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Graph-based topological approximation of saddle-node bifurcation in maps

Dedicated to Professor Yasumasa Nishiura for his sixtieth birthday.

By

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§ 1. Introduction

An efficient computational framework for obtaining a rigorous combinatorial description of the global dynamics over a large range of parameter values of multiparameter nonlinear systems was presented in [1]. Because the resulting information is easily queryable and provides important information about the qualitative dynamics, we refer to the output as a database for the dynamics. Such a procedure can only involve a finite number of computations and thus the dynamics can only be represented down to a fixed scale. However, the theory of dynamical systems indicates that nonlinear systems can exhibit different structures at all scales in phase space and that bifurcations of the

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structures can occur on all scales in parameter space. A consequence of this is that the descriptions presented in [1] are only indirectly related to fundamental notions in dynamical systems such as structural stability. The goal of this paper is to address this relationship in the simplest possible context, that of a saddle-node bifurcation.

To provide an overview of the results presented here consider the archetypical example of a map which undergoes a saddle-node bifurcation $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by

$$f(x,\lambda) = x + x^2 - \lambda. \tag{1.1}$$

The fixed points of this map are solutions to

$$\varphi(x,\lambda) = f(x,\lambda) - x = 0$$

and are given by $x^{\pm}(\lambda) = \pm\sqrt{\lambda}$ for $\lambda \geq 0$. We are only interested in the dynamics associated with the saddle-node bifurcation and hence we restrict our attention to $X \times \Lambda := [-1,1]\times[\Lambda_-,\Lambda_+]$ where $-1 < \Lambda_- < \Lambda_+ < 1$.

Conley's topological approach to dynamics [2, 6] provides the theoretical basis for [1]. As such the global dynamics is described in terms of Conley-Morse graphs which codify the information associated with Morse decompositions of the maximal invariant set $S_\lambda := \operatorname{Inv}(X, f_\lambda)$ in $X = [-1,1]$ as a function of $\lambda \in \Lambda$. For this system we make use of the following Conley-Morse graphs (we follow the notation presented in [1]):

$$\mathrm{CMG}(A) = \emptyset \tag{1.2}$$

$$\mathrm{CMG}(B) = \boxed{p_0 : 0} \tag{1.3}$$

$$\mathrm{CMG}(C) = \boxed{p_1 : 1 \to \{1\}} \longrightarrow \boxed{p_0 : 0 \to \{1\}} \tag{1.4}$$

To understand the information provided by these Conley-Morse graphs we recall that for a dynamical system generated by a map the Conley index of an isolated invariant set K is given by the shift equivalence class of an induced map on homology (see [3] for details). The full theory is not necessary for the purposes of this introduction. It is sufficient to recall that if $K = \emptyset$, then the index map is nilpotent, in which case we say that the index is trivial. The converse is not true: a trivial index does *not* imply that the associated invariant set is empty. In fact, the failure of the converse is the driving force behind this paper. Furthermore, if K is a hyperbolic fixed point with real eigenvalues, n of which are greater than 1 and all the rest has modulus less than 1, then the induced map on homology is nilpotent on dimensions different from n and is shift equivalent to the identity map on $\mathbb{Z}$ in dimension n. This is indicated by $n \to \{1\}$ where n is the dimension on which the nontrivial homology map acts and $\{1\}$ are the non-zero eigenvalues of the homology map.

Returning to the above mentioned Conley-Morse graphs, $\mathrm{CMG}(A)$ is a valid Conley-Morse graph at parameter values λ for which $S_\lambda = \emptyset$ and hence there is no non-empty Morse set. If $\mathrm{CMG}(B)$ is a valid Conley-Morse graph, then there exists a Morse decomposition of S_λ which consists of a single Morse set. This Morse set is indexed by p_0 and the induced index map is nilpotent, thus it has no nonzero eigenvalues. If $\mathrm{CMG}(C)$ is a valid Conley-Morse graph, then there exists a Morse decomposition $\mathcal{M} = \{M(p_0), M(p_1) \mid p_1 > p_0\}$ of S_λ. Furthermore, the index of the Morse set $M(p_1)$ is that of a hyperbolic fixed point with a one-dimensional unstable manifold, while the index of $M(p_0)$ is that of an attracting hyperbolic fixed point.

Since the dynamics of the saddle-node bifurcation is completely understood analytically, we can assign the Conley Morse graphs to parameter values as follows: $\mathrm{CMG}(A)$ is valid for $\lambda \in [\Lambda_-, 0)$, $\mathrm{CMG}(B)$ is valid for $\lambda = 0$, and $\mathrm{CMG}(C)$ is valid for $\lambda \in (0, \Lambda_+]$.

The construction of the database can involve at most a finite number of computations. This is done by discretizing both the phase space and the parameter space into compact intervals. For the phase space X, this is denoted by

$$\mathcal{X} := \{G_i = [x_j, x_{j+1}] \mid j = 0, \dots, J\} \tag{1.5}$$

where $x_{j+1} - x_j = \delta$. The discretization of the parameter space Λ is given by

$$\mathcal{Q} := \{Q_i = [\lambda_i, \lambda_{i+1}] \mid i = 0, \dots, I\} \tag{1.6}$$

where $\lambda_{i+1} - \lambda_i = \nu$. The approximation of the nonlinear dynamical system is geometric in nature. In particular, for each $Q \in \mathcal{Q}$ a multivalued map $\mathcal{F}_Q \colon \mathcal{X} \rightrightarrows \mathcal{X}$ is defined by

$$\mathcal{F}_Q(G) := \{G_j \in \mathcal{X} \mid f(G, Q) \cap G_j \neq \emptyset\} \tag{1.7}$$

and all the information expressed in the database is obtained via these maps. The grids $\mathcal{X}$ and $\mathcal{Q}$ define the level of resolution on which the computations are being performed. A simple consequence of this is that the Morse graphs used to describe the global dynamics must be valid over the intervals of parameter space Q_i.

Recall that the database is presented in the form of a continuation graph. This is a graph with the following properties. To each vertex V in the graph there is associated a Conley-Morse graph $\mathrm{CMG}(V)$ and a connected region in parameter space $\mathcal{Q}(V) \subset \mathcal{Q}$ such that for each $Q \in \mathcal{Q}(V)$, $\mathrm{CMG}(V)$ is a valid Conley-Morse graph for f_λ for all $\lambda \in Q$. There is an edge between two vertices V and V' if there exist $Q \in \mathcal{Q}(V)$ and $Q' \in \mathcal{Q}(V')$ such that $Q \cap Q' \neq \emptyset$. In the context of our idealized example where we have an analytic understanding of the dynamics the continuation graph can take the form

$$A \text{ ------ } B \text{ ------ } C \tag{1.8}$$

The associated Conley-Morse graphs are given by (1.2), (1.3), and (1.4). If we assume that $\lambda_{i_0} < 0 < \lambda_{i_0+1}$, then the associated parameter regions are $\mathcal{Q}(A) := \{Q_i \mid i < i_0\}$, $\mathcal{Q}(B) := \{Q_{i_0}\}$, and $\mathcal{Q}(C) := \{Q_i \mid i > i_0\}$.

Of course, in applications our knowledge of the dynamics is derived from the multivalued maps (1.7) which are defined in terms of the elements of $\mathcal{X}$ and $\mathcal{Q}$. To be more precise, recall [1, Section 3.1] that for each $Q \in \mathcal{Q}$ the multivalued map $\mathcal{F}_Q$ can be viewed as a directed graph, the vertices are the elements of the grid $\mathcal{X}$ and the edges are determined by the images of $\mathcal{F}_Q$. The strongly connected path components of this graph define the nodes of the Morse graph associated with Q. The ordering on the Morse graph is determined by the acyclic quotient graph defined by collapsing each connected path component to a node. Finally, the set of grid elements in each strongly connected path component defines an isolating neighborhood for the respective Morse set. The action of $\mathcal{F}_Q$ on this isolating neighborhood is used to compute the Conley index of the Morse set [1, Section 4.3].

Observe that for fixed f the Conley Morse graphs are completely determined by the parameters δ and ν used to define $\mathcal{X}$ and $\mathcal{Q}$. Numerical artifacts of this procedure occur at nodes for which the associated Morse set under f_λ, $\lambda \in Q$, is the empty set. As is indicated above, the Conley index of an empty Morse set is trivial. However, our computations provide us with a Morse set with trivial Conley index, which is not sufficient to conclude that the Morse set is empty. One possibility of handling a Morse set with trivial index correctly is the Conley-Morse graph reduction test, which is explained in [1, Section 4.5].

As is indicated above, for each $Q \in \mathcal{Q}$ the strongly connected path components of $\mathcal{F}_Q$ define the nodes of the Conley-Morse graph. In the context of a saddle-node bifurcation the number of nodes in this graph can vary dramatically as a function of $\mathcal{Q}$, δ, ν and the non-linearity f. We use the following concept to simplify the Conley-Morse graphs.

Definition 1.1. Given $Q \in \mathcal{Q}$ let $\{\mathcal{M}_Q(p) \mid p \in \mathsf{P}\}$ denote the set of strongly connected path components of $\mathcal{F}_Q$. We call $\mathcal{M}_Q(p)$ and $\mathcal{M}_Q(q)$ *adjacent* if

$$|\mathcal{M}_Q(p)| \cap |\mathcal{M}_Q(q)| \neq \emptyset.$$

By extending the adjacency transitively, we can define an equivalence relation on $\{\mathcal{M}_Q(p) \mid p \in \mathsf{P}\}$, which is also called the adjacency.

If $p_* \subset \mathsf{P}$ indexes an *adjacency class*, an equivalence class of the adjacency equivalence relation, of $\{\mathcal{M}_Q(p) \mid p \in \mathsf{P}\}$, then

$$\bar{\mathcal{M}}_Q(p_*) := \bigcup_{p \in p_*} \mathcal{M}_Q(p).$$

By construction $\left|\bar{\mathcal{M}}_Q(p_*)\right| \subset X$ is a closed interval.

In the next section, we shall prove that, in a saddle-node neighborhood, the adjacency classes of the Morse sets indeed form a coarser Morse decomposition. We call the Morse decomposition and the corresponding Conley-Morse graph *adjacency reduced.*

The following theorem indicates that, as far as the adjacency reduced Conley-Morse graphs are concerned, the computational procedure of [1] is capable of recovering the continuation graph (1.8).

Theorem 1.2. *Assume that $X \times \Lambda$ is a saddle-node neighborhood for f. For sufficiently small $\delta, \nu > 0$, (1.8) is the associated continuation graph of the adjacency reduced Conley-Morse graphs for f over $X \times \Lambda$.*

The parameter region for which the adjacency reduced Conley-Morse graph is $CMG(B)$ limits to the point at which the saddle-node bifurcation occurs, as $\delta, \nu \to 0$.

Furthermore, the geometric realization of the union of the adjacency reduced Morse sets in $X \times \Lambda$ converges to the saddle-node bifurcation diagram as $\delta, \nu \to 0$, while δ/ν remains bounded from above and below by some positive constants.

§ 2. Notation and Statement of Results

This section provides precise statements that relate the continuation graph information to the invariant sets associated with a saddle-node bifurcation. With this in mind we assume that

$$\begin{aligned} f\colon \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, \lambda) &\mapsto f(x, \lambda) = f_\lambda(x) \end{aligned}$$

is C^2 and satisfies the following two conditions:

SN1 $f(0,0) = 0$ and $f_x(0,0) = 1$;

SN2 $f_\lambda(0,0) < 0$ and $f_{xx}(0,0) > 0$.

Note that a similar result as in the following holds in the case of $f_\lambda(0,0) > 0$ or $f_{xx}(0,0) < 0$. Define

$$\varphi(x, \lambda) := f(x, \lambda) - x. \tag{2.1}$$

It is a classical result that **SN1** and **SN2** imply the existence of a saddle-node bifurcation [4, 7] at the point $(0,0)$ and that the following conditions hold.

Lemma 2.1. *Given* **SN1** *and* **SN2** *there exist compact intervals $X \subset \mathbb{R}$ and $\Lambda \subset \mathbb{R}$ such that the following conditions hold.*

(i) *There exist positive constants* $a_0 \leq |f_\lambda(x,\lambda)| \leq a_1$ *and* $b_0 \leq |f_{xx}(x,\lambda)| \leq b_1$ *for all* $(x,\lambda) \in X \times \Lambda$.

(ii) *There exists a unique* C^1 *function* $\zeta\colon X \to \Lambda$ *such that*

$$f(x,\zeta(x)) = x, \quad \zeta(0) = 0.$$

Furthermore, $\lambda = \zeta(x)$ *is parabola-like in the sense that* $\zeta(0) = 0, \zeta'(0) = 0$, *and* $\zeta''(0) > 0$.

(iii) *For* φ *defined by (2.1) there exists a unique* C^1 *function* $\xi\colon \Lambda \to X$ *such that*

$$\varphi_x(\xi(\lambda),\lambda) \equiv 0, \quad \xi(0) = 0. \tag{2.2}$$

Furthermore, for $\lambda > 0$, *there exist unique* C^1 *curves* $x^\pm$ *such that* $x^-(\lambda) < \xi(\lambda) < x^+(\lambda)$ *and* $\zeta(x^\pm(\lambda)) = \lambda$.

(iv) *For all* $(x,\lambda) \in X \times \Lambda$,

$$f_x(x,\lambda) > 0.$$

Definition 2.2. Let $X = [X_-, X_+] \subset \mathbb{R}$ and $\Lambda = [\Lambda_-, \Lambda_+] \subset \mathbb{R}$ be intervals satisfying the conditions of Lemma 2.1. Note that this implies that $X_- < 0 < X_+$ and $\Lambda_- < 0 < \Lambda_+$. The region $X \times \Lambda$ is a *saddle-node neighborhood* if the following additional constraints are satisfied.

(i) For each $\lambda \in \Lambda$, X is an isolating neighborhood for f_λ.

(ii) If $\lambda \in [\Lambda_-, 0)$, then $\mathrm{Inv}\,(X, f_\lambda) = \emptyset$.

(iii) If $\lambda = 0$, then $\mathrm{Inv}\,(X, f_\lambda) = 0$.

(iv) If $\lambda \in (0, \Lambda_+]$, then $\mathrm{Inv}\,(X, f_\lambda)$ consists of $x^\pm(\lambda)$ and heteroclinic orbits from $x^+(\lambda)$ to $x^-(\lambda)$.

To obtain a continuation graph we need an understanding of the parameter values associated with these Conley-Morse graphs. Bounds on the local dynamics as a function of the parameter values will be obtained via the following sets

$$T_0^\pm(\delta,\nu) := \{(x,\lambda) \in X \times \Lambda \mid \pm\varphi(x,\lambda) > 0\} \tag{2.3}$$

$$T^+(\delta,\nu) := \{(x,\lambda) \in X \times \Lambda \mid \varphi(x,\lambda+\nu) > \delta\} \tag{2.4}$$

$$T^-(\delta,\nu) := \{(x,\lambda) \in X \times \Lambda \mid \varphi(x+\delta,\lambda) < -\delta\}\,. \tag{2.5}$$

The boundary of $T_0^\pm(\delta,\nu)$ is given by the equation $\varphi(x,\lambda) = 0$, which is clearly parabola-like in the sense of Lemma 2.1 (ii) by the implicit function theorem. The same conclusion holds for the boundary curves of $T^\pm(\delta,\nu)$.

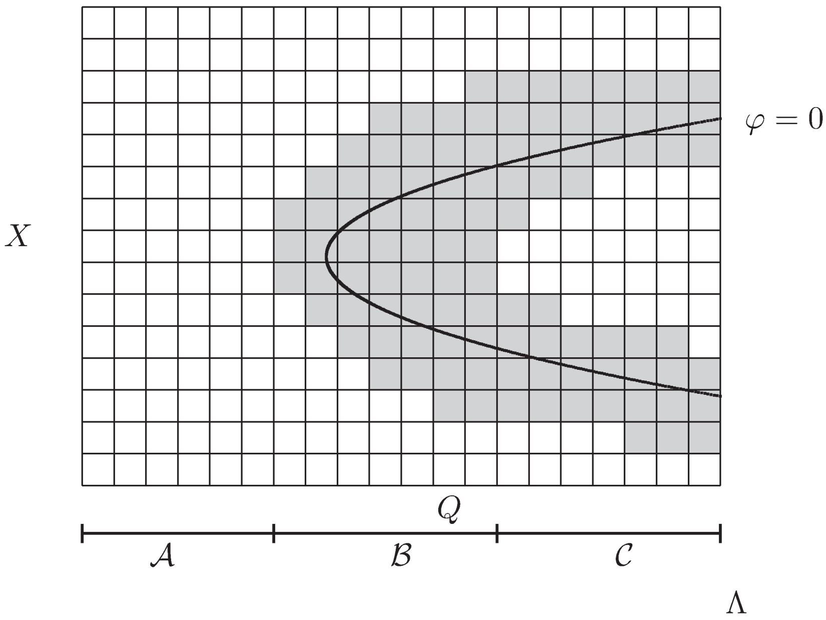

Figure 1. The grid defined by $\mathcal{X} \times \mathcal{Q}$. Given $Q \in \mathcal{Q}$, the shaded set of squares $\{G_j \times Q\}$ represents the set of grid element $\{G_j\}$ which lie in strongly connected path components of $\mathcal{F}_Q$. Theorem 2.8 implies that there are precisely three types of parameter regions in this figure. Theorem 2.9 implies that for each Q the collection of shaded squares is empty, connected or consists of exactly two distinct components. It also guarantees that these components can be used to define Conley-Morse graphs. Together Theorems 2.11 and 2.12 imply that the shaded region converges to the curve of equilibria as δ and ν tend to 0.

Lemma 2.3. *The boundary curves of* $T_0^{\pm}(\delta,\nu)$ *and* $T^{\pm}(\delta,\nu)$ *are all parabola-like.*

From now on we restrict our attention to the saddle-node neighborhood. Define the grids $\mathcal{X}$, $\mathcal{Q}$, and the multivalued maps $\mathcal{F}_Q$ as in (1.5), (1.6) and (1.7), respectively. Figure 1 is included to provide intuition for the results that are presented in this section.

To emphasize the combinatorial nature of the computations we define the following sets of grid points:

$$\mathcal{P}_{\mathcal{X}}(\delta,\nu) := \{x_j \mid G = [x_j, x_j+\delta] \in \mathcal{X}\}\,, \tag{2.6}$$

$$\mathcal{P}_{\mathcal{Q}}(\delta,\nu) := \{\lambda_i \mid Q = [\lambda_i, \lambda_i+\nu] \in \mathcal{Q}\}\,, \tag{2.7}$$

$$\mathcal{P}_{\mathcal{Z}}(\delta,\nu) := \{(x_j,\lambda_i) \mid x_j \in \mathcal{P}_{\mathcal{X}}, \lambda_i \in \mathcal{P}_{\mathcal{Q}}\}\,. \tag{2.8}$$

For $G = [x_j, x_j+\delta] \in \mathcal{X}$ and $Q = [\lambda_i, \lambda_i+\nu] \in \mathcal{Q}$, define $p(G) = x_j$, $p(Q) = \lambda_i$, and $p(Z) = (x_j, \lambda_i)$ where $Z = G \times Q \in \mathcal{Z} = \mathcal{X} \times \mathcal{Q}$. These are the grid points associated with the grid elements in the phase space and parameter space. Also define the sets of grid points in the regions $T_0^{\pm}(\delta,\nu)$ and $T^{\pm}(\delta,\nu)$ as follows:

$$\mathcal{T}_0^{\pm}(\delta,\nu) := \mathcal{P}_{\mathcal{Z}}(\delta,\nu) \cap T_0^{\pm}(\delta,\nu), \qquad \mathcal{T}^{\pm}(\delta,\nu) := \mathcal{P}_{\mathcal{Z}}(\delta,\nu) \cap T^{\pm}(\delta,\nu)$$

Lemma 2.4. *In $X \times \Lambda$ and for $Z \in \mathcal{Z}$, if $p(Z) \in \mathcal{T}^+(\delta,\nu)$, then $|Z| \subset T_0^+(\delta,\nu)$.*

Proof: Observe that $(x,\lambda) \in T^+(\delta,\nu)$ is equivalent to $\varphi(x,\lambda+\nu) > \delta$, which means $f(x,\lambda+\nu) > x+\delta$. Suppose $[x,x+\delta] \times \{\lambda+\nu\}$ is not contained in $T_0^+(\delta,\nu)$, then there is an $x' \in [x,x+\delta]$ satisfying $\varphi(x',\lambda+\nu) \le 0$, which means $f(x',\lambda+\nu) \le x'$. Hence there must be an $x'' \in [x,x+\delta]$ with $\frac{\partial f}{\partial x}(x'',\lambda+\nu) \le 0$, which contradicts to the assumption. ■

Similarly we have:

Lemma 2.5. *In $X \times \Lambda$, and for $Z \in \mathcal{Z}$, if $p(Z) \in \mathcal{T}^-(\delta,\nu)$, then $|Z| \subset T_0^-(\delta,\nu)$.*

Corollary 2.6. *In $X \times \Lambda$, $T^+(\delta,\nu) \subset T_0^+(\delta,\nu)$ and $T^-(\delta,\nu) \subset T_0^-(\delta,\nu)$.*

Now we are ready to define the sets of parameter grid elements corresponding to different phases of the saddle-node bifurcation.

Definition 2.7.

$$\mathcal{A} := \{Q \in \mathcal{Q} \mid \forall G \in \mathcal{X}, G \notin \mathcal{F}_Q(G)\}, \tag{2.9}$$

$$\mathcal{C} := \left\{Q \in \mathcal{Q} \mid \exists G \in \mathcal{X} \text{ s.t. } |G \times Q| \subset T_0^-(\delta,\nu),\ G \notin \mathcal{F}_Q(G)\right\}, \tag{2.10}$$

$$\mathcal{B} := \mathcal{Q} \setminus (\mathcal{A} \cup \mathcal{C}). \tag{2.11}$$

Let $|\mathcal{A}| = \cup_{Q\in\mathcal{A}} Q$, $|\mathcal{B}| = \cup_{Q\in\mathcal{B}} Q$, $|\mathcal{C}| = \cup_{Q\in\mathcal{C}} Q$ are the corresponding regions in the parameter space.

Recall from Lemma 2.1 that we can assume

$$a_0 \le |f_\lambda(x,\lambda)| \le a_1, \qquad b_0 \le |f_{xx}(x,\lambda)| \le b_1 \tag{H}$$

for any $(x,\lambda) \in X \times \Lambda$.

Theorem 2.8. *The regions $|\mathcal{A}|$, $|\mathcal{B}|$ and $|\mathcal{C}|$ are intervals. Furthermore,*

$$|\mathcal{A}| = \left[\Lambda_-, \Lambda^a_{\delta,\nu}\right], \quad |\mathcal{B}| = \left[\Lambda^a_{\delta,\nu}, \Lambda^c_{\delta,\nu}\right], \quad |\mathcal{C}| = \left[\Lambda^c_{\delta,\nu}, \Lambda_+\right],$$

where

$$\Lambda^a_{\delta,\nu} = \max\left\{\lambda+\nu \in \mathcal{P}_{\mathcal{Q}} \mid \forall x \in \mathcal{P}_{\mathcal{X}},\ (x,\lambda) \in T^+(\delta,\nu)\right\},$$
$$\Lambda^c_{\delta,\nu} = \min\left\{\lambda \in \mathcal{P}_{\mathcal{Q}} \mid \exists x \in \mathcal{P}_{\mathcal{X}},\ (x,\lambda) \in T^-(\delta,\nu)\right\}.$$

We shall show that these sets correspond to the parameter intervals A, B, C in the saddle-node continuation graph. In order to prove this, we shall recall the notion of adjacency introduced in Definition 1.1.

The next theorem shows that the structure of the Conley-Morse graph is essentially what we have expected.

Theorem 2.9. *Let $X \times \Lambda$ be a saddle-node neighborhood with grids $\mathcal{X}$, $\mathcal{Q}$ and multivalued maps $\mathcal{F}_Q$ as defined in (1.5), (1.6) and (1.7), respectively. For $Q \in \mathcal{Q}$, let $\bar{\mathsf{P}}_Q$ denote the set of adjacency classes of strongly connected path components. Then $\bar{\mathsf{P}}_Q$ is either the empty set, a singleton set $\{p_0\}$, or contains exactly two elements $\{p_0, p_1\}$. More precisely,*

(i) *If $Q \in \mathcal{A}$, then $\bar{\mathsf{P}}_Q = \emptyset$ and $\bar{\mathcal{M}}_Q = \emptyset$.*

(ii) *If $Q \in \mathcal{B}$, then $\bar{\mathsf{P}}_Q = \{p_0\}$, and $\bar{\mathcal{M}}_Q(p_0) = \mathcal{S}_Q$ with $\mathrm{Con}(\mathcal{S}_Q) = 0$.*

(iii) *If $Q \in \mathcal{C}$, then $\bar{\mathsf{P}}_Q = \{p_0, p_1\}$, and $\bar{\mathcal{M}}_Q(p_0) = \mathcal{A}_Q$, $\bar{\mathcal{M}}_Q(p_1) = \mathcal{R}_Q$, where $\{\mathcal{A}_Q, \mathcal{R}_Q\}$ forms a combinatorial attractor-repeller pair for $\mathcal{F}_Q$.*

Define $\bar{\mathsf{P}}(A) = \emptyset$, $\bar{\mathsf{P}}(B) = \{p_0\}$, $\bar{\mathsf{P}}(C) = \{p_0, p_1\}$. If $\bar{\mathsf{P}}_Q = \bar{\mathsf{P}}(V)$, where $V = A$, B, or C, then there is a Conley-Morse graph valid over Q of the form $CMG(V)$ as given by (1.2), (1.3), or (1.4). Furthermore, if $\bar{\mathsf{P}}(V) \neq \emptyset$, then the node p_*, $* = 0, 1$ corresponds to the Morse set defined by the isolating neighborhood $\left|\bar{\mathcal{M}}_Q(p_*)\right|$.

Definition 2.10. We refer to the Conley-Morse graphs of Theorem 2.9 as *adjacency reduced.* An adjacency reduced continuation graph is a continuation graph in which all the Conley-Morse graphs are adjacency reduced.

Theorem 2.9 provides a description of possible Conley-Morse graphs associated with the saddle-node bifurcations. The following Theorem implies that the adjacency reduced continuation graph for the saddle-node bifurcation takes the form of (1.8).

Theorems 2.8 and 2.9 guarantee that the shape of the shaded region in Figure 1 is correct. To show that the database construction can be used to identify the location of the bifurcation point we prove the following theorem from which it follows that

$$\lim_{\delta,\nu \to 0} \Lambda^a_{\delta,\nu} = 0 \quad \text{and} \quad \lim_{\delta,\nu \to 0} \Lambda^c_{\delta,\nu} = 0$$

under the assumption that while taking the limit the ratio of δ and ν remains bounded.

Theorem 2.11. *The boundary points $\Lambda^a_{\delta,\nu}$ and $\Lambda^c_{\delta,\nu}$ are estimated as follows:*

(i)
$$-\left(\frac{\delta}{a_0 - K_1\nu} + \frac{\nu}{1 - K_2\nu}\right) \leq \Lambda^a_{\delta,\nu} \leq 0$$

for some $K_1, K_2 > 0$.

(ii)

$$\Lambda^c_{\delta,\nu} \leq \frac{\delta}{a_0}\left(1 + \frac{b_1\delta}{8}\right)$$

Finally, we shall give estimates of the sizes of each of the above reduced Morse sets. Theorem 1.2 clearly follows from this and obove theorems.

Theorem 2.12. *Let $Q = [\lambda_0, \lambda_0 + \nu] \in \mathcal{Q}$, and choose $\delta > 0$ and $\nu > 0$ sufficiently small and satisfying $1/L < \delta/\nu < L$ for some $L > 1$.*

(i) *If $\lambda_0 + \nu \leq 2\delta/a_0$, especially, if $Q \in \mathcal{B}$, then*

$$\ell(|\mathcal{S}_Q|) \leq K_3\sqrt{\delta}$$

for some $K_3 > 0$, where $\ell(I)$ stands for the length of an interval I.

(ii) *If $\lambda_0 \geq 2\delta/a_0$, then, $Q \in \mathcal{C}$ and*

$$\max\{\ell(|\mathcal{R}_Q|), \ell(|\mathcal{A}_Q|)\} \leq K_4\sqrt{\delta}$$

for some $K_4 > 0$.

§ 3. Proof of Convergence of Adjacency Reduced Continuation Graphs

In this section, we shall give proofs of Theorems in §2. Let $G = [x_0, x_0 + \delta] \in \mathcal{X}$ and $Q = [\lambda_0, \lambda_0 + \nu] \in \mathcal{Q}$ be grid elements in the phase space and the parameter space, respectively.

Proof of Theorem 2.8:

(i) Clearly, $|\mathcal{A}| \subset [\Lambda_-, 0)$, since otherwise there must be a fixed point and hence there must exist some G and Q with $\mathcal{F}(G, Q) \cap G \neq \emptyset$, which is a contradiction.

It follows from $f(x, \lambda) > x$ for any $(x, \lambda) \in T_0^+(\delta, \nu)$ that $Q = [\lambda_0, \lambda_0 + \nu] \in \mathcal{A}$ is equivalent to

$$\min f(x_0, Q) > x_0 + \delta \quad \text{for} \quad \forall\, G = [x_0, x_0 + \delta] \in \mathcal{X}.$$

Note that $\min f(x_0, Q) = f(x_0, \lambda_0 + \nu)$ since $\frac{\partial f}{\partial \lambda} < 0$ on $X \times \Lambda$, and therefore, $Q \in \mathcal{A}$ is equivalent to $\varphi(x_0, \lambda_0 + \nu) > \delta$ for all $x_0 \in \mathcal{P}_{\mathcal{X}}$, which is equivalent to $(x_0, \lambda_0) \in T^+(\delta, \nu)$ for all $x_0 \in \mathcal{P}_{\mathcal{X}}$. By the definition of the parameter value $\Lambda^a_{\delta,\nu}$, we obtain the conclusion.

(ii) Clearly $|\mathcal{C}| \subset [0, \Lambda_+]$, since $T_0^-(\delta, \nu) \subset X \times [0, \Lambda_+]$. The condition $Q = [\lambda_0, \lambda_0 + \nu] \in \mathcal{C}$ is equivalent to the existence of $G = [x_0, x_0 + \delta] \in \mathcal{X}$ satisfying that $|G \times Q| \subset T_0^-(\delta, \nu)$ and that $\max f(x_0 + \delta, Q) = f(x_0 + \delta, \lambda_0) < x_0$, which is then equivalent to the existence of $G \in \mathcal{X}$ with $|G \times Q| \subset T_0^-(\delta, \nu)$ and $(x_0, \lambda_0) \in T^-(\delta, \nu)$.

By Lemma 2.5, the latter condition $(x_0, \lambda_0) \in T^-(\delta, \nu)$ implies $G \times \{\lambda_0\} \subset T_0^-(\delta, \nu)$. Moreover, $(x_0, \lambda_0) \in T^-(\delta, \nu)$ implies $(x_0, \lambda) \in T^-(\delta, \nu)$ for all $\lambda > \lambda_0$, since $\frac{\partial f}{\partial \lambda} < 0$, which then implies $|G \times Q| \subset T_0^-(\delta, \nu)$. Therefore, $Q \in \mathcal{C}$ is equivalent to the existence of $G \in \mathcal{X}$ with $|G \times Q| \subset T_0^-(\delta, \nu)$.

(iii) is obvious from (i) and (ii), and the definition of $\mathcal{B}$. ■

Proof of Theorem 2.11:

(i) We need to study the extremal value of the projection of the boundary of $T^+(\delta, \nu)$ to the λ-axis, and thus consider the equations

$$\begin{cases} \varphi(x, \lambda + \nu) - \delta = 0 \\ \varphi_x(x, \lambda) = 0. \end{cases}$$

These are the equations of $(x, \lambda, \delta, \nu)$ which can be solved for (x, λ) around the origin by the implicit function theorem to obtain

$$(x, \lambda) = (X(\delta, \nu), \Lambda(\delta, \nu)) \quad \text{with} \quad (X(0), \Lambda(0)) = (0, 0).$$

Observe that $\Lambda(\delta, \nu) \leq \Lambda^a_{\delta,\nu}$, since $Q = [\lambda_0, \lambda_0 + \nu] \subset [\Lambda_-, \Lambda(\delta, \nu)]$ implies $|G \times Q| \subset T^+(\delta, \nu)$ for any $G = [x_0, x_0 + \delta] \in \mathcal{X}$, which then implies $(x_0, \lambda_0) \in T^+(\delta, \nu)$ for any $x_0 \in \mathcal{P}_{\mathcal{X}}$, and hence $Q \in \mathcal{A}$. Note that the interval $[\Lambda_-, \Lambda(\delta, \nu)]$ is not necessarily equal to $|\mathcal{A}|$.

Now recall $x = \xi(\lambda)$ is given in Lemma 2.1 (iii). By the the implicit function theorem, the derivative $\xi'(\lambda)$ can be computed as

$$\xi'(\lambda) = -\frac{\varphi_{x\lambda}(\xi(\lambda), \lambda)}{\varphi_{xx}(\xi(\lambda), \lambda)},$$

and hence $\xi'(\lambda)$ is bounded over the parameter space Λ. By the definition of the function $\lambda = \Lambda(\delta, \nu)$ given above, $\lambda = \Lambda(\delta, \nu)$ is a function which is implicitly given by solving the equation $\varphi(\xi(\lambda), \lambda + \nu) - \delta = 0$. Therefore, again by the implicit function theorem, we can compute

$$\frac{\partial \Lambda}{\partial \delta}(\delta, \nu) = \frac{1}{\varphi_x(\xi(\lambda), \lambda + \nu) \cdot \xi'(\lambda) + \varphi_\lambda(\xi(\lambda), \lambda + \nu)}.$$

Using $\varphi_x(\xi(\lambda), \lambda + \nu) = \varphi_x(\xi(\lambda), \lambda) + \varphi_{xx}(\xi(\lambda), \bar{\lambda})\nu$ for some $\bar{\lambda}$ together with $\varphi_x(\xi(\lambda), \lambda) = 0$ and $\varphi_{xx}(x, \lambda) \neq 0$ for $(x, \lambda) \in X \times \Lambda$, we have

$$\left| \frac{\partial \Lambda}{\partial \delta}(\delta, \nu) \right| \leq \frac{1}{a_0 - K_1 \nu},$$

where $K_1 = \max_{X \times \Lambda} |\varphi_{xx}(x, \lambda)| \cdot \max_\Lambda |\xi'(\lambda)|$.

Similarly we obtain

$$\left|\frac{\partial\Lambda}{\partial\nu}(\delta,\nu)\right| \le \frac{1}{1-K_2\nu}$$

for some $K_2 > 0$. Therefore, we obtain

$$\Lambda^a_{\delta,\nu} \ge \Lambda(\delta,\nu) = \Lambda(0) + \frac{\partial\Lambda}{\partial\delta}(*)\delta + \frac{\partial\Lambda}{\partial\nu}(*)\nu \ge -\left(\frac{\delta}{a_0-K_1\nu} + \frac{\nu}{1-K_2\nu}\right).$$

(ii) Consider the equation for the boundary of $T^-(\delta,\nu)$ given by $\varphi(x+\delta,\lambda)+\delta = 0$. Similarly to Lemma 2.1 (iii), there exist functions $x^{\pm}_{\delta}(\lambda)$ which give the two solutions of $\varphi(x^{\pm}_{\delta}(\lambda)+\delta,\lambda)+\delta = 0$ with $x^+_\delta(\lambda) > x^-_\delta(\lambda)$ for any $\lambda \in |\mathcal{C}|$. Now define $\Lambda^2(\delta)$ to be the minimum of λ for which $x^+_\delta(\lambda) - x^-_\delta(\lambda) \ge \delta$.

It holds that $\Lambda^2(\delta) > \Lambda^c_{\delta,\nu}$, since $Q = [\lambda_0, \lambda_0+\nu] \subset [\Lambda^2(\delta), \Lambda_+]$ implies the existence of $G = [x_0, x_0+\delta] \in \mathcal{X}$ with $(x_0,\lambda_0) \in T^-(\delta,\nu)$, hence $Q \in \mathcal{C}$. Thus we need to give estimates of $x^+_\delta(\lambda) - x^-_\delta(\lambda)$ and $\Lambda^2(\delta)$.

From the definition of $x^{\pm}_\delta(\lambda)$, we have

$$\begin{aligned}
-\delta &= \varphi(x^{\pm}_\delta(\lambda)+\delta,\lambda)\\
&= \varphi(\xi(\lambda),\lambda) + \varphi_x(\xi(\lambda),\lambda)\cdot(x^{\pm}_\delta(\lambda)+\delta-\xi(\lambda))\\
&\qquad + \frac{1}{2}\varphi_{xx}(*)\cdot(x^{\pm}_\delta(\lambda)+\delta-\xi(\lambda))^2\\
&= \{\varphi_c(\xi(\lambda),\lambda)\cdot\xi'(\lambda) + \varphi_\lambda(*)\cdot\lambda\}\\
&\qquad + \frac{1}{2}\varphi_{xx}(*)\cdot(x^{\pm}_\delta(\lambda)+\delta-\xi(\lambda))^2\\
&= \varphi_\lambda(*)\lambda + \frac{1}{2}\varphi_{xx}(*)\cdot(x^{\pm}_\delta(\lambda)+\delta-\xi(\lambda))^2.
\end{aligned}$$

It follows from above that

$$\frac{2(a_0\lambda-\delta)}{b_1} \le (x^{\pm}_\delta(\lambda)+\delta-\xi(\lambda))^2 = \frac{-\varphi_\lambda(*)\lambda-\delta}{\frac{1}{2}\varphi_{xx}(*)} \le \frac{2(a_1\lambda+\delta)}{b_0},$$

which shows

$$\sqrt{\frac{2(a_0\lambda-\delta)}{b_1}} \le x^+_\delta(\lambda)+\delta-\xi(\lambda) \le \sqrt{\frac{2(a_1\lambda+\delta)}{b_0}}$$

and

$$\sqrt{\frac{2(a_0\lambda-\delta)}{b_1}} \le -\{x^-_\delta(\lambda)+\delta-\xi(\lambda)\} \le \sqrt{\frac{2(a_1\lambda+\delta)}{b_0}}.$$

Adding these inequalities termwise, we obtain

$$2\sqrt{\frac{2(a_0\lambda-\delta)}{b_1}} \le x^+_\delta(\lambda) - x^-_\delta(\lambda) \le 2\sqrt{\frac{2(a_1\lambda+\delta)}{b_0}},$$

and thus a sufficient condition for $\lambda \geq \Lambda^2(\delta)$ is $2\sqrt{\dfrac{2(a_0\lambda - \delta)}{b_1}} \geq \delta$, which is equivalent to $\lambda \geq \dfrac{\delta}{a_0}\left(1 + \dfrac{b_1\delta}{8}\right)$. Therefore, we have shown that $\Lambda^c_{\delta,\nu} \leq \Lambda^2(\delta) \leq \dfrac{\delta}{a_0}\left(1 + \dfrac{b_1\delta}{8}\right)$. ■

Proof of Theorem 2.9:

(i) is trivial from the definition of $\mathcal{A}$.

(ii) Define, for $Q \in \mathcal{B}$, $\mathcal{S}_Q = \{G \in \mathcal{X} \mid G \in \mathcal{F}_Q(G)\}$. From the definition of $\mathcal{B}$, clearly $\mathcal{S}_Q \neq \emptyset$. We claim the set $|\mathcal{S}_Q|$ which is defined by $\cup_{G\in\mathcal{S}_Q}|G|$ is an interval. Suppose $|\mathcal{S}_Q|$ is not connected, there must exist a $G_0 = [x_0, x_0 + \delta] \in \mathcal{X}$ such that $|G_0|$ is contained in the hull of $|\mathcal{S}_Q|$ but $G_0 \notin \mathcal{S}_Q$.

There are two possibilities: either $f(G_0, Q) < \min G_0$ or $f(G_0, Q) > \max G_0$. In the former case, for any $\lambda \in Q$, it must be $f(x_0+\delta) < x_0$, or equivalently, $\varphi(x_0+\delta, \lambda) < -\delta$, hence $(x_0, \lambda) \in T^-(\delta, \nu)$ for all $\lambda \in Q$, which means $\lambda_0 \geq \Lambda^c_{\delta,\nu}$, which is a contradiction to $Q \in \mathcal{B}$.

In the latter case, we similarly have $(x_0, \lambda - \nu) \in T^+(\delta, \nu)$ for all $\lambda \in Q$. Since $T^+(\delta, \nu)$ has at most two connected components, either $(x_0, Q - \nu) \subset T^+(\delta, \nu)$ for any $x > x_0$ or $(x_0, Q-\nu) \subset T^+(\delta, \nu)$ for any $x < x_0$ must hold. This shows that, if $G_0 \notin \mathcal{S}_Q$, then either $G \notin \mathcal{S}_Q$ for any $G > G_0$ or $G \notin \mathcal{S}_Q$ for any $G < G_0$ must hold. Therefore $|\mathcal{S}_Q|$ has to be an interval.

Clearly $|\mathcal{S}_Q|$ is an isolating neighborhood with trivial Conley index, and $\mathcal{S}_Q$ is an adjacency class of $\mathcal{M}_Q$. Since any $G \notin \mathcal{S}_Q$ is not recurrent, we conclude that $\bar{\mathcal{M}}_Q = \{\mathcal{S}_Q\}$.

(iii) For a $Q = [\lambda_0, \lambda_0 + \nu] \in \mathcal{C}$, define $\mathcal{S}_Q$ as above. We claim that $\mathcal{S}_Q$ can be decomposed into disjoint sets $\mathcal{R}_Q$ and $\mathcal{A}_Q$ such that both $|\mathcal{R}_Q|$ and $|\mathcal{A}_Q|$ are disjoint intervals. To show this, it follows from $Q \in \mathcal{C}$ that there exists a $G_* \in \mathcal{X}$ for which $|G_* \times Q| \subset T_0^-(\delta, \nu)$ and $G_* \notin \mathcal{F}_Q(G_*)$. Define

$$\mathcal{R}_Q = \{G \in \mathcal{S}_Q \mid G > G_*\} \quad \mathcal{A}_Q = \{G \in \mathcal{S}_Q \mid G < G_*\},$$

then clearly $\mathcal{S}_Q = \mathcal{R}_Q \cup \mathcal{A}_Q$.

Suppose $|\mathcal{R}_Q|$ is not an interval, then there must exist a $G_0 = [x_0, x_0 + \delta] \in \mathcal{X}$ such that $|G_0|$ is contained in the hull of $|\mathcal{R}_Q|$ but $f(G_0 \times Q) \cap \mathrm{int}G_0 = \emptyset$. There are two possibilities in this case: either $f(G_0 \times Q) > \max G_0$ or $f(G_0 \times Q) < \min G_0$, which are equivalent, respectively, to either $f(x_0, Q) > x_0 + \delta$ or $f(x_0, Q) < x_0$. In the former case, we have $(x_0, \lambda_0) \in T^+(\delta, \nu)$ and $x_0 > x^+_\delta(\lambda_0)$, and hence, for any $G > G_0$, $G \times Q \subset T^+(\delta, \nu)$, which is a contradiction, since it would mean there is no $G \in \mathcal{R}_Q$ with $G > G_0$. In the latter case, we have $(x_0, \lambda_0) \in T^-(\delta, \nu)$, hence $G \times Q \subset T^-(\delta, \nu)$ for any G with $G_* < G < G_0$, which is again a contradiction. Therefore $|\mathcal{R}_Q|$ must be an interval. Similarly for $|\mathcal{A}_Q|$. It is also clear from the definition that $|\mathcal{R}_Q| \cap |\mathcal{A}_Q| = \emptyset$, and that the sets $\mathcal{R}_Q$ and $\mathcal{A}_Q$ are adjacency reduced Morse sets which form an attractor-repeller pair. Thus we have shown that $\bar{\mathcal{M}}_Q = \{\mathcal{R}_Q, \mathcal{A}_Q\}$. This completes the proof. ■

Proof of Theorem 2.12:

(i) Observe first that, if $Q = [\lambda_0, \lambda_0 + \nu] \in \mathcal{B}$, then, from Theorem 2.11 (ii), we have $\lambda_0 + \nu \leq \dfrac{1}{a_0}\left(1 + \dfrac{b_1\delta}{8}\right) \leq \dfrac{2\delta}{a_0}$.

From the proof of Theorem 2.9 (ii), we have shown, for $Q = [\lambda_0, \lambda_0 + \nu] \in \mathcal{Q}$, that

$$|\mathcal{S}_Q| \subset [\tilde{x}^-_{\delta,\nu}(\lambda_0), \tilde{x}^+_{\delta,\nu}(\lambda_0) + \delta]$$

where $x = \tilde{x}^{\pm}_{\delta,\nu}(\lambda)$ are defined as two solutions of the equation for the boundary of $T^+(\delta, \nu)$, namely $\varphi(x, \lambda + \nu) = \delta$.

Similarly to the proof of Theorem 2.11 (ii), we have

$$\tilde{x}^+_{\delta,\nu}(\lambda_0) - \tilde{x}^-_{\delta,\nu}(\lambda_0) \leq 2\sqrt{\frac{2\{a_1(\lambda_0 + \nu) + \delta\}}{b_0}},$$

and thus

$$\ell(|\mathcal{S}_Q|) \leq (\tilde{x}^+_{\delta,\nu}(\lambda_0) + \delta) - \tilde{x}^-_{\delta,\nu}(\lambda_0) \leq 2\sqrt{\frac{2\{a_1(\lambda_0 + \nu) + \delta\}}{b_0}} + \delta.$$

Since we assume $\lambda_0 + \nu \leq 2\delta/a_0 = O(\sqrt{\delta})$, we therefore obtain the conclusion for sufficiently small $\delta > 0$.

(ii) In the proof of Theorem 2.11 (ii), we have shown

$$x^+_\delta(\lambda) - \xi(\lambda) \geq \sqrt{\frac{2(a_0\lambda - \delta)}{b_1}} - \delta.$$

Note that, from the assumption $\lambda_0 \geq 2\delta/a_0$, this implies $x^+_\delta(\lambda) - \xi(\lambda) \geq \sqrt{2\delta/b_1}$ for sufficiently small $\delta > 0$. Note also that the assumption $\lambda_0 \geq 2\delta/a_0$ implies $Q \in \mathcal{C}$.

Recall, from the proof of Theorem 2.9 (iii), that

$$|\mathcal{R}_Q| \subset [x^+\delta(\lambda_0), \tilde{x}^+_{\delta,\nu}(\lambda_0) + \delta], \quad |\mathcal{A}_Q| \subset [\tilde{x}^-_{\delta,\nu}(\lambda_0), x^-_\delta(\lambda_0) + \delta].$$

We shall obtain the upper estimate of $U = \tilde{x}^+_{\delta,\nu}(\lambda) - x^+\delta(\lambda)$ for $\lambda \geq 2\delta/a_0$.

By definition, $\varphi(\tilde{x}^+_{\delta,\nu}(\lambda), \lambda + \nu) = \delta$, and hence

$$\begin{aligned}\delta &= \varphi(\tilde{x}^+_{\delta,\nu}(\lambda), \lambda + \nu)\\ &= \varphi(x^+_\delta(\lambda), \lambda + \nu) + \varphi_x(x^+_\delta(\lambda), \lambda + \nu) \cdot U + \frac{1}{2}\varphi_{xx}(*) \cdot U^2\end{aligned}$$

Let $F = \frac{1}{2}\varphi_{xx}(*)$, $E = \varphi_x(x_\delta^+(\lambda), \lambda+\nu)$, and $D = \varphi(x_\delta^+(\lambda), \lambda+\nu) - \delta$ in the above, then we obtain a quadratic equation $FU^2 + EU + D = 0$ for U, or equivalently,

$$U = \frac{-E + \sqrt{E^2 - 4FD}}{2F} = \frac{-4FD}{2F(E + \sqrt{E^2 - 4FD}} = -\frac{2D}{E + \sqrt{E^2 - 4FD}}.$$

We give estimates of D, E, F. Firstly, by assumption, $0 < b_0/2 \le F \le b_1/2$ over $X \times \Lambda$. Secondly,

$$\begin{aligned} E = \varphi_x(x_\delta^+(\lambda), \lambda+\nu) \ &= \ \varphi_x(x_\delta^+(\lambda), \lambda) + \varphi_{x\lambda}(*)\nu \\ &= \varphi_x(\xi(\lambda), \lambda) + \varphi_{xx}(*) \cdot \{x_\delta^+(\lambda) - \xi(\lambda)\} + \varphi_{x\lambda}(*)\nu \\ &\ge \varphi_{xx}(*)\sqrt{\frac{2\delta}{b_1}} + \varphi_{x\lambda}(*)\nu \end{aligned}$$

Since we choose $\delta, \nu > 0$ sufficiently small and satisfying $1/L < \delta/\nu < L$ for some $L > 1$, it is clear that $E \ge K_5\sqrt{\delta}$ for some $K_5 > 0$. Finally,

$$\begin{aligned} D = \varphi(x_\delta^+(\lambda), \lambda+\nu) - \delta \ &= \ \varphi(x_\delta^+(\lambda), \lambda) + \varphi_\lambda(*)\nu - \delta \\ &= \varphi(x_\delta^+(\lambda) + \delta, \lambda) + \varphi_x(*)\delta + \varphi_\lambda(*)\nu - \delta \\ &= -\delta + \varphi_x(*)\delta + \varphi_\lambda(*)\nu - \delta \ = \ -2\delta + \varphi_x(*)\delta + \varphi_\lambda(*)\nu, \end{aligned}$$

and hence $D \ge -K_6\delta$ for some $K_6 > 0$. Putting all together, we thus obtain $U \le -D/E = O(\sqrt{\delta})$ for sufficiently small $\delta > 0$. Therefore we obtain $\ell(|\mathcal{R}_Q|) \le U + \delta < K_4\sqrt{\delta}$ for some $K_4 > 0$.

An estimate for $\ell(|\mathcal{A}_Q|)$ can be obtained similarly. Thus we have completed the proof. ∎

§ 4. Comments on Saddle-Node Bifurcations in Higher Dimensional Systems

On a qualitative level the results in the previous sections remain valid for higher dimensional maps. As is indicated below this follows from the fact that the essential dynamics associated with a non-degenerate saddle-node bifurcation of maps can be reduced to one space dimension with one parameter([7]). However, to obtain precise estimates, as in the previous sections, requires extra information. For example, one would need to obtain bounds on the effect of the change of coordinates on the grid elements that arise from the nonlinear change of variables used to bring the original system into a "normal form." Since such a calculation is fairly technical, but the number of additional grid elements is only changed by a linear factor as compared to the one-dimensional case, we forego the attempt to provide precise estimates in the general case.

Consider a one-parameter family of smooth maps $F(u,\mu)$ with $u \in \mathbb{R}^n,\ \mu \in \mathbb{R}$ that undergoes a saddle-node bifurcation at (u_0,μ_0), namely u_0 is a fixed point of $F(\cdot,\mu_0)$ whose first derivative $D_uF(u_0,\mu_0)$ has the unity as its simple eigenvalue with modulus one, and all the other eigenvalues are of modulus different from one. Without loss of generality, one may assume $u_0 = 0,\ \mu_0 = 0$, and that $D_uF(0,0)$ takes of the form

$$D_uF(0,0) = \begin{pmatrix} 1 & O \\ O & A_0 \end{pmatrix},$$

where A_0 is an $(n-1)$-dimensional hyperbolic linear map. From the center manifold theory and the partial linearization theorem (Takens [8]), there is a smooth change of coordinates which brings the map into a normal form

$$u = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x,\mu) \\ A(\mu)y \end{pmatrix} \qquad (x \in \mathbb{R},\ y \in \mathbb{R}^{n-1}), \tag{4.1}$$

where the smooth function $f(x,\mu)$ satisfies

$$f(0,0) = 0, \quad f_x(0,0) = 1, \tag{4.2}$$

while $A(\mu)$ is a hyperbolic linear map with $A(0) = A_0$ that depends smoothly on μ. Moreover the non-degeneracy conditions of the saddle-node bifurcation can be formulated as

$$f_\mu(0,0) = a \neq 0, \quad f_{xx}(0,0) = b \neq 0 \tag{4.3}$$

In the case of multi-dimensional parameter $\lambda \in \mathbb{R}^k$, there is a smooth co-dimension one hypersurface as the bifurcation set for the saddle-node bifurcation and, for any curve transverse to the bifurcation hypersurface in $\mathbb{R}^k$, exactly the same bifurcation takes place for the maps. Locally the bifurcation hypersurface is expressed as a graph of a function $\mu_1 = \Sigma(\tilde{\mu})$ where λ is decomposed into $\mu_1 \in \mathbb{R}$ and $\tilde{\mu} \in \mathbb{R}^{k-1}$, say $\lambda = (\mu_1, \tilde{\mu})$, and thus, for a fixed $\tilde{\mu}$, μ_1 can be considered as the bifurcation parameter as discussed in the above. We therefore obtain essentially the same conclusion as in Theorems 1.2, as well as the estimates given in Theorems 2.8, 2.9, 2.11, 2.12, even for the multi-parameter family.

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