{a.s.}$$

By letting $\varepsilon \to 0$ and by noting noting (2.5), we have

$$\varlimsup_{M\to\infty} \phi^{-1}(l_M)\left|\sum_{k=1}^M T_k\right| = \varlimsup_{M\to\infty} \phi^{-1}(l_M)\left|\sum_{k=1}^M Y_k\right| = \sigma(g,\theta), \quad \text{a.s.} \tag{2.13}$$

Since this conclusion is valid over the product probability space with probability 1, by applying Fubini's theorem we see that it is valid over the original space a.e.

By applying (1.4) and Koksma's inequality and by noting $\sum_{i=1}^M [1 + 9\log_{|\theta|} i] = O(M\log M)$ we have $\left|\sum_{k=1}^M T'_k\right| = O\big(\sqrt{M\log M \log\log(M\log M)}\big) = o(\sqrt{l_M})$. Therefore, by $\varlimsup_{M\to\infty} \phi^{-1}(l_M)\left|\sum_{k=1}^M T'_k\right| = 0$, $M^+ = l_M + \sum_{i=1}^M [1 + 9\log_{|\theta|} i] \sim l_M$, and (2.13), we have $\varlimsup_{M\to\infty} \phi^{-1}(M^+)\left|\sum_{i=1}^M \sum_{k\in\Delta'_i\cup\Delta_i} g(\theta^k x)\right| = \sigma(g,\theta)$, a.e. Moreover we have $\sum_{k\in\Delta'_M\cup\Delta_M} \|g(\theta^k\,\cdot\,)\|_\infty = o(\phi(M^+))$, and hence we have (2.1).

Acknowledgement The author thanks the referee for having corrected errors in the original manuscript. He is also grateful to Professor Y. Yamada for his encouragement.

References

[1] Aistleitner, C., On the law of the iterated logarithm for the discrepancy of $\langle n_k x\rangle$ with multidimensional indices, *Unif. Distrib. Theory*, **2** (2007), 89–104.

[2] Aistleitner, C., On the law of the iterated logarithm for the discrepancy of lacunary sequences, *Trans. Amer. Math. Soc.*, **362** (2010), 5967–5982.

[3] Aistleitner, C., Irregular discrepancy behavior of lacunary series, *Monatsh. Math.*, **160** (2010), 1–29.

[4] Aistleitner, C., The law of the iterated logarithm for $\sum c_k f(n_k x)$, *Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp*, Eds. Berkes, I., Bradley, R., Dehling, H., Peligrad, M., and Tichy, R., Kendrick press, 2010, pp. 21–34.

[5] Aistleitner, C., Private communication, (2010).

[6] Aistleitner, C., Diophantine equations and the LIL for the discrepancy of sublacunary sequences, *Illinois J. Math.*, **53** (2009), 785–815.

[7] Aistleitner, C., Irregular discrepancy behavior of lacunary series II, *Monatsh. Math.*, **161** (2010) 255–270.

[8] Aistleitner, C. and Berkes, I., On the law of the iterated logarithm for the discrepancy of $\langle n_k x\rangle$, *Monatsh. Math.*, **156** (2009), 103–121.

[9] Aistleitner, C., Berkes, I. and Tichy, R., Lacunary sequences and permutations, The law of the iterated logarithm for $\sum c_k f(n_k x)$, *Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp*, Eds. Berkes, I., Bradley, R., Dehling, H., Peligrad, M., and Tichy, R., Kendrick press, 2010, pp. 35–50.

[10] Aistleitner, C., Berkes, I. and Tichy, R., On the asymptotic behaviour of weakly lacunary series, *Proc. Amer. Math. Soc.*, **139** (2011) 2505–2517.

[11] Aistleitner, C., Berkes, I. and Tichy, R., On permutations of Hardy-Littlewood-Pólya sequences, *Trans. Amer. Math. Soc.*, **363** (2011) 6219–6244.

[12] Aistleitner, C., Berkes, I. and Tichy, R., *On the law of the iterated logarithm for permuted lacunary sequences*, Proc. Steklov Inst. Math., (to appear)

[13] Aistleitner, C., Berkes, I. and Tichy, R., *Permutations and the subsequence principle*, (preprint).

[14] Berkes, I., On the asymptotic behaviour of $\sum f(n_k x)$. I. Main Theorems, II. Applications, *Z. Wahrsch. verw. Gebiete*, **34** (1976), 319–345, 347–365.

[15] Berkes, I., A central limit theorem for trigonometric series with small gaps, *Z. Wahr. verw. Geb.*, **47** (1979), 157–161.

[16] Berkes, I. and Philipp, W., The size of trigonometric and Walsh series and uniform distribution mod 1, *J. London Math. Soc.* (2), **50** (1994), 454–464.

[17] Berkes, I., Philipp, W. and Tichy, R., Empirical processes in probabilistic number theory: The LIL for the discrepancy of $(n_k\omega)$ mod 1. *Illinois J. Math.*, **50** (2006), 107–145.

[18] Berkes, I., Philipp, W. and Tichy, R., Metric discrepancy results for sequence $\{n_k x\}$ and diophantine equations, *Diophantine Approximations*, Eds. Tichy, R,, Schlickewei, H., Schmidt, K., *Festschrift for Wolfgang Schmidt, Development in Mathematics*, **16**, Springer, 2008, pp. 95–105.

[19] Berkes, I. and Dehling, H., Dependence in probability, analysis and number theory: the mathematical work of Walter Philipp, *Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp*, Eds. Berkes, I., Bradley, R., Dehling, H., Peligrad, M., and Tichy, R., Kendrick Press, 2010, pp. 1–19.

[20] Bohl, P., Über ein in der Theorie der säkulären Störungen vorkommendes Problem, *Jour. Reine Angew. Math.*, **135** (1909), 189–283.

[21] Chung, K., An estimate concerning the Kolmogorov limit distibution, *Trans. Amer. Math. Soc.*, **67** (1949), 36–50.

[22] Dhompongsa, S., Almost sure invariance principles for the empirical process of lacunary sequences, *Acta Math. Hungar.*, **49** (1987), 83–102.

[23] Erdős, P. and Gál, I., On the law of the iterated logarithm I, II, *Nederl. Akad. Wetensch. Proc. Ser. A* **58** (*Indag. Math.***17**) (1955), 65–76, 77–84.

[24] Fukuyama, K., The central limit theorem for Riesz-Raikov sums, *Prob. Theory Related Fields*, **100** (1994), 57–75.

[25] Fukuyama, K., The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$, *Acta Math. Hungar.*, **118** (2008), 155–170.

[26] Fukuyama, K., The law of the iterated logarithm for the discrepancies of a permutation of $\{n_k x\}$, *Acta Math. Hungar.*, **123** (2009), 121–125.

[27] Fukuyama, K., A central limit theorem and a metric discrepancy result for sequence with bounded gaps, *Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp*, Eds. Berkes, I., Bradley, R., Dehling, H., Peligrad, M., and Tichy, R., Kendrick press, 2010, pp. 233–246

[28] Fukuyama, K., A law of the iterated logarithm for discrepancies: non-constant limsup. *Monatsh. Math.*, **160** (2010), 143–149.

[29] Fukuyama, K., A metric discrepancy result for a lacunary sequence with small gaps, *Monatsh. Math.*, **162** (2011) 277–288

[30] Fukuyama, K. and Hiroshima, N., Metric discrepancy results for subsequences of $\{\theta^k x\}$,

Monatsh. Math., (to appear).

[31] Fukuyama, K. and Miyamoto, S., Metric discrepancy results for Erdős-Fortet sequence, *Studia Sci. Math. Hungar.*, (to appear).

[32] Fukuyama, K. and Nakata, K., A metric discrepancy result for the Hardy-Littlewood-Pólya sequences. *Monatsh. Math.*, **160** (2010), 41–49.

[33] Fukuyama, K. and Mitsuhata, Y., Bounded law of the iterated logarithm for discrepancies of permutations of lacunary sequences, RIMS Kôkyûroku Bessatsu, (submitted).

[34] Fukuyama, K., Murakami K, Ohno, R. and Ushijima, S., The law of the iterated logarithm for discrepancies of three variations of geometric progressions, RIMS Kôkyûroku Bessatsu, (submitted).

[35] Fukuyama, K. and Watada, T., A metric discrepancy result for lacunary sequences, *Proc. Amer. Math. Soc.*, **140** (2012) 749–754.

[36] Gál, L. and Gál, S., The discrepancy of the sequence $\{2^n x\}$, *Indag. Math.*, **26** (1964), 129–143.

[37] Kesten, H., The discrepancy of random sequences $\{kx\}$, *Acta Arith.*, **10** (1964/1965), 183–213.

[38] Khintchine, A., Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.*, **92** (1924), 115–125.

[39] Kronecker, L., Näherungweise ganzzahlige Auflösung linearer Gleichungen, *Monatsberichte Königl. Preuß. Akad. Wiss. Berlin*, (1884), 1179-1193 and 1271–1299

[40] Monrad, D. and Philipp, W., Nearby variables with nearby conditional laws and a strong approximation theorem for Hilbert space valued martingales, *Probab. Theory rel. Fields*, **88** (1991), 381–404.

[41] Philipp, W., Mixing sequences of random variables and probabilistic number theory, *Memoirs Amer. Math. Soc.*, **114** (1971).

[42] Philipp, W. and Stout, W., Almost sure invariance principles for partial sums of weakly dependent random variables, *Memoirs Amer. Math. Soc.*, **161** (1975).

[43] Philipp, W., Limit theorems for lacunary series and uniform distribution mod 1, *Acta Arith.* **26** (1975), 241–251

[44] Philipp, W., A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables. *Ann. Probab.*, **5** (1977), 319–350

[45] Philipp, W., Empirical distribution functions and strong approximation theorems for dependent random variables. A problem of Baker in probabilistic number theory. *Trans. Amer. Math. Soc.*, **345** (1994), 705–727.

[46] Strassen, V., Almost sure behavior of sums of independent random variables and martingales, *Fifth Berkeley Symp. Math. Stat. Prob.* Vol II, Part I, (1967), 315–343.

[47] Takahashi, S., An asymptotic property of a gap sequence, *Proc. Japan Acad.*, **38** (1962), 101–104.

[48] Sierpiński, W., Sur la valuer asymptotique d'une certaine somme, *Bull. Int. Acad. Polon. Sci. (Cracovie) A*, **1910** (1910), 9–11.

[49] Smirnov, N., Approximate variables from empirical data (Russian), *Uspehi. Mat. Nauk.*, **10** (1944), 36–50.

[50] Weber, M., Discrepancy of randomly sampled sequences of reals, *Math. Nachr.*, **271** (2004), 105–110.

[51] Weyl, H., Über die Gleichverteilung von Zahlen mod Eins, *Math. Ann.*, **77** (1916), 313–352.